Uppsala University
Department of Information Technology
Systems and Control
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Final exam: Automatic Control II (Reglerteknik II, 1TT495)

Date: August 19, 2011

Responsible examiner: Torsten Söderström

Preliminary grades: 3 = 23–32p, 4 = 33–42p, 5 = 43–50p.

Instructions

The solutions to the problems can be given in Swedish or in English.

Problem 1c-e is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 1c-e you will be accounted for the best performance of the homework assignments and Problem 1c-e.)

Solve each problem on a separate page.

Write your code on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding material: Textbooks in automatic control (such as 'Reglerteori – flervariabla och olinjära metoder', 'Reglerteknik – Grundläggande teori', and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators, copies of OH transparencies. Note that the following is **not allowed**: Exempelsamling med lösningar.

Good luck!

Problem 1

Consider the scalar system

$$\begin{aligned} \dot{x} &= -x + u + v \\ y &= x + e \end{aligned}$$

where the process noise v and the measurement noise e both have constant intensities $\phi(\omega) \equiv 1$.

a) Assume that the state $x(t)$ is estimated using a standard observer

$$\dot{\hat{x}} = -\hat{x} + u + K(y - \hat{x})$$

with a constant gain K. Determine the stationary variance, say V, of the estimation error $\tilde{x} = x - \hat{x}$ as a function of K. **2 points**

(b) Determine what value of the observer gain that minimizes V. Let K^* denote this value of the gain. What is the minimum value of V? **2 points**

(c) What is the solution to the associated Riccati equation? **2 points**

(d) Assume next that the gain K^* is used, but that the observation process is improved by using a more accurate sensor, so that the measurement noise has intensity $\Phi_e(\omega) \equiv 1/3$. What is then the variance of the estimation error? **2 points**

(e) How much lower value of V can be obtained by re-optimizing the observer gain for the case treated in part (e)? **2 points**

Problem 2

(a) Consider a multivariable system with the transfer function

$$G(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+2} \\ \frac{3}{s+1} & \frac{-2}{s+1} \end{pmatrix}$$

Show that the system can be represented in state space form as

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 0 \\ 0 & -2 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} x \end{aligned}$$

2 points

(b) Is the system controllable? If not, compute the controllable subspace. **4 points**

(c) Is the system observable? If not, compute the non-observable subspace. **4 points**

Problem 3

Consider a rocket in space, which we model as a double integrator for the movement in one direction. The acceleration is due to some process noise, so with the state variables $x_1 = y,\ x_2 = \dot{y}$ the state space model is

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + e \end{aligned}$$

We assume for the time being that the position y is measured with some error e.

Assume that the spectra of the noise sources are

$$\phi_v = R_1 = \gamma^4, \quad \phi_e = R_2 = 1$$

The process noise and the measurement noise are independent.

(a) Determine the Kalman filter. Determine also the covariance matrix of the estimation error

$$P = E\tilde{x}(t)\tilde{x}^T(t)$$

4 points

(b) Assume next that a second sensor is added, so that also the velocity $\dot{y}$ is measured. The output equation is then changed to

$$y = x + e$$

where now e is a two-dimensional vector with spectrum

$$\phi_e = R_2 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix}$$

Determine the optimal Kalman filter in this case. Show that in this case the covariance matrix of the estimation error is

$$P = E\tilde{x}(t)\tilde{x}^T(t) = \frac{1}{2} \begin{pmatrix} \sqrt{3}\gamma & \gamma^2 \\ \gamma^2 & \sqrt{3}\gamma^3 \end{pmatrix}$$

3 points

(c) Compare $P = P_a$ in part (a) and $P = P_b$ in part (b). Will more accurate estimates be obtained in case (b)? If so, in what sense? **2 points**

Problem 4

Consider a double integrator. The acceleration is due both to an input u and to some white process noise, so with the state variables $x_1 = y$, $x_2 = \dot{y}$ the state space model is

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v$$

(a) Assume that the aim of control is to minimize the variance of the position $y = x_1$. Consider first the noise-free case and find a full state feedback controller, $u(t) = -Lx(t)$, so that the criterion

$$V = \int_0^\infty [x_1^2(t) + \rho^4 u^2(t)]dt$$

is minimized. **3 points**

(b) Determine the closed loop poles when the state feedback from (a) is applied. **3 points**

