Exam in Automatic Control II
Reglerteknik II 5hp (1RT495)

Date: October 25, 2017

Venue: Bergsbrunnagatan 15, sal 1

Responsible teacher: Hans Rosth.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

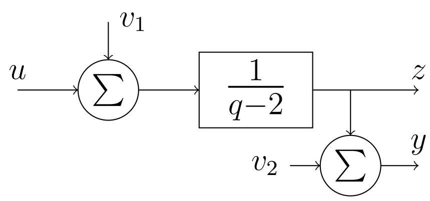
Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments from the autumn semester 2017. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Please use English in your solutions when possible, that would be appreciated!

Good luck!

Problem 1 The block diagram below shows an unstable discrete-time system.



The system is affected by the process noise v_1 and the measurement noise v_2. The noise processes, v_1 and v_2, are zero mean, uncorrelated white noise processes with intensities $R_1 = 24$ and $R_2 = 30$.

The system is to be stabilized by the standard LQG controller

$$u(k) = -L\hat{x}(k|k-1), \tag{1}$$

that minimizes the cost function

$$V = E\left[\rho z^2 + u^2\right], \qquad \rho \geq 0. \tag{2}$$

(a) Give the "standard form" state space representation for the system. **(1p)**
(b) Determine the Kalman filter that provides the estimate $\hat{x}(k|k-1)$. **(3p)**
(c) Determine the feedback gain L in (1). **(3p)**
(d) What are the poles of the closed loop system? **(2p)**
(e) Determine the value of the cost function V in (2) when $\rho = 0$. **(3p)**

Problem 2 Consider the continuous-time system

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t). \end{cases} \tag{3}$$

The zero-order-hold (ZOH) sampled version of (3) is a discrete-time model for which the state vector is identical to $x(t)$ in (3) in the sampling instants, provided that the input $u(t)$ is constant between the sampling instants.

(a) Give the ZOH sampled state space model of (3), expressed in the sampling interval h. **(4p)**
(b) Is the ZOH sampled model in (a) a minimal realization? **(2p)**
(c) Assume that the sampling interval is chosen to $h = \log 2 \Leftrightarrow e^h = 2$. Determine the zero of the ZOH model in (a) for this particular h. **(1p)**

Problem 3

(a) When producing windpower at a wind farm it is important to have good knowledge about the wind variations. For that reason the spectral density of the (short term) wind fluctuations was measured at a potential future wind farm site. It was found that

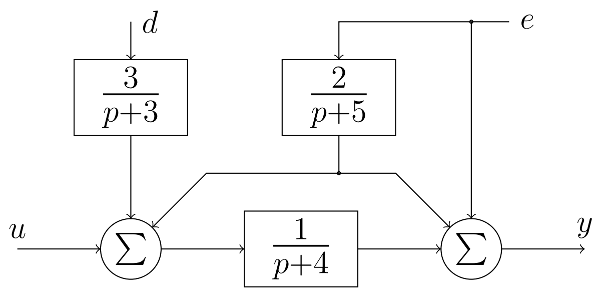
$$\Phi_w(\omega) = \frac{9\omega^2 + 81}{\omega^4 + 29\omega^2 + 100} \tag{4}$$

was a good approximation of the spectral density. Find a stable, minimum phase transfer operator $G_w(p)$ such that the stochastic process

$$w(t) = G_w(p)v(t), \qquad Ev(t) = 0, \quad \Phi_v(\omega) = 1,$$

as a model of the fluctuations, has the spectral density $\Phi_w(\omega)$ in (4). **(4p)**

(b) The block diagram below shows a continuous-time system.



