Final exam
Automatic Control II
Reglerteknik II 5hp

Date: January 10, 2012

Venue: Bergsbrunnagatan 15, room 1

Responsible teacher: Hans Norlander.

Aiding material: Textbooks (by Glad/Ljung), calculator, mathematical handbooks.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Your solutions can be given in Swedish or in English.

Good luck!

Problem 1 The system with transfer function

$$G(s) = \begin{bmatrix} \frac{2s+1}{s(s+1)} & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \quad \text{has} \quad \begin{cases} \dot{x}(t) & = Ax(t) + Bu(t), \\ y(t) & = Cx(t) \end{cases}$$

as state space representation, where the matrices A, B and C is one combination of the following matrices:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) Which combination of the matrices A_1–A_3, B_1–B_4 and C_1–C_3 constitutes a state space representation of $G(s)$? A full motivation is required. **(4p)**

(b) Is the state space representation associated with A_2, B_2 and C_2 above a minimal realisation? **(2p)**

(c) The system in (b) is sampled using zero-order hold (i.e. the input u is constant between the sampling instants), and a discrete-time state space model $x(kh+h) = Fx(kh)+Gu(kh)$, $y(kh) = Hx(kh)$ is obtained. Determine the matrix F, expressed in the sampling period h. **(2p)**

Problem 2 Consider the system

$$y(t) = \frac{4}{p+4}\left(\frac{1}{p}u(t) + w(t)\right).$$

Here $w(t)$ is a stochastic process that can be modeled as

$$w(t) = \frac{p+b}{p^2 + a_1 p + a_2} v(t), \qquad b, a_1, a_2 > 0,$$

where $v(t)$ is zero mean white noise with intensity $\Phi_v(\omega) = R_v$.

(a) Give a state space representation for the system with y as output, u as input and v as system noise. Let $x_1 = y$.
Hint: First determine one state space representation with y as output and u and w as inputs, and another with w as output and v as input. Then combine these two into one total state space representation. **(4p)**

(b) The spectral density of $w(t)$ is

$$\Phi_w(\omega) = \frac{\omega^2 + 4}{\omega^4 + 4}.$$

Determine the spectral density of the output, $\Phi_y(\omega)$, when $u(t) \equiv 0$. **(2p)**

(c) Determine b, a_1, a_2 and R_v. **(4p)**

Problem 3 A discrete-time system has the state space representation

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_1(k),$$
$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + v_2(k),$$

where v_1 and v_2 are zero mean white noise with

$$E\left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \right\} = \begin{bmatrix} 1 & 0 \\ 0 & 2r \end{bmatrix}.$$

(a) Show that the system is equivalent with the difference equation

$$y(k) = -v_1(k-1) + v_1(k-2) + v_2(k).$$

(2p)

(b) Determine the Kalman filter for obtaining the optimal state estimate $\hat{x}(k|k-1)$. **(4p)**

(c) What is the covariance matrix $E\tilde{x}\tilde{x}^T$ for the state estimation error $\tilde{x} = \tilde{x}(k|k-1) = x(k) - \hat{x}(k|k-1)$? **(2p)**

Problem 4 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true" / "false" are considered!

(a) It is always preferable to have the Nyquist frequency greater than the sampling frequency.

(b) A drawback with MPC is that it is only applicable for systems with sufficiently small time constants.

(c) MPC yields a linear time-invariant controller.

(d) LQG yields a linear time-invariant controller.

(e) LQG handles control constraints better than MPC does.

(f) The weighting matrices $\{Q_1, Q_2\}$ and $\{kQ_1, kQ_2\}$ (with $k > 0$) give identical LQG controllers.

(g) White noise is always periodic.

Each correct answer scores $+1$, each incorrect answer scores -1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(7p)**

Problem 5 A specific space probe *(sv: rymdsond)*, traveling in the interplanetary space, is primarily driven by the solar wind. The deviation from its planned motion (in one dimension) is modeled as

$$\dot{x}(t) = u(t) + v_1(t),$$
$$z(t) = x(t),$$
$$y(t) = x(t) + v_2(t).$$

Here u is the thrust *(sv: dragkraft)* from a rocket engine, z is the velocity and y is the measured velocity (deviations). The velocity is also affected by v_1, representing variations in the solar wind, which is assumed to be white noise. The measurement noise v_2 is also assumed to be white noise. The velocity is controlled using an LQG controller, $u(t) = -L\hat{x}(t) + L_r z_{ref}(t)$, minimizing the cost function

$$V = Ez(t)^2 + \rho Eu(t)^2, \qquad \rho > 0.$$

The noise intensities are assumed to be $R_1 = 1$, $R_2 = r$ and $R_{12} = 0$. The reference is nominally $z_{ref} = 0$, and L_r is chosen to give unit static gain from z_{ref} to z.

