Uppsala University
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Systems and Control
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Final exam: Automatic Control II (Reglerteknik II, 1TT495)

Date: October 22, 2010

Responsible examiner: Torsten Söderström

Preliminary grades: 3 = 23–32p, 4 = 33–42p, 5 = 43–50p.

Instructions

The solutions to the problems can be given in Swedish or in English, except for Problem 2 that **must** be solved in English.

Problem 1 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 1 you will be accounted for the best performance of the homework assignments and Problem 1.)

Solve each problem on a separate page.

Write your code on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding material: Textbooks in automatic control (such as 'Reglerteori – flervariabla och olinjära metoder', 'Reglerteknik – Grundläggande teori', and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators, copies of OH transparencies. Note that the following is **not allowed**: Exempelsamling med lösningar.

Good luck!

Problem 1

Consider the system

$$\begin{aligned} x(t+1) &= \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} x(t) \end{aligned}$$

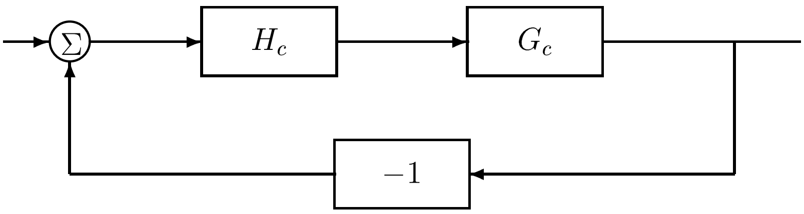
(a) Is the system controllable? **2 points**

(b) Is the system observable? **2 points**

(c) Is the system controllable from the input u_1? **2 points**

Problem 2

Consider the system $G_c(s)$ controlled with the regulator $H_c(s)$ according to the block diagram.

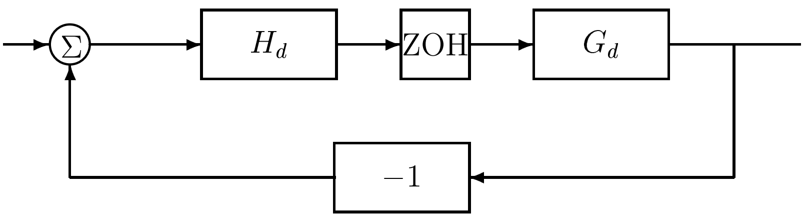


In this case, the system is a simple DC motor. Set

$$G_c(s) = \frac{1}{s(s+1)}, \quad H_c(s) = K$$

(a) For which values of K is the closed-loop system stable? **2 points**

(b) Assume the system is computer-controlled, so that the closed loop system actually can be represented as in the block diagram below. Let the sampling interval be equal to h.



Determine the discrete-time transfer function $G_d(z)$ that corresponds to sampling $G_c(s)$. Set $\alpha = e^{-h}$. **4 points**

Hint. It may be an advantage to first split $G_c(s)$ as a sum of two first order systems: $G_c(s) = 1/s - 1/(s+1)$.

(c) Assume that the discrete-time regulator is a proportional regulator, so that that $H_d(q) = K$. Show that the closed-loop system can be unstable for some values of K. **3 points**

Problem 3

Consider a random process in discrete-time $y(t)$ with covariance function

$$r(\tau) = 5a^{|\tau|}, \quad (0 < a < 1)$$

(a) Assume that we want to make prediction k steps ahead of the process. A very simple estimator of the $y(t+k)$ from data available at time t would be to take

$$\hat{y}(t+k) = y(t)$$

that is to take the last available measurement. How large will the prediction error variance

$$V = E\left[y(t+k) - \hat{y}(t+k)\right]^2$$

become? **2 points**

(b) A somewhat more sophisticated predictor would be to multiply the last available measurement with a constant, say

$$\hat{y}(t+k) = \alpha y(t)$$

Determine how the prediction error variance depends on the parameter α, and how it can be minimized with respect to α. **3 points**

(c) Determine the spectrum of the process. **4 points**

(d) Use the spectrum to derive a time-domain model of the process on state-space form

$$\begin{aligned} x(t+1) &= Ax(t) + Bv(t) \\ y(t) &= Cx(t) + e(t) \end{aligned}$$

and specify also the variances of $v(t)$ and $e(t)$. **3 points**

Problem 4

Consider a first order system given on polynomial form

$$(1 + aq^{-1})y(t) = bq^{-1}u(t) + (1 + cq^{-1})e(t)$$

where $e(t)$ is white noise, with zero mean and variance λ^2. Assume that $|c| < 1$ holds.