Here d and e are uncorrelated white noise processes, with intensities R_d and R_e respectively. Give a state space representation of the system in the "standard form", that is, find the matrices and vectors A, B, N and C in

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nv_1(t), & Ev_1(t) = 0, \quad \Phi_1(\omega) = R_1, \\ y(t) = Cx(t) + v_2(t), & Ev_2(t) = 0, \quad \Phi_2(\omega) = R_2. \end{cases}$$

You should also specify v_1 and v_2, and give the noise intensities R_1 and R_2, as well as the cross-intensity R_{12}. **(5p)**

Problem 4 A discrete-time system has the state space representation

$$\begin{cases} x(k+1) = \begin{bmatrix} 2 & -0.64 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k), \\ y(k) = \begin{bmatrix} 1 & 0.2 \end{bmatrix} x(k). \end{cases} \tag{5}$$

(a) For which $K \in \mathbb{R}$ is the closed loop system stable, when the proportional control $u(k) = K(r(k) - y(k))$ is used on (5)? **(3p)**

(b) Assume that the state feedback $u(k) = -\begin{bmatrix} 1 & -0.4 \end{bmatrix} \hat{x}(k) + mr(k)$ is used on (5). Determine the gain m so that $y(k) = r(k)$ in stationarity. **(2p)**

(c) The control law in (b) can also be written as

$$u(k) = F_r(q)mr(k) - F_y(q)y(k).$$

Determine $F_r(q)$ and $F_y(q)$ when the observer gain is $K = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$. **(3p)**

Problem 5 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true" / "false" are considered!

(a) Stability is always preserved under zero-order-hold sampling.

(b) Observability is always preserved under zero-order-hold sampling.

(c) A Kalman filter is an observer.

(d) For a Kalman filter, based on a correct model, the *output innovations* are white noise.

(e) MPC is typically implemented as PID controllers.

(f) In MPC the *prediction/output horizon* is typically longer than the *control/input horizon*.

(g) An advantage with MPC is that it can account for bounds and constraints, for example of the type $|u| \leq U_{max}$.

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(7p)**

Problem 6 *The HW bonus points (from the autumn 2017) are exchangeable for this problem.*

(a) Determine the spectral density, $\Phi_y(\omega)$, for the output $y(k)$ of the system

$$\begin{cases} x(k+1) = \begin{bmatrix} 1 & 1 \\ -0.25 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} v_1(k), \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + v_2(k). \end{cases}$$

Here $v_1(k)$ and $v_2(k)$ are two uncorrelated, zero mean white noise processes, with intensities $R_1 = 2$ and $R_2 = 1$. **(4p)**

(b) Control design is in general an iterative process. A senior engineering student is designing a position servo as one part of a bigger project. She is using a DC motor, modeled as $y(t) = \frac{k}{p(1+\tau p)} u(t)$ and controlled by pure state feedback. The student uses the (deterministic) LQ design technique to compute the feedback gain, with the cost function

$$V = \int_0^\infty (Q_1 y^2 + Q_2 u^2) dt.$$

Initially she chooses the weightings $Q_1 = Q_2 = 1$, and then she evaluates the resulting controller in a step response experiment (from reference to output). She then finds that the step response is too slow (the rise time is too long). Suggest new (numerical) values for Q_1 and Q_2 to be used in a new control design, resulting in an improvement when evaluated in the step response experiment. Your choice should be thoroughly motivated! **(3p)**

Solutions to the exam in Automatic Control II, 2017-10-25:

1. (a) From the block diagram we have

$$z = \frac{1}{q-2}(u+v_1) \quad \Leftrightarrow \quad qz = 2z + u + v_1, \quad \text{and} \quad y = z + v_2.$$

A natural choice of state variable is $x = z$, which gives

$$\begin{cases} qx = 2x + u + v_1, \\ z = x, \\ y = x + v_2. \end{cases}$$

(b) The Kalman filter is $q\hat{x} = F\hat{x}+Gu+K(y-H\hat{x})$, with $K = FPH^T(HPH^T + R_2)^{-1}$ where $P = P^T \geq 0$ solves the DARE $P = FPF^T + NR_1N^T - FPH^T(HPH^T + R_2)^{-1}HPF^T$ (here $R_{12} = 0$ since v_1 and v_2 are uncorrelated). Here we have $F = 2$, $G = N = M = H = 1$ so the DARE spells out as

$$P = 2^2P + 24 - \frac{2^2P^2}{P+30} \quad \Leftrightarrow \quad P^2 - 114P - 720 = 0 \quad \Leftrightarrow \quad P = 120.$$

Hence, $K = \frac{2P}{P+30} = \frac{240}{150} = 1.6$.