(a) The control law can be expressed as

$$u(t) = L_r F_r(p) z_{ref}(t) - F_y(p) y(t).$$

Determine the transfer operators $F_r(p)$ and $F_y(p)$ expressed in r and ρ. **(4p)**

(b) Determine the closed loop tranfer functions from z_{ref} and v_2 to the output z, respectively, both expressed in r, ρ and L_r. **(3p)**

(c) Determine L_r. **(1p)**

(d) Explain how r and ρ affect the properties of the closed loop system (the two transfer functions in (b)). In particular, how is the speed of the responses in z, from changes in z_{ref} and v_2, affected? **(2p)**

Problem 6 *The HW bonus points are exchangeable for this problem.*
Consider the continuous-time system

$$Y(s) = \frac{s+1}{s^2+1} U(s).$$

(a) Assume that a continuous-time proportional controller $U(s) = K(Y(s) - R(s))$ is used. For which $K \in \mathbb{R}$ is the closed loop system stable? **(2p)**

(b) The system is to be controlled by a sampling controller using zero-order hold, i.e. the input u is constant between the sampling instants. Determine the transfer operator $G(q)$ for the sampled, discrete-time model of the system, expressed in the sampling interval h. **(3p)**

(c) Discuss how the possibilities to stabilize the continuous-time system, with *any* sampling controller, depend on the sampling interval h. **(2p)**

Solutions to the exam in Automatic Control II, 2012-01-10:

1. (a) The system $G(s)$ has two inputs, two outputs and poles in the origin and in -1. These fact excludes A_2, B_1, B_3, C_1 and C_2. From the B- and C-matrices it is clear that it is a third order system, why also A_3 is excluded. Thus the A-matrix must be A_1, and the C-matrix is C_3. The B-vector is either B_2 or B_4.

$$C_3(sI - A_1)^{-1}B_2 = G(s) \quad \text{while } C_3(sI - A_1)^{-1}B_4 \neq G(s),$$

and hence $\{A_1, B_2, C_3\}$ is the correct triple.
(b) The controllability and observability matrices are

$$\mathcal{S} = \begin{bmatrix} B_2 & A_2B_2 & A_2^2B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 4 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} C_2 \\ C_2A_2 \\ C_2A_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 4 \end{bmatrix}.$$

Both $\mathcal{S}$ and $\mathcal{O}$ are rank deficient, and thus the system is neither controllable, nor observable. Hence it is not a minimal realisation.
(c)

$$F = e^{A_2h} = \begin{bmatrix} e^{-h} & 0 & 0 \\ 0 & e^{-h} & 0 \\ 0 & 0 & e^{-2h} \end{bmatrix}.$$

2. (a) With $x_1 = y$ we have $(p+4)x_1 = 4(\frac{1}{p}u + w) \Leftrightarrow \dot{x}_1 = -4x_1 + 4\frac{1}{p}u + 4w$. Now set (for example) $x_2 = \frac{1}{p}u \Rightarrow \dot{x}_2 = u$. Using e.g. the controller canonical form for describing w we get

$$\begin{aligned} \dot{x}_3 &= -a_1x_3 - a_2x_4 + v, \\ \dot{x}_4 &= x_3, \\ w &= x_3 + bx_4, \end{aligned}$$

and in total we get

$$\begin{aligned} \dot{x}_1 &= -4x_1 + 4x_2 + 4x_3 + 4bx_4, \\ \dot{x}_2 &= u, \\ \dot{x}_3 &= -a_1x_3 - a_2x_4 + v, \\ \dot{x}_4 &= x_3, \\ y &= x_1. \end{aligned}$$

In vector form this is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -4 & 4 & 4 & 4b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & -a_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} v, \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x \end{aligned}$$

(b) With $u = 0$ we have $y = \frac{4}{p+4}w$, and since $\Phi_y(\omega) = |G(i\omega)|^2\Phi_w(\omega)$ we get

$$\Phi_y(\omega) = \frac{4^2}{|i\omega + 4|^2}\Phi_w(\omega) = \frac{16(\omega^2 + 4)}{(\omega^2 + 16)(\omega^4 + 4)}.$$

(c) Use that $\Phi_w(\omega) = |G_w(i\omega)|^2\Phi_v(s)$:

$$\begin{aligned}|G_w(i\omega)|^2\Phi_v(s) &= \frac{|i\omega + b|^2}{|(i\omega)^2 + ia_1\omega + a_2|^2}R_v \\ &= \frac{\omega^2 + b^2}{(a_2 - \omega^2)^2 + a_1^2\omega^2}R_v = \frac{\omega^2 + b^2}{\omega^4 + (a_1^2 - 2a_2)\omega^2 + a_2^2}R_v\end{aligned}$$