(a) Show that the system can be written in standard state-space form as follows

$$\begin{aligned} x(t+1) &= \begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} b \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ c \end{pmatrix} v(t) \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) \end{aligned}$$

How are v and e related? **2 points**

(b) Assume both states can be measured. Determine the optimal feedback

$$u(t) = -Lx(t)$$

that minimizes $E\sum_t y^2(t)$. **4 points**

(c) Determine the Kalman filter (the predictor form) giving $\hat{x}(t|t-1)$. **5 points**

Problem 5

Consider a first order process $x(t)$ observed with some noise $e(t)$. This may be modelled as

$$\begin{aligned} \dot{x} &= -ax + v \\ y &= x + e \end{aligned}$$

where v and e are white noise processes, with intensities r_1 and r_2, respectively. Also, assume that $a > 0$.

(a) Determine the spectrum and the variance of the signal $x(t)$. **2 points**

(b) As the signal $x(t)$ is not measured directly, consider estimating it with an observer

$$\dot{\hat{x}} = -a\hat{x} + K\left(y - \hat{x}\right)$$

Show that this can be written as

$$\hat{x}(t) = G(p)y(t)$$

and determine the transfer function operator $G(p)$, and sketch the character of its Bode plot, that is sketch how $|G(i\omega)|$ varies with ω. Let K be a fixed parameter. **2 points**

(c) Examine the estimation error $\tilde{x} = x - \hat{x}$ as a function of v and e. Show that the error can be written in the form

$$\tilde{x}(t) = H_1(p)v(t) + H_2(p)e(t)$$

and determine the two transfer function operators $H_1(p)$ and $H_2(p)$. **2 points**

(d) Determine the variances of the two error terms in part (c), that is the variances of $H_1(p)v(t)$ and of $H_2(p)e(t)$. **2 points**

(e) Determine the observer gain K that minimizes the total error variance

$$V(K) = E\left[H_1(p)v(t)\right]^2 + E\left[H_2(p)e(t)\right]^2$$

2 points

(f) Determine the Kalman filter related to the state space model set up in this problem. What is the variance $E\tilde{x}^2(t)$ when the Kalman filter is used?

2 points

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Answers and brief solutions

Problem 1

(a) As B is nonsingular, the controllability matrix has full rank. The system is controllable.

(b) As C is nonsingular, the observability matrix has full rank. The system is observable.

(c) As the element $B_{11} = 0$ and the system is in diagonal form, the state x_1 is not controllable. The system is not controllable using u_1.

Problem 2

(a) The closed loop characteristic equation becomes

$$1 + G_c(s)H_c(s) = 0 \Rightarrow s^2 + s + K = 0$$

The poles will lie in the left half plane for all positive values of K.

(b)

$$G_c(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

The sampled-data system can be written as follows. Using $\alpha = e^{-h}$,

$$G_d(z) = \frac{h}{z-1} - \frac{1-\alpha}{z-\alpha} = \frac{hz - h\alpha - (z-1)(1-\alpha)}{(z-1)(z-\alpha)} = \frac{z(h-1+\alpha) + (1-\alpha-h\alpha)}{(z-1)(z-\alpha)}$$

(c) The closed loop characteristic equation becomes

$$1 + G_d(z)H_d(z) = 0$$

leading to

$$(z-1)(z-\alpha) + K\left[z(h-1+\alpha) + (1-\alpha-h\alpha)\right] = 0$$

or yet

$$z^2 + z\underbrace{(-1-\alpha+K(h-1+\alpha))}_{a_1} + \underbrace{\alpha + K(1-\alpha-h\alpha)}_{a_2}$$

A necessary condition for stability is

$$|a_1| < 2, \quad |a_2| < 1$$

Both these two inequalities are violated if K is large enough.