(c) We have $L = (G^TSG+Q_2)^{-1}G^TSF$, where $S = S^T \geq 0$ solves the DARE $S = F^TSF+M^TQ_1M-F^TSG(G^TSG+Q_2)^{-1}G^TSF$. Here the DARE spells out as

$$S = 2^2S+\rho-\frac{2^2S^2}{S+1} \quad \Leftrightarrow \quad S^2-(3+\rho)S-\rho = 0 \quad \Leftrightarrow \quad S = \frac{3+\rho+\sqrt{9+10\rho+\rho^2}}{2},$$

and thus

$$L = \frac{2S}{S+1} = \frac{3+\rho+\sqrt{9+10\rho+\rho^2}}{\frac{3+\rho+\sqrt{9+10\rho+\rho^2}}{2}+1} = \frac{2\left(3+\rho+\sqrt{9+10\rho+\rho^2}\right)}{5+\rho+\sqrt{9+10\rho+\rho^2}}$$
$$= 2 - \frac{4}{5+\rho+\sqrt{9+10\rho+\rho^2}} = 2 - \frac{5+\rho-\sqrt{9+10\rho+\rho^2}}{4}.$$

(d) The poles of the closed loop system are the the zeros of the pole polynomial

$$\begin{aligned} 0 &= \det(qI - F + GL)\det(qI - F + KH) \\ &= \left(q - 2 + 2 - \frac{5+\rho-\sqrt{9+10\rho+\rho^2}}{4}\right)(q - 2 + 1.6) \\ &= \left(q - \frac{5+\rho-\sqrt{9+10\rho+\rho^2}}{4}\right)(q - 0.4), \end{aligned}$$

so the poles are $\frac{5+\rho-\sqrt{9+10\rho+\rho^2}}{4}$ and (the observer pole) 0.4.
(e) Compute $V = Eu^2$. A state space model of the closed loop system is

$$\begin{cases} qx = Fx + Gu + Nv_1 = Fx - GL\hat{x} + Nv_1, \\ q\hat{x} = F\hat{x} + Gu + K(y - H\hat{x}) = KHx + (F - GL - KH)\hat{x} + Kv_2, \end{cases}$$

or, in block matrix form,

$$\underbrace{q\begin{bmatrix} x \\ \hat{x} \end{bmatrix}}_{=x_c} = \underbrace{\begin{bmatrix} F & -GL \\ KH & F - GL - KH \end{bmatrix}}_{=F_c} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \underbrace{\begin{bmatrix} N & 0 \\ 0 & K \end{bmatrix}}_{=N_c} \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{=v_c}.$$

Then the covariance matrix $\Pi_c = Ex_cx_c^T$ solves the discrete-time Lyapunov equation $\Pi_c = F_c\Pi_cF_c^T + N_cR_cN_c^T$, where $R_c = Ev_cv_c^T$. As $u = -L\hat{x} = -\begin{bmatrix} 0 & L \end{bmatrix} x_c = -L_cx_c$ we get $Euu^T = EL_cx_cx_c^TL_c^T = L_c\Pi_cL_c^T$.

With $\rho = 0$ we have $L = 2 - \frac{5-\sqrt{9}}{4} = 1.5$ and $K = 1.6$, and thus

$$F_c = \begin{bmatrix} 2 & -1.5 \\ 1.6 & -1.1 \end{bmatrix}, \quad N_c = \begin{bmatrix} 1 & 0 \\ 0 & 1.6 \end{bmatrix}, \quad \text{and} \quad R_c = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} = \begin{bmatrix} 24 & 0 \\ 0 & 30 \end{bmatrix}.$$

The Lyapunov equation then is

$$\begin{aligned} \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} &= \begin{bmatrix} 2 & -1.5 \\ 1.6 & -1.1 \end{bmatrix} \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} \begin{bmatrix} 2 & 1.6 \\ -1.5 & -1.1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1.6 \end{bmatrix} \begin{bmatrix} 24 & 0 \\ 0 & 30 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1.6 \end{bmatrix} \\ &= \begin{bmatrix} 4\pi_1 - 6\pi_{12} + 2.25\pi_2 + 24 & 3.2\pi_1 - 4.6\pi_{12} + 1.65\pi_2 \\ 3.2\pi_1 - 4.6\pi_{12} + 1.65\pi_2 & 2.56\pi_1 - 3.52\pi_{12} + 1.21\pi_2 + 76.8 \end{bmatrix}, \end{aligned}$$

which is solved by

$$\Pi_c = \begin{bmatrix} 632 & 512 \\ 512 & 512 \end{bmatrix} \quad \Rightarrow \quad Eu^2 = L_c\Pi_cL_c^T = 1.5^2 \cdot 512 = 1152.$$