(c) Let the feedback from part (a) be applied. Determine the covariance matrix of the state vector, that is compute

$$P = Ex(t)x^T(t)$$

4 points

Problem 5

(a) Given a discrete-time system with the pulse transfer function

$$H(q) = \frac{1}{q+1}$$

Can this system occur by sampling a first order system with the sampling interval being h? Motivate your answer! **2 points**

(b) Consider a harmonic oscillator with transfer function

$$G(s) = \frac{\omega_o^2}{s^2 + \omega_o^2}$$

Represent it in state space form as

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ -\omega_o^2 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \omega_o^2 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

Determine the state space form of the sampled version of this system, when the sampling interval equals h. **3 points**

(c) Determine the pulse transfer function of the sampled system of part (b). **2 points**

(d) Given a discrete-time system with the pulse transfer function

$$H(q) = \frac{1}{q+1}$$

Can this system occur by sampling a *second order* system with the sampling interval being h? Motivate your answer! **4 points**

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Automatic control II, August 19, 2011
Answers and brief solutions

Problem 1

To treat the general case, let the measurement noise have intensity r_2. The estimation error $\tilde{x}$ satisfies

$$\dot{\tilde{x}} = (-1-K)\tilde{x} - Ke + v$$

Applying the Lyapunov equation then gives easily

$$V(K) = \frac{K^2 r_2 + 1}{2(1+K)}$$

(a) Setting $r_2 = 1$ gives

$$V(K) = \frac{K^2 + 1}{2(1+K)}$$

(b) The minimizing element of $V(K)$ is found to be $K^* = -1 + \sqrt{2}$. Further, the minimal value turns out to be $V = \sqrt{2} - 1 \approx 0.414$.

(c) The associated Riccati equation is

$$0 = -P - P + 1 - P^2 \times 1^2/r_2$$

which leads to

$$P^2 + 2r_2 P - r_2 = 0$$

with the solution $P = -r_2 \pm \sqrt{r_2^2 + r_2}$. As $r_2 = 1$ in part (c), the positive solution is $P = \sqrt{2} - 1$, as V in part (b).

(d) The variance using the fixed observer gain K^* for the noise intensity $r_2 = 1/3$ becomes

$$V = \frac{(K^*)^2/3 + 1}{2(1+K^*)} = \frac{\sqrt{2}}{2} - \frac{1}{3} \approx 0.374$$

(e) The minimal variance of the estimation error, when $r_2 = 1/3$, is given by the solution to the Riccati equation

$$0 = -P - P + 1 - P^2/(1/3)$$

which is $P = 1/3 \approx 0.333$.

Problem 2

(a) Laplace transforming the state space equations give directly

$$X_1(s) = \frac{1}{s+1}U_1(s), \quad X_2(s) = \frac{2}{s+2}U_2(s),$$
$$X_3(s) = \frac{3}{s+1}U_1(s), \quad X_4(s) = \frac{-2}{s+1}U_2(s)$$

and then

$$Y_1(s) = X_1(s) + X_2(s), \quad Y_2(s) = X_3(s) + X_4(s)$$

which directly leads to $Y(s) = G(s)U(s)$.

(b) The controllability matrix becomes

$$\begin{aligned} \mathcal{C}(A,B) &= \begin{pmatrix} B & AB & A^2B & A^3B \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -4 & 0 & 8 & 0 & -16 \\ 3 & 0 & -3 & 0 & 3 & 0 & -3 & 0 \\ 0 & -2 & 0 & 2 & 0 & -2 & 0 & 2 \end{pmatrix} \end{aligned}$$

The matrix has rank $= 3$. The first and the third row vectors are parallel. The system is hence not controllable. The controllable subspace is spanned by the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -4 \\ 0 \\ 2 \end{pmatrix},$$

and alternatively by the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(c) The observability matrix becomes

$$\mathcal{O}(C,A) = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -8 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