Comparing coefficients with $\Phi_w(\omega) = \frac{\omega^2+4}{\omega^4+4}$ gives

$$\begin{cases} b^2 &= 4 \\ a_1^2 - 2a_2 &= 0 \\ a_2^2 &= 4 \\ R_v &= 1 \end{cases} \Rightarrow \begin{cases} b &= 2 \\ a_1 &= 2 \\ a_2 &= 2 \\ R_v &= 1 \end{cases}$$

3. (a) Use e.g. $y(k) = H(qI - F)^{-1}Nv_1(k) + v_2(k)$:

$$\begin{aligned}y(k) &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}q & -1 \\ 0 & q\end{bmatrix}^{-1}\begin{bmatrix}-1 \\ 1\end{bmatrix}v_1(k) + v_2(k) = \frac{-q+1}{q^2}v_1(k) + v_2(k) \\ &= (-q^{-1} + q^{-2})v_1(k) + v_2(k) = -v_1(k-1) + v_1(k-2) + v_2(k)\end{aligned}$$

(b) The Kalman filter is

$$\hat{x}(k+1|k) = F\hat{x}(k|k-1) + K(y(k) - H\hat{x}(k|k-1)),$$

with the Kalman gain $K = FPH^T(HPH^T + R_2)^{-1}$. $P = P^T \geq 0$ is the solution to the associated DARE:

$$P = FPF^T + NR_1N^T - FPH^T(HPH^T + R_2)^{-1}HPF^T$$

Evaluating the individual terms in the DARE gives

$$P = \begin{bmatrix}p_1 & p_{12} \\ p_{12} & p_2\end{bmatrix} = \begin{bmatrix}p_2 & 0 \\ 0 & 0\end{bmatrix} + \begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix} - \begin{bmatrix}\frac{p_{12}^2}{p_1+2r} & 0 \\ 0 & 0\end{bmatrix}.$$

Thus $p_{12} = -1$ and $p_2 = 1$, and p_1 is the solution ≥ 1 ($\Leftrightarrow P \geq 0$) to

$$p_1 = p_2 + 1 - \frac{p_{12}^2}{p_1 + 2r} = 2 - \frac{1}{p_1 + 2r} \quad \Leftrightarrow \quad p_1^2 + (2r-2)p_1 + 1 - 4r = 0,$$

i.e. $p_1 = 1 - r + \sqrt{r^2 + 2r}$. Thus

$$P = \begin{bmatrix}1 - r + \sqrt{r^2+2r} & -1 \\ -1 & 1\end{bmatrix} \quad \Rightarrow \quad K = \begin{bmatrix}\frac{p_{12}}{p_1+2r} \\ 0\end{bmatrix} = \begin{bmatrix}-\frac{1}{1+r+\sqrt{r^2+2r}} \\ 0\end{bmatrix}.$$

(c) $E\tilde{x}\tilde{x}^T = P = \begin{bmatrix} 1 - r + \sqrt{r^2 + 2r} & -1 \\ -1 & 1 \end{bmatrix}$.

4. **(a)** False (nonsense — per definition $\omega_n = 0.5\omega_s$.); **(b)** False (rather the other way around); **(c)** False (MPC = nonlinear and time-varying); **(d)** True; **(e)** False (the other way around); **(f)** True (only the realtive sizes of Q_1 and Q_2 matter); **(g)** False (for white noise $Ew(t)w(s) = 0$ for all $s \neq t$.).

5. **(a)** The Kalman filter is $\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) = u + K(y - \hat{x})$, where $K = PC^T R_2^{-1} = P/r$ and $P > 0$ is the solution to the CARE

$$0 = AP + PA^T + NR_1N^T - PC^T R_2^{-1} CP = 1 - P^2/r.$$

Thus $P = \sqrt{r}$ and $K = 1/\sqrt{r}$. The state feedback gain is $L = Q_2^{-1}B^T S = S/\rho$, where $S > 0$ is the solution to the CARE

$$0 = A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S = 1 - S^2/\rho,$$

and thus $S = \sqrt{\rho}$ and $L = 1/\sqrt{\rho}$. The controller is

$$\begin{cases} \dot{\hat{x}} &= u - K(y - \hat{x}), \\ u &= -L\hat{x} + L_r z_{ref}, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dot{\hat{x}} &= -(K + L)\hat{x} + L_r z_{ref} + Ky, \\ u &= -L\hat{x} + L_r z_{ref}, \end{cases}$$

so the control law is

$$u = L_r \left(1 - \frac{L}{p + K + L}\right) z_{ref} - \frac{KL}{p + K + L} y = L_r \frac{p + K}{p + K + L} z_{ref} - \frac{KL}{p + K + L} y.$$

