Uppsala University
Department of Information Technology
Systems and Control
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Final exam: Automatic Control II (Reglerteknik II, 1TT495)

Date: January 10, 2011

Responsible examiner: Torsten Söderström

Preliminary grades: 3 = 23–32p, 4 = 33–42p, 5 = 43–50p.

Instructions

The solutions to the problems can be given in Swedish or in English.

Problem 1 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 1 you will be accounted for the best performance of the homework assignments and Problem 1.)

Solve each problem on a separate page.

Write your code on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding material: Textbooks in automatic control (such as 'Reglerteori – flervariabla och olinjära metoder', 'Reglerteknik – Grundläggande teori', and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators, copies of OH transparencies. Note that the following is **not allowed**: Exempelsamling med lösningar.

Good luck!

Problem 1

Consider the system

$$\begin{aligned} x(t+1) &= \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} x(t) \end{aligned}$$

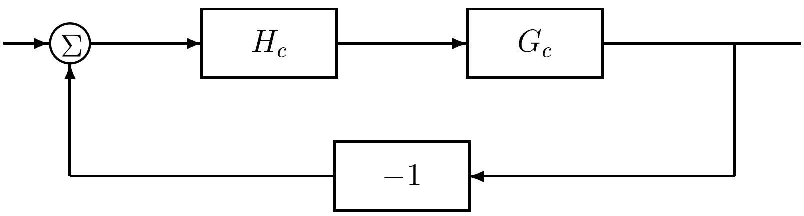
(a) Is the system controllable? **2 points**

(b) Is the system observable? **2 points**

(c) Is the system controllable from the input u_1? **2 points**

Problem 2

Consider the system $G_c(s)$ controlled with the regulator $H_c(s)$ according to the block diagram.

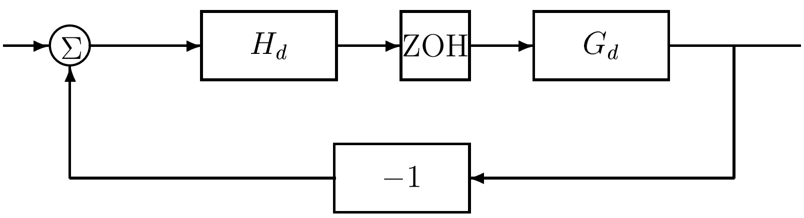


In this case, the system is a simple integrator. Set

$$G_c(s) = \frac{1}{s}, \quad H_c(s) = K$$

(a) Determine the poles of the closed loop system. Denote the result $\{p_{c,i}\}_{i=1}^{n}$. **1 point**

(b) Assume the system is computer-controlled, so that the closed loop system actually can be represented as in the block diagram below. Let the sampling interval be equal to h.



Determine the discrete-time transfer function $G_d(z)$, that corresponds to sampling $G_c(s)$. **1 point**

(c) Assume that the discrete-time regulator is a proportional regulator, and that $H_d(q) = K$. Determine the closed loop poles of the closed-loop discrete-time system. Will the system always be stable? **2 points**

(d) Assume that it is desired that the closed loop system for the continuous-time system have all poles in $s = -\alpha$. What value, say $K = K_c$, should the gain then take?

Next, assume that it is desired that the closed loop system for the discrete-time system have all poles in $z = e^{-\alpha h}$. What value, say $K = K_d$, should the gain then take?

Compare the gains K_c and K_d. Is one of them always larger than the other? **4 points**

Problem 3

Consider a random process in discrete-time $y(t)$ with covariance function

$$r(\tau) = 5a^{|\tau|}, \quad (0 < a < 1)$$

(a) Assume that we want to make prediction k steps ahead of the process. A very simple estimator of the $y(t+k)$ from data available at time t would be to take

$$\hat{y}(t+k) = y(t)$$

that is to take the last available measurement. How large will the prediction error variance

$$V = E\left[y(t+k) - \hat{y}(t+k)\right]^2$$

become? **2 points**

(b) A somewhat more sophisticated predictor would be to multiply the last available measurement with a constant, say

$$\hat{y}(t+k) = \alpha y(t)$$

Determine how the prediction error variance depends on the parameter α, and how it can be minimized with respect to α. **3 points**