The matrix has rank $= 3$. The third and the fourth column vectors are equal. The system is hence not observable. The nonobservable subspace is spanned by vectors obeying $\mathcal{O}x = 0$, that is

$$x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

Problem 3

(a) The Riccati equation gives

$$0 = AP + PA^T + \gamma^4 NN^T - PC^T CP$$

or

$$\begin{aligned}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &\quad + \gamma^4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \end{pmatrix}\end{aligned}$$

which leads to the equations

$$\begin{aligned} 0 &= 2p_{12} - p_{11}^2 \\ 0 &= p_{22} - p_{11}p_{12} \\ 0 &= \gamma^4 - p_{12}^2 \end{aligned}$$

As P must be positive definite, the solution is

$$P = \begin{pmatrix} \sqrt{2}\gamma & \gamma^2 \\ \gamma^2 & \sqrt{2}\gamma^3 \end{pmatrix}$$

The Kalman gain becomes

$$K = PC^T = \begin{pmatrix} \sqrt{2}\gamma \\ \gamma^2 \end{pmatrix}$$

The Kalman filter will be

$$\dot{\hat{x}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\hat{x} + \begin{pmatrix} \sqrt{2}\gamma \\ \gamma^2 \end{pmatrix}(y - \hat{x}_1)$$

(b) The Riccati equation becomes in this case

$$\begin{aligned} 0 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma^4 \end{pmatrix} \\ &\quad - \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \end{aligned}$$

Simplifying the equation, and evaluating the different elements leads to the following system of equations

$$\begin{aligned} 0 &= 2p_{12} - p_{11}^2 - \frac{1}{\gamma^2}p_{12}^2 \\ 0 &= p_{22} - p_{11}p_{12} - \frac{1}{\gamma^2}p_{12}p_{22} \\ 0 &= \gamma^4 - p_{12}^2 - \frac{1}{\gamma^2}p_{22}^2 \end{aligned}$$

It is straightforward to see that the claimed solution does indeed satisfy these three equations.

The Kalman gain becomes

$$K = PC^T R_2^{-1} = \frac{1}{2}\begin{pmatrix} \sqrt{3}\gamma & \gamma^2 \\ \gamma^2 & \sqrt{3}\gamma^3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} \sqrt{3}\gamma & 1 \\ \gamma^2 & \sqrt{3}\gamma \end{pmatrix}$$

The Kalman filter can be written elementwise as

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + 0.5\sqrt{3}\gamma(y_1 - \hat{x}_1) + 0.5(y_2 - \hat{x}_2) \\ \dot{\hat{x}}_2 &= 0.5\gamma^2(y_1 - \hat{x}_1) + 0.5\sqrt{3}\gamma(y_2 - \hat{x}_2) \end{aligned}$$

(c) The difference between the two covariance matrices becomes

$$\begin{aligned} P_a - P_b &= \begin{pmatrix} \sqrt{2}\gamma & \gamma^2 \\ \gamma^2 & \sqrt{2}\gamma^3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \sqrt{3}\gamma & \gamma^2 \\ \gamma^2 & \sqrt{3}\gamma^3 \end{pmatrix} \\ &= \begin{pmatrix} (\sqrt{2} - \sqrt{3}/2)\gamma & 0.5\gamma^2 \\ 0.5\gamma^2 & (\sqrt{2} - \sqrt{3}/2)\gamma^3 \end{pmatrix} \end{aligned}$$

which turns out to be positive definite. Hence, any combination of the state space variables will be more accurately estimated in case (b), which is intuitively reasonable.

Problem 4

(a) The Riccati equation becomes

$$\begin{aligned} 0 &= A^T S + SA + Q_1 - SBQ_2^{-1}B^T S \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} + \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{1}{\rho^4}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \end{aligned}$$

which leads to the following system of equations

$$\begin{aligned} 0 &= 1 - s_{12}^2/\rho^4 \\ 0 &= s_{11} - s_{12}s_{22}/\rho^4 \\ 0 &= 2s_{12} - s_{22}^2/\rho^4 \end{aligned}$$

The solution is readily found to be

$$S = \begin{pmatrix} \sqrt{2}\rho & \rho^2 \\ \rho^2 & \sqrt{2}\rho^3 \end{pmatrix}$$

The corresponding feedback vector is

$$\begin{aligned} L &= \frac{1}{\rho^4}B^T S \\ &= \frac{1}{\rho^4}\begin{pmatrix} s_{12} & s_{22} \end{pmatrix} \\ &= \begin{pmatrix} \rho^2 & \sqrt{2}\rho^3 \end{pmatrix}/\rho^4 \end{aligned}$$