Problem 3

(a) In this case

$$\begin{aligned} V &= E\left[y(t+k) - \hat{y}(t+k)\right]^2 \\ &= E\left[y(t+k) - y(t)\right]^2 = Ey^2(t+k) + Ey^2(t) - 2Ey(t+k)y(t) \\ &= r(0) + r(0) - 2r(k) = 10 - 10a^k = 10(1-a^k) \end{aligned}$$

(b) In this case

$$\begin{aligned} V(\alpha) &= E\left[y(t+k) - \hat{y}(t+k)\right]^2 \\ &= E\left[y(t+k) - \alpha y(t)\right]^2 = r(0) + \alpha^2 r(0) - 2\alpha r(k) \end{aligned}$$

Minimizing $V(\alpha)$ with respect to α gives that the best value of α is $\alpha = r(k)/r(0)$, and

$$\min_{\alpha} V(\alpha) = r(0) - r^2(k)/r(0) = 5 - 5a^{2k} = 5(1-a^{2k})$$

which is always smaller than $10(1-a^k)$.

(c) The spectrum becomes directly from the definition

$$\begin{aligned} \phi(\omega) &= \sum_{k=-\infty}^{\infty} r(k)e^{-i\omega k} \\ &= \sum_{k=0}^{\infty} r(k)e^{-i\omega k} + \sum_{k=-\infty}^{0} r(k)e^{-i\omega k} - \sum_{k=0}^{0} r(k)e^{-i\omega k} \\ &= \sum_{k=0}^{\infty} 5(ae^{-i\omega})^k + \sum_{k=-\infty}^{0} 5(a^{-1}e^{-i\omega})^k - 5 \\ &= \frac{5}{1-ae^{-i\omega}} + \frac{5}{1-ae^{i\omega}} - 5 \\ &= \frac{5}{(1-ae^{-i\omega})(1-ae^{i\omega})}\left[1 - ae^{i\omega} + 1 - ae^{-i\omega} - 1 - a^2 + ae^{-i\omega} + ae^{i\omega}\right] \\ &= \frac{5(1-a^2)}{1+a^2-2a\cos(\omega)} \end{aligned}$$

(d) The process is apparently an AR(1) process, and can be represented as a first order system with

$$A = a, \quad C = 1, \quad R_1 = 5(1-a^2), \quad C = 1, \quad R_2 = 0$$

Problem 4

(a) The transfer function operators from $u(t)$ and $v(t)$ to $y(t)$ appear from the calculations

$$\begin{aligned}
y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} q+a & -1 \\ 0 & q \end{pmatrix}^{-1} \left[\begin{pmatrix} b \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ c \end{pmatrix} v(t) \right] \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{q(q+a)} \begin{pmatrix} q & 1 \\ 0 & q+a \end{pmatrix} \left[\begin{pmatrix} b \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ c \end{pmatrix} v(t) \right] \\
&= \frac{1}{q(q+a)} \begin{pmatrix} q & 1 \end{pmatrix} \left[\begin{pmatrix} b \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ c \end{pmatrix} v(t) \right] \\
&= \frac{b}{(q+a)} u(t) + \frac{q+c}{q(q+a)} v(t)
\end{aligned}$$

This describes precisely the given system if $v(t) = e(t+1)$.

(b) In this case $Q_1 = C^T C,\ Q_2 = 0$. The Riccati equation becomes

$$S = A^T S A + C^T C - A^T S B \left(B^T S B \right)^{-1} B^T S A$$

Set

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}$$

Spelling out the Riccati equation elementwise leads to

$$\begin{aligned}
\begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} &= \begin{pmatrix} -a \\ 1 \end{pmatrix} s_{11} \begin{pmatrix} -a & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad - \begin{pmatrix} -a \\ 1 \end{pmatrix} \frac{s_{11}}{s_{11}} \begin{pmatrix} -a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

The feedback gain becomes

$$\begin{aligned}
L &= (B^T S B)^{-1} B^T S A \\
&= \frac{b s_{11}}{b^2 s_{11}} \begin{pmatrix} -a & 1 \end{pmatrix} = \frac{1}{b} \begin{pmatrix} -a & 1 \end{pmatrix}
\end{aligned}$$

(c) Set

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

The Riccati equation is in this case

$$\begin{aligned}
P &= A P A^T + N \lambda^2 N^T - A P C^T \left(C P C^T \right)^{-1} C P A^T \\
&= \begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix} P \begin{pmatrix} -a & 0 \\ 1 & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} \\
&\quad - \begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix} P \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{p_{11}} \begin{pmatrix} 1 & 0 \end{pmatrix} P \begin{pmatrix} -a & 0 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} -a & 1 \end{pmatrix} \left[P - \begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} \frac{1}{p_{11}} \begin{pmatrix} p_{11} & p_{12} \end{pmatrix} \right] \begin{pmatrix} -a \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\
&= \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} -a & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(p_{22} - p_{12}^2/p_{11} \right) \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -a \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\
&= \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \left(p_{22} - p_{12}^2/p_{11} \right)
\end{aligned}
$$