(c) Determine the spectrum of the process. **4 points**

(d) Use the spectrum to derive a time-domain model of the process on state-space form

$$\begin{aligned} x(t+1) &= Ax(t) + Bv(t) \\ y(t) &= Cx(t) + e(t) \end{aligned}$$

and specify also the variances of $v(t)$ and $e(t)$. **3 points**

Problem 4

Consider a first order system with a delay given on polynomial form

$$(1 + aq^{-1})y(t) = bq^{-1}(1 + \beta q^{-1})u(t) + (1 + cq^{-1})e(t)$$

where $e(t)$ is white noise, with zero mean and variance λ^2. Assume that $|c| < 1$ and $0 < |\beta| < 1$ hold.

(a) Show that the system can be written in state-space form as follows

$$
\begin{aligned}
x(t+1) &= \begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} b \\ b\beta \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ c \end{pmatrix} v(t) \\
y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)
\end{aligned}
$$

How are v and e related? **2 points**

(b) Assume both states can be measured. Determine the optimal feedback

$$u(t) = -Lx(t)$$

that minimizes $E\sum_t y^2(t)$. **5 points**

(c) Determine the Kalman filter (the predictor form) giving $\hat{x}(t|t-1)$. **5 points**

Problem 5

Consider the stochastic process $s(t)$ given by

$$(1 - 0.5q^{-1})s(t) = v(t)$$

where $v(t)$ is white noise with zero mean and variance 1.35. Assume that process is observed with some measurement noise, so that

$$y(t) = s(t) + e(t)$$

where $e(t)$ is white noise with zero mean and variance 1.

(a) Determine the spectrum of the observed signal $y(t)$. **2 points**

(b) Use spectral factorization and show that the observed signal can be modelled as

$$(1 - 0.5q^{-1})y(t) = w(t) + cw(t-1)$$

where $|c| < 1$ and $w(t)$ is a white noise process of zero mean and variance λ^2. Determine the numerical values of c and λ^2. **2 points**

(c) Represent $y(t)$ on state space form as a second order system. Set $e_1(t) = w(t+1)$. Show that $y(t)$ admits the representation

$$
\begin{aligned}
x(t+1) &= Ax(t) + Ne_1(t) \\
y(t) &= Cx(t)
\end{aligned}
$$

Note that there is no measurement noise. Determine $A,\ N,\ C,\ R_1$. **2 points**

(d) Determine the Kalman filter

$$\hat{x}(t+1) = A\hat{x}(t) + K\left(y(t) - C\hat{x}(t)\right)$$

What is the value of K? **4 points**

(e) Write the prediction error related to the Kalman filter in part (d) as

$$\varepsilon(t) = y(t) - C\hat{x}(t)$$

Show that the observations admit a representation as

$$\begin{aligned} \hat{x}(t+1) &= A\hat{x}(t) + K\varepsilon(t) \\ y(t) &= C\hat{x}(t) + \varepsilon(t) \end{aligned}$$

2 points

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Answers and brief solutions

Problem 1

(a) As B is nonsingular, the controllability matrix has full rank. The system is controllable.

(b) As C is nonsingular, the observability matrix has full rank. The system is observable.

(c) As the element $B_{11} = 0$ and the system is in diagonal form, the state x_1 is not controllable. The system is not controllable using u_1.

Problem 2

(a) The closed loop characteristic equation becomes

$$1 + G_c(s)H_c(s) = 0 \Rightarrow 1 + K/s = 0 \Rightarrow s = -K$$

This leads to that the closed loop pole (there is only one) is

$$p_c = -K$$

(b)

$$G_c(s) = \frac{1}{s} \Rightarrow G_d(z) = \frac{h}{z-1}$$

(c) The closed loop characteristic equation becomes

$$1 + G_d(z)H_d(z) = 0 \Rightarrow 1 + \frac{h}{z-1}K = 0 \Rightarrow z - 1 + hK = 0 \Rightarrow z = 1 - hK$$

The system is stable if

$$0 < K < \frac{2}{h}$$

(d)

$$-K_c = -\alpha \Rightarrow K_c = \alpha$$

$$e^{-\alpha h} = 1 - hK_d \Rightarrow K_d = \frac{1 - e^{-\alpha h}}{h}$$

For small h, $K_d \approx \alpha = K_c$. In general,

$$K_d > K_c \Leftrightarrow 1 - e^{-\alpha h} > \alpha h \Leftrightarrow 1 - \alpha h > e^{-\alpha h}$$

which is always true.