(b) The characteristic equation for the closed loop system is

$$\begin{aligned} 0 &= \det(sI - A + BL) \\ &= \det \begin{pmatrix} s & -1 \\ \rho^{-2} & s + \sqrt{2}\rho^{-1} \end{pmatrix} = s^2 + \sqrt{2}\rho^{-1}s + \rho^{-2} \end{aligned}$$

which has its solutions in

$$s = \frac{\sqrt{2}}{2}(-1 \pm i)\frac{1}{\rho}$$

(c) The closed loop system is given by the state space model

$$\dot{x} = (A - BL)x + Nv$$

or more explicitly,

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\rho^{-2} & -\sqrt{2}\rho^{-1} \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v$$

Assume that the process noise have unit spectrum. The covariance matrix

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

is found from the Lyapunov equation

$$0 = \begin{pmatrix} 0 & 1 \\ -\rho^{-2} & -\sqrt{2}\rho^{-1} \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & -\rho^{-2} \\ 1 & -\sqrt{2}\rho^{-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Evaluating the elements leads to the following system of equations

$$\begin{aligned} 0 &= 2p_{12} \\ 0 &= p_{22} - \rho^{-2}p_{11} - \sqrt{2}\rho^{-1}p_{12} \\ 0 &= -2\rho^{-2}p_{12} - 2\sqrt{2}\rho^{-1}p_{22} + 1 \end{aligned}$$

The solution to these equations is found to be

$$P = \frac{1}{2\sqrt{2}} \begin{pmatrix} \rho^3 & 0 \\ 0 & \rho \end{pmatrix}$$

Problem 5

(a) A first order continuous-time system

$$\begin{aligned} \dot{x} &= ax + bu \\ y &= x \end{aligned}$$

will always lead to a discrete time system with a pole in e^{ah}. However, the equation

$$-1 = e^{ah}$$

cannot have any real-valued solution a.

(b) Compute first the matrix exponential.

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1}\begin{pmatrix} s & -1 \\ \omega_o^2 & s \end{pmatrix}^{-1} = \mathcal{L}^{-1}\frac{1}{s^2+\omega_o^2}\begin{pmatrix} s & 1 \\ -\omega_o^2 & s \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \cos(\omega_o t) & \frac{1}{\omega_o}\sin(\omega_o t) \\ -\omega_o \sin(\omega_o t) & \cos(\omega_o t) \end{pmatrix} \end{aligned}$$

Set

$$C = \cos(\omega_o h), \qquad S = \sin(\omega_o h)$$

Then

$$\begin{aligned} F &= \begin{pmatrix} C & \frac{1}{\omega_o}S \\ -\omega_o S & C \end{pmatrix} \\ G &= \int_0^h e^{As}B ds = \int_0^h \begin{pmatrix} \omega_o \sin(\omega_o s) \\ \omega_o^2 \cos(\omega_o s) \end{pmatrix} ds = \begin{pmatrix} 1-C \\ \omega_o S \end{pmatrix} \end{aligned}$$

(c)

$$\begin{aligned} H(q) &= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} q-C & -\frac{1}{\omega_o}S \\ \omega_o S & q-C \end{pmatrix}^{-1}\begin{pmatrix} 1-C \\ \omega_o S \end{pmatrix} \\ &= \frac{1}{(q-C)^2+S^2}\begin{pmatrix} q-C & \frac{1}{\omega_o}S \end{pmatrix}\begin{pmatrix} 1-C \\ \omega_o S \end{pmatrix} \\ &= \frac{(1-C)(q+1)}{q^2-2qC+1} \end{aligned}$$

(d) Use the results of parts (b) and (c). Consider a continuous-time system with the transfer function

$$G(s) = \frac{K\omega_o^2}{s^2+\omega_o^2}$$

(where the parameters K and ω_o are to be determined). The pulse transfer function of this system is, according to part (b)

$$H(q) = \frac{K(1-C)(q+1)}{q^2-2Cq+1}$$

Now choose in particular

$$K = 0.5, \quad \omega_o = \pi/h$$

Then, $C = -1$ and

$$H(q) = \frac{0.5 \times 2(q+1)}{q^2+2q+1} = \frac{1}{q+1}$$