2. (a) Theorem 4.1 states that

$$\begin{cases} x(k+1) = Fx(k) + Gu(k), \\ y(k) = Hx(k), \end{cases} \quad \text{with} \quad \begin{aligned} &F = e^{Ah}, \quad G = \int_0^h e^{At}Bdt, \\ &H = C. \end{aligned}$$

Use that $e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$. Here we get

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+2} \end{bmatrix} \quad \Rightarrow \quad e^{At} = \begin{bmatrix} e^{-t} & 0 \\ e^{-t} - e^{-2t} & e^{-2t} \end{bmatrix}$$

Thus,

$$F = \begin{bmatrix} e^{-h} & 0 \\ e^{-h} - e^{-2h} & e^{-2h} \end{bmatrix}, \quad G = \int_0^h \begin{bmatrix} e^{-t} \\ e^{-t} - e^{-2t} \end{bmatrix} = \begin{bmatrix} 1 - e^{-h} \\ 0.5(1 + e^{-2h}) - e^{-h} \end{bmatrix},$$

and $H = C = \begin{bmatrix} 0 & 1 \end{bmatrix}$.
(b) Minimal realization $\Leftrightarrow$ both controllable and observable. Check the controllability and observability matrices:

$$\mathcal{S} = \begin{bmatrix} G & FG \end{bmatrix} = \begin{bmatrix} 1 - e^{-h} & e^{-h}(1 - e^{-h}) \\ 0.5(1 + e^{-2h}) - e^{-h} & e^{-h} - 1.5e^{-2h} + 0.5e^{-4h} \end{bmatrix}$$
$$\Rightarrow \quad \det \mathcal{S} = 0.5e^{-h}(1 - e^{-h})^2(1 - e^{-2h}) > 0 \quad \forall h > 0.$$

$$\mathcal{O} = \begin{bmatrix} C \\ CF \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ e^{-h} - e^{-2h} & e^{-2h} \end{bmatrix} \Rightarrow \quad \det \mathcal{O} = -e^{-h}(1 - e^{-h}) < 0 \quad \forall h > 0.$$

Both controllable and observable for all $h \Rightarrow$ a minimal realization.
(c) With $h = \log 2 \Leftrightarrow e^h = 2$ we get

$$F = \begin{bmatrix} 0.5 & 0 \\ 0.25 & 0.25 \end{bmatrix}, \quad G = \begin{bmatrix} 0.5 \\ 0.125 \end{bmatrix} \quad \Rightarrow \quad G(q) = C(qI - F)^{-1}G$$
$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} q - 0.5 & 0 \\ -0.25 & q - 0.25 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 0.125 \end{bmatrix} = \frac{0.125(q + 0.5)}{(q - 0.5)(q - 0.25)}.$$

The zero is in -0.5.
3. (a) The spectrum $\Phi_w(\omega)$ indicates that

$$G_w(p) = \frac{b_1 p + b_2}{p^2 + a_1 p + a_2}$$

is a candidate for transfer operator. We need to impose that b_1, b_2, a_1 and a_2 all should be non-negative (for stability and minimum phase). The spectral density for the model becomes

$$\Phi_w(\omega) = |G_w(i\omega)|^2 \Phi_v(\omega) = \frac{ib_1\omega + b_2}{-\omega^2 + a_2 + ia_1\omega} \cdot \frac{-ib_1\omega + b_2}{-\omega^2 + a_2 - ia_1\omega}$$
$$= \frac{b_1^2\omega^2 + b_2^2}{(-\omega^2 + a_2)^2 + a_1^2\omega^2} = \frac{b_1^2\omega^2 + b_2^2}{\omega^4 + (a_1^2 - 2a_2)\omega^2 + a_2^2}$$