Problem 3

(a) In this case

$$\begin{aligned} V &= E\left[y(t+k) - \hat{y}(t+k)\right]^2 \\ &= E\left[y(t+k) - y(t)\right]^2 = Ey^2(t+k) + Ey^2(t) - 2Ey(t+k)y(t) \\ &= r(0) + r(0) - 2r(k) = 10 - 10a^k = 10(1-a^k) \end{aligned}$$

(b) In this case

$$\begin{aligned} V(\alpha) &= E\left[y(t+k) - \hat{y}(t+k)\right]^2 \\ &= E\left[y(t+k) - \alpha y(t)\right]^2 = r(0) + \alpha^2 r(0) - 2\alpha r(k) \end{aligned}$$

Minimizing $V(\alpha)$ with respect to α gives that the best value of α is $\alpha = r(k)/r(0)$, and

$$\min_{\alpha} V(\alpha) = r(0) - r^2(k)/r(0) = 5 - 5a^{2k} = 5(1-a^{2k})$$

which is always smaller than $10(1-a^k)$.

(c) The spectrum becomes directly from the definition

$$\begin{aligned} \phi(\omega) &= \sum_{k=-\infty}^{\infty} r(k)e^{-i\omega k} \\ &= \sum_{k=0}^{\infty} r(k)e^{-i\omega k} + \sum_{k=-\infty}^{0} r(k)e^{-i\omega k} - \sum_{k=0}^{0} r(k)e^{-i\omega k} \\ &= \sum_{k=0}^{\infty} 5(ae^{-i\omega})^k + \sum_{k=-\infty}^{0} 5(a^{-1}e^{-i\omega})^k - 5 \\ &= \frac{5}{1-ae^{-i\omega}} + \frac{5}{1-ae^{i\omega}} - 5 \\ &= \frac{5}{(1-ae^{-i\omega})(1-ae^{i\omega})}\left[1 - ae^{i\omega} + 1 - ae^{-i\omega} - 1 - a^2 + ae^{-i\omega} + ae^{i\omega}\right] \\ &= \frac{5(1-a^2)}{1+a^2-2a\cos(\omega)} \end{aligned}$$

(d) The process is apparently an AR(1) process, and can be represented as a first order system with

$$A = a, \quad C = 1, \quad R_1 = 5(1-a^2), \quad C = 1, \quad R_2 = 0$$

Problem 4

(a) The transfer function operators from $u(t)$ and $v(t)$ to $y(t)$ appear from the calculations

$$\begin{aligned} y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} q+a & -1 \\ 0 & q \end{pmatrix}^{-1} \left[\begin{pmatrix} b \\ b\beta \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ c \end{pmatrix} v(t) \right] \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{q(q+a)} \begin{pmatrix} q & 1 \\ 0 & q+a \end{pmatrix} \left[\begin{pmatrix} b \\ b\beta \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ c \end{pmatrix} v(t) \right] \\ &= \frac{1}{q(q+a)} \begin{pmatrix} q & 1 \end{pmatrix} \left[\begin{pmatrix} b \\ b\beta \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ c \end{pmatrix} v(t) \right] \\ &= \frac{b(q+\beta)}{q(q+a)} u(t) + \frac{q+c}{q(q+a)} v(t) \end{aligned}$$

This describes precisely the given system if $v(t) = e(t+1)$.