Hence

$$P = \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} + \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$$

where γ is the remaining unknown to be determined. It holds

$$\gamma = p_{22} - p_{12}^2/p_{11} \Rightarrow \gamma = \lambda^2 c^2 - \frac{(\lambda^2 c)^2}{\lambda^2 + \gamma}$$

leading to

$$\gamma\lambda^2 + \gamma^2 = \lambda^4 c^2 + \lambda^2 c^2 \gamma - \lambda^4 c^2 \Rightarrow \gamma \left[\gamma + \lambda^2 - \lambda^2 c^2 \right] = 0$$

with the two solutions

$$\gamma_1 = 0, \;\; \gamma_2 = \lambda^2 (c^2 - 1) < 0$$

and hence γ_1 must be chosen. Therefore,

$$P = \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix}$$

The Kalman gain becomes

$$
\begin{aligned}
K &= APC^T \left(CPC^T \right)^{-1} \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} -a & 1 \end{pmatrix} \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\lambda^2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (c - a)
\end{aligned}
$$

The Kalman filter will be

$$
\begin{aligned}
\hat{x}(t+1|t) &= (A - KC)\hat{x}(t|t-1) + Bu(t) + Ky(t) \\
&= \begin{pmatrix} -c & 1 \\ 0 & 0 \end{pmatrix} \hat{x}(t|t-1) + \begin{pmatrix} b \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} c - a \\ 0 \end{pmatrix} y(t)
\end{aligned}
$$

It even follows that $\hat{x}_2(t+1|t) = 0$. As this applies at all times, we also have

$$\hat{x}_1(t+1|t) = -c\hat{x}_1(t|t-1) + bu(t) + (c-a)y(t)$$

Problem 5

(a)

$$x(t) = \frac{1}{p+a}v(t) \Rightarrow \phi_x(\omega) = \frac{r_1}{\omega^2 + a^2}$$

The variance of $x(t)$ can be obtained by integrating the spectrum, but even easier by solving a Lyapunov equation,

$$0 = -2aP + r_1 \Rightarrow P = Ex^2(t) = \frac{r_1}{2a}$$

(b) One gets directly

$$G(p) = \frac{K}{p+a+K}$$

which is a first order, lowpass, filter. The static gain is $K/(a+K)$.

(c) Set $\tilde{x} = x - \hat{x}$. Then

$$\begin{aligned} \dot{\tilde{x}} &= (-ax + v) - (-a\hat{x} + Ky - K\hat{x}) \\ &= (-a - K)\tilde{x} + v - Ke \end{aligned}$$

It follows that

$$H_1(p) = \frac{1}{p+a+K}, \quad H_2(p) = \frac{-K}{p+a+K}$$

(d) Using part (a),

$$E\left[H_1(p)v(t)\right]^2 = \frac{r_1}{2(a+K)}, \quad E\left[H_2(p)v(t)\right]^2 = \frac{K^2 r_2}{2(a+K)},$$

(e)

$$V(K) = \frac{r_1 + K^2 r_2}{2(a+K)},$$

$V'(K) = 0 \Rightarrow (a+K) \times 2Kr_2 - (r_1 + K^2 r_2) \times 1 = 0$
$\Rightarrow K^2 r_2 + 2Kar_2 - r_1 = 0 \Rightarrow K = -a \pm \sqrt{a^2 + r_1/r_2} = K = -a + \sqrt{a^2 + r_1/r_2}$

The positive sign is chosen as $a + K > 0$ must hold to guarantee stability.

(f) The Riccati equation gives in this case

$$\begin{aligned} &0 = -aP - Pa + r_1 - P^2/r_2 \\ &P^2 + 2ar_2P - r_1r_2 = 0 \\ &P = -ar_2 + \sqrt{a^2r_2^2 + r_1r_2} \\ &K = P/r_2 = -a + \sqrt{a^2 + r_1/r_2} \end{aligned}$$

which is the same result as in part (e).