Comparing the coefficients gives the equation system

$$\begin{cases} b_1^2 = 9, \\ b_2^2 = 81, \\ a_1^2 - 2a_2 = 29, \\ a_2^2 = 100 \end{cases} \Leftrightarrow \begin{cases} b_1 = 3, \\ b_2 = 9, \\ a_1 = 7, \\ a_2 = 10 \end{cases} \Rightarrow \quad G_w(p) = \frac{3p + 9}{p^2 + 7p + 10}.$$

(b) From the block diagram we have

$$y = \underbrace{\frac{1}{p+4}\left(u + \overbrace{\frac{3}{p+3}d}^{=x_2} + \overbrace{\frac{2}{p+5}e}^{=x_3}\right)}_{=x_1} + \overbrace{\frac{2}{p+5}e}^{=x_3} + e.$$

Third order system ⇒ we need three state variables. One possible choice is indicated above. With this particular choice we get

$$\begin{aligned}
x_1 &= \frac{1}{p+4}(u + x_2 + x_3) &&\Leftrightarrow \quad px_1 = -4x_1 + x_2 + x_3 + u,\\
x_2 &= \frac{3}{p+3}d &&\Leftrightarrow \quad px_2 = -3x_2 + 3d,\\
x_3 &= \frac{2}{p+5}e &&\Leftrightarrow \quad px_3 = -5x_3 + 2e,\\
& && \qquad y = x_1 + x_3 + e
\end{aligned}$$

By introducing the state vector $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ we can write this as

$$\begin{cases} \dot{x} = \begin{bmatrix} -4 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} d \\ e \end{bmatrix}, \\ y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x + e. \end{cases}$$

We see that $v_1 = \begin{bmatrix} d & e \end{bmatrix}^T$ and $v_2 = e$, and hence

$$R_1 = \begin{bmatrix} R_d & 0 \\ 0 & R_e \end{bmatrix}, \quad R_2 = R_e, \quad R_{12} = \begin{bmatrix} 0 \\ R_e \end{bmatrix}.$$

4. **(a)** The state space representation is on controller canonical form, so we can directly get the transfer operator as $G(q) = \frac{q+0.2}{q^2-2q+0.64}$. With the proportional feedback the closed loop system will be

$$G_c(q) = \frac{KG(q)}{1+KG(q)} = \frac{K\frac{q+0.2}{q^2-2q+0.64}}{1+K\frac{q+0.2}{q^2-2q+0.64}} = \frac{K(q+0.2)}{q^2-2q+0.64+K(q+0.2)}.$$

For stability the poles must be inside the unit circle, and for a quadratic polynomial $q^2 + aq + b$ the zeros lies inside the unit circle exactly when $|a| - 1 < b < 1$. Here the pole polynomial is $q^2 - 2q + 0.64 + K(q + 0.2) = q^2 + (K-2)q + 0.64 + 0.2K$. The stability conditions then gives

$$\begin{aligned}
b < 1: &\quad 0.64 + 0.2K < 1 &&\Leftrightarrow K < \frac{0.36}{0.2} = 1.8,\\
a - 1 < b: &\quad (K-2) - 1 < 0.64 + 0.2K &&\Leftrightarrow K < \frac{3.64}{0.8} = 4.55,\\
-a - 1 < b: &\quad -(K-2) - 1 < 0.64 + 0.2K &&\Leftrightarrow K > \frac{0.36}{1.2} = 0.3,
\end{aligned}$$

and the closed loop system is thus stable for $0.3 < K < 1.8$.
(b) The closed loop system will be $y(k) = G_c(q)mr(k)$, where $G_c(q)$ can be computed by $G_c(q) = H(qI - F + GL)^{-1}G$, or from the knowledge that the numerator is the same as for $G(q)$ (known from (a)) and that the pole

polynomial is $\det(qI - F + GL)$, or by use of the controller canonical form ... Here we get $G_c(q) = \frac{q+0.2}{q^2-q+0.24}$. We want unit staic gain $\Leftrightarrow 1 = G_c(1)m = \frac{1.2m}{0.24}$, so choose $m = 0.2$.