(b) In this case $Q_1 = C^T C,\ Q_2 = 0$. The Riccati equation becomes

$$S = A^T S A + C^T C - A^T S B \left(B^T S B \right)^{-1} B^T S A$$

Set

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}$$

Spelling out the Riccati equation elementwise leads to

$$\begin{aligned} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} &= \begin{pmatrix} -a \\ 1 \end{pmatrix} s_{11} \begin{pmatrix} -a & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} -a \\ 1 \end{pmatrix} \frac{(s_{11} + \beta s_{12})^2}{s_{11} + \beta^2 s_{22} + 2\beta s_{12}} \begin{pmatrix} -a & 1 \end{pmatrix} \end{aligned}$$

We can therefore write the solution as

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} = \begin{pmatrix} 1 + a^2\sigma & -a\sigma \\ -a\sigma & \sigma \end{pmatrix}$$

where the unknown σ is introduced as

$$\sigma = s_{11} - \frac{(s_{11} + \beta s_{12})^2}{s_{11} + \beta^2 s_{22} + 2\beta s_{12}} = \frac{\beta^2 (s_{11} s_{22} - s_{12}^2)}{s_{11} + \beta^2 s_{22} + 2\beta s_{12}}$$

This parameter is thus the positive solution to

$$\sigma \left(1 + a^2\sigma + \beta^2\sigma - 2a\beta\sigma \right) = \beta^2 \left(\sigma + a^2\sigma^2 - a^2\sigma^2 \right)$$

or

$$\sigma \left(1 - \beta^2 + (a-\beta)^2 \sigma \right) = 0$$

leading to two solutions,

$$\sigma_1 = 0, \quad \sigma_2 = \frac{\beta^2 - 1}{(a-\beta)^2} < 0$$

As $\sigma_2 < 0$, this solution would lead to an indefinite matrix S. The solution σ_1 must be chosen. Hence

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and the feedback gain becomes

$$L = (B^T S B)^{-1} B^T S A = \frac{b s_{11}}{b^2 s_{11}} \begin{pmatrix} -a & 1 \end{pmatrix} = \frac{1}{b} \begin{pmatrix} -a & 1 \end{pmatrix}$$

(c) Set

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

The Ricatti equation is in this case

$$\begin{aligned}
P &= APA^T + N\lambda^2 N^T - APC^T \left(CPC^T\right)^{-1} CPA^T \\
&= \begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix} P \begin{pmatrix} -a & 0 \\ 1 & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} \\
&\quad - \begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix} P \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{p_{11}} \begin{pmatrix} 1 & 0 \end{pmatrix} P \begin{pmatrix} -a & 0 \\ 1 & 0 \end{pmatrix} \\
&= \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} -a & 1 \end{pmatrix} \left[P - \begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} \frac{1}{p_{11}} \begin{pmatrix} p_{11} & p_{12} \end{pmatrix} \right] \begin{pmatrix} -a \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\
&= \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} -a & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(p_{22} - p_{12}^2/p_{11}\right) \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -a \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\
&= \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \left(p_{22} - p_{12}^2/p_{11}\right)
\end{aligned}$$

Hence

$$P = \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} + \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$$

where γ is the remaining unknown to be determined. It holds

$$\gamma = p_{22} - p_{12}^2/p_{11} \Rightarrow \gamma = \lambda^2 c^2 - \frac{(\lambda^2 c)^2}{\lambda^2 + \gamma}$$

leading to

$$\gamma\lambda^2 + \gamma^2 = \lambda^4 c^2 + \lambda^2 c^2 \gamma - \lambda^4 c^2 \Rightarrow \gamma \left[\gamma + \lambda^2 - \lambda^2 c^2\right] = 0$$

with the two solutions

$$\gamma_1 = 0, \;\; \gamma_2 = \lambda^2 (c^2 - 1) < 0$$

and hence γ_1 must be chosen. Therefore,

$$P = \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix}$$

The Kalman gain becomes

$$\begin{aligned} K &= APC^T \left(CPC^T\right)^{-1} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} -a & 1 \end{pmatrix} \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\lambda^2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (c-a) \end{aligned}$$

The Kalman filter will be

$$\begin{aligned} \hat{x}(t+1|t) &= (A-KC)\hat{x}(t|t-1) + Bu(t) + Ky(t) \\ &= \begin{pmatrix} -c & 1 \\ 0 & 0 \end{pmatrix} \hat{x}(t|t-1) + \begin{pmatrix} b \\ b\beta \end{pmatrix} u(t) + \begin{pmatrix} c-a \\ 0 \end{pmatrix} y(t) \end{aligned}$$