(c) We have that $F_y(q) = L(qI - F + GL + KH)^{-1}K$ and $F_r(q) = 1 - L(qI - F + GL + KH)^{-1}G$ (see eg. eq. (8.26)). Here we have $L = \begin{bmatrix}1 & -0.4\end{bmatrix}$ and $K = \begin{bmatrix}2 & 1\end{bmatrix}^T$, and F, G and H as in the given state space model. It is convenient to start with the common part in the two transfer operators:

$$
\begin{aligned}
&L(qI - F + GL + KH)^{-1} = \\
&\quad \begin{bmatrix}1 & -0.4\end{bmatrix} \left(\begin{bmatrix}q & 0\\ 0 & q\end{bmatrix} - \begin{bmatrix}2 & -0.64\\ 1 & 0\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & -0.4\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}1 & 0.2\end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix}1 & -0.4\end{bmatrix} \begin{bmatrix}q+1 & 0.64\\ 0 & q+0.2\end{bmatrix}^{-1} = \frac{1}{(q+1)(q+0.2)} \begin{bmatrix}q+0.2 & -0.4q - 1.04\end{bmatrix}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
F_y(q) &= \frac{1}{(q+1)(q+0.2)} \begin{bmatrix}q+0.2 & -0.4q - 1.04\end{bmatrix} \begin{bmatrix}2\\1\end{bmatrix} = \frac{1.6q - 0.64}{(q+1)(q+0.2)}, \\
F_r(q) &= 1 - \frac{1}{(q+1)(q+0.2)} \begin{bmatrix}q+0.2 & -0.4q - 1.04\end{bmatrix} \begin{bmatrix}1\\0\end{bmatrix} \\
&= 1 - \frac{q+0.2}{(q+1)(q+0.2)} = \frac{q^2 + 0.2q}{(q+1)(q+0.2)} = \frac{q}{q+1}.
\end{aligned}
$$

5. **(a)** True (The left half plane is mapped onto the unit disc); **(b)** False (For some systems observability is lost for certain sampling intervals); **(c)** True; **(d)** True (Theorem 5.5); **(e)** False (MPC requires numerical optimization); **(f)** True; **(g)** True (This is handled by the numerical optimization)

6. **(a)** Use that $\Phi_y(\omega) = G(e^{i\omega})\Phi_v(\omega)G^*(e^{i\omega})$ (see eq. (5.85)). Here we have

$$
y(k) = \frac{q+0.5}{q^2 - q + 0.25} v_1(k) + v_2(k) = \begin{bmatrix}\frac{q+0.5}{(q-0.5)^2} & 1\end{bmatrix} \begin{bmatrix}v_1(k)\\ v_2(k)\end{bmatrix} = G(q)v(q).
$$

(Notice the observer canonical form.) Furthermore,

$$
\Phi_v(\omega) = \begin{bmatrix}R_1 & 0\\ 0 & R_2\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}.
$$

Hence,

$$
\begin{aligned}
\Phi_y(\omega) &= \begin{bmatrix}\frac{e^{i\omega}+0.5}{(e^{i\omega}-0.5)^2} & 1\end{bmatrix} \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix} \begin{bmatrix}\frac{e^{-i\omega}+0.5}{(e^{-i\omega}-0.5)^2}\\ 1\end{bmatrix} = 2\cdot \frac{(e^{i\omega}+0.5)(e^{-i\omega}+0.5)}{[(e^{i\omega}-0.5)(e^{-i\omega}-0.5)]^2} + 1 \\
&= \frac{1.25 + \cos\omega}{(1.25 - \cos\omega)^2} + 1 = \frac{1.25 + \cos\omega}{2.0625 - 2.5\cos\omega + 0.5\cos 2\omega} + 1 \\
&\qquad = \frac{4.5625 - 0.5\cos\omega + 0.5\cos 2\omega}{2.0625 - 2.5\cos\omega + 0.5\cos 2\omega}.
\end{aligned}
$$

(b) It is only the ratio Q_1/Q_2 that matters, and this represents the balance between keeping the output, or the control error (penalized by Q_1) small, and keeping the input (penalized by Q_2) small. To increase the response speed one should increase Q_1, or decrease Q_2, and and in order to see any significant effect the change should be of an order of magnitude (eg. ten times). Thus, for instance

$$Q_1 = 10, \quad Q_2 = 1, \qquad \text{or} \qquad Q_1 = 1, \quad Q_2 = 0.1$$

would be plausible choices.