As the last equation gives

$$\hat{x}_2(t+1)|t) = b\beta u(t)$$

we may also write

$$\hat{x}_1(t+1|t) = -c\hat{x}_1(t|t-1) + bu(t) + b\beta u(t-1) + (c-a)y(t)$$

Problem 5

(a) It holds

$$\phi_s(\omega) = \frac{\lambda_v^2}{|1-0.5e^{i\omega}|^2} = \frac{1.35}{1.25-\cos(\omega)}$$

$$\begin{aligned} \phi_y(\omega) &= \phi_s\omega) + \phi_e(\omega) \\ &= \frac{1.35}{1.25-\cos(\omega)} + 1 = \frac{2.6-\cos(\omega)}{1.25-\cos(\omega)} \end{aligned}$$

(b) The unknowns c and λ^2 must satisfy

$$\begin{matrix} \lambda^2(1+c^2) = 2.6 \\ \lambda^2 c = -0.5 \end{matrix} \Rightarrow \frac{c}{1+c^2} = \frac{-0.5}{2.6} \Rightarrow c = -0.2,\ \lambda^2 = 2.5$$

(c) It apparently holds

$$y(t) = \frac{1.0q^{-1} - 0.2q^{-2}}{1-0.5q^{-1}} e_1(t)$$

Using observable canonical form, this can be written as

$$\begin{aligned} x(t+1) &= \begin{pmatrix} 0.5 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ -0.2 \end{pmatrix} e_1(t) \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) \end{aligned}$$

Here $R_1 = Ee_1^2(t) = Ew^2(t) = 2.5$ and $R_2 = 0$.

(d) The Riccati equation becomes

$$
\begin{aligned}
P &= APA^T + NR_1N^T - APC^T\left(CPC^T\right)^{-1}CPA^T \\
&= NR_1N^T + A\left[P - PC^T\left(CPC^T\right)^{-1}CP\right]A^T \\
&= 2.5\begin{pmatrix} 1 \\ -0.2 \end{pmatrix}\begin{pmatrix} 1 & -0.2 \end{pmatrix} + \begin{pmatrix} 0.5 & 1 \\ 0 & 0 \end{pmatrix} \\
&\quad \times\left[\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} - \begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix}\frac{1}{p_{11}}\begin{pmatrix} p_{11} & p_{12} \end{pmatrix}\right]\begin{pmatrix} 0.5 & 0 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2.5 & -0.5 \\ -0.5 & 0.1 \end{pmatrix} + \begin{pmatrix} p_{22} - p_{12}^2/p_{11} & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

Set $\alpha = p_{22} - p_{12}^2/p_{11}$. It then holds

$$
\alpha = 0.1 - \frac{0.5^2}{2.5+\alpha} \Rightarrow 2.5\alpha + \alpha^2 = 0.25 + 0.1\alpha - 0.25 \Rightarrow \alpha\,(2.5 + \alpha - 0.1) = 0
$$

and we can conclude that $\alpha = 0$.
The Kalman gain vector becomes

$$
K = \begin{pmatrix} 0.5 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2.5 & -0.5 \\ -0.5 & 0.1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{1}{2.5} = \begin{pmatrix} 1.25 - 0.5 \\ 0 \end{pmatrix}\frac{1}{2.5} = \begin{pmatrix} 0.3 \\ 0 \end{pmatrix}
$$

The Kalman filter will be

$$
\begin{aligned}
\hat{x}(t+1|t) &= (A - KC)\hat{x}(t|t-1) + Ky(t) \\
&= \begin{pmatrix} 0.2 & 1 \\ 0 & 0 \end{pmatrix}\hat{x}(t|t-1) + \begin{pmatrix} 0.3 \\ 0 \end{pmatrix}y(t)
\end{aligned}
$$

(e) Using the definition of $\varepsilon(t)$ it follows directly that

$$
\begin{aligned}
\hat{x}(t+1) &= A\hat{x}(t) + K[y(t) - C\hat{x}(t)] \\
&= A\hat{x}(t) + K\varepsilon(t) \\
y(t) &= C\hat{x}(t) + \varepsilon(t)
\end{aligned}
$$