Hence (with $K = 1/\sqrt{r}$ and $L = 1/\sqrt{\rho}$)

$$F_r(p) = \frac{L_r(p + 1/\sqrt{r})}{p + 1/\sqrt{r} + 1/\sqrt{\rho}} \quad \text{and} \quad F_y(p) = \frac{1/\sqrt{r\rho}}{p + 1/\sqrt{r} + 1/\sqrt{\rho}}.$$

(b) The closed loop system: $z = \frac{1}{p}(u + v_1)$, $u = F_r(p) z_{ref} - F_y(p)(z + v_2) \Rightarrow$

$$\begin{aligned} z &= \frac{\frac{1}{p}}{1 + \frac{1}{p} F_y(p)} \left(F_r(p) z_{ref} + v_1 - F_y(p) v_2\right) \\ &= \frac{1}{p + \frac{KL}{p+K+L}} \left(\frac{L_r(p + K)}{p + K + L} z_{ref} + v_1 - \frac{KL}{p + K + L} v_2\right) \\ &= \frac{L_r(p + K)}{(p + K)(p + L)} z_{ref} + \frac{p + K + L}{(p + K)(p + L)} v_1 - \frac{KL}{(p + K)(p + L)} v_2 \\ &= \frac{L_r}{p + L} z_{ref} + \frac{p + K + L}{(p + K)(p + L)} v_1 - \frac{KL}{(p + K)(p + L)} v_2 \end{aligned}$$

Thus the transfer function for $z_{ref} \mapsto z$ is $G_c(s) = \frac{L_r}{s + 1/\sqrt{\rho}}$, and for $v_2 \mapsto z$ it is $-T(s) = -\frac{1/\sqrt{r\rho}}{(s + 1/\sqrt{r})(s + 1/\sqrt{\rho})}$.

(c) To get unit static gain one should choose $L_r = 1/\sqrt{\rho}$.

(d) $G_c(s)$ has one pole in $-1/\sqrt{\rho}$, while $T(s)$ has two poles, in $-1/\sqrt{\rho}$ and $-1/\sqrt{r}$. The speed in the response depends on the distance to the origin for the poles. Thus, as $G_c(s)$ will have a faster response if ρ is decreased, while the response of $T(s)$ will be faster if the greatest one of ρ and r is decreased (corresponding to the dominating pole).

6. **(a)** The loop gain is $G_o(s) = K\frac{s+1}{s^2+1}$, and the poles of the closed loop system are the roots of

$$0 = 1+G_o(s) = 1+K\frac{s+1}{s^2+1} \quad \Leftrightarrow \quad 0 = s^2+1+K(s+1) = s^2+Ks+1+K.$$

For stability all poles must lie in the left half plane $\Rightarrow K > 0$.

(b) Put the system on state space form, e.g. controller canonical form:

$$\begin{aligned} \dot{x} &= Ax + Bu = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\ y &= Cx = \begin{bmatrix} 1 & 1 \end{bmatrix} x. \end{aligned}$$

The sampled system is then

$$\begin{aligned} x(kh+h) &= Fx(kh) + Gu(kh), \\ y(kh) &= Cx(kh), \end{aligned}$$

where $F = e^{Ah}$ and $G = \int_0^h e^{At} B dt$. Here

$$e^{At} = \mathcal{L}^{-1}[(sI-A)^{-1}] = \mathcal{L}^{-1}\left\{ \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}^{-1} \right\} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix},$$

so

$$F = \begin{bmatrix} \cos h & -\sin h \\ \sin h & \cos h \end{bmatrix}, \quad G = \int_0^h \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} dt = \begin{bmatrix} \sin h \\ 1-\cos h \end{bmatrix}.$$

For brevity, set $\alpha = \sin h$ and $\beta = \cos h$. The transfer operator is

$$\begin{aligned} G(q) = C(qI-F)^{-1}G &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} q-\beta & \alpha \\ -\alpha & q-\beta \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ 1-\beta \end{bmatrix} \\ &= \frac{1}{q^2-2\beta q+\alpha^2+\beta^2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} q-\beta & -\alpha \\ \alpha & q-\beta \end{bmatrix} \begin{bmatrix} \alpha \\ 1-\beta \end{bmatrix} \\ &= \frac{1}{q^2-2\beta q+\alpha^2+\beta^2} \begin{bmatrix} q+\alpha-\beta & q-\alpha-\beta \end{bmatrix} \begin{bmatrix} \alpha \\ 1-\beta \end{bmatrix} \\ &= \frac{(1+\alpha-\beta)q+\alpha^2+\beta^2-\alpha-\beta}{q^2-2\beta q+\alpha^2+\beta^2}. \end{aligned}$$

Since $\alpha = \sin h$ and $\beta = \cos h$, $\alpha^2+\beta^2 = 1$ and

$$G(q) = \frac{(1+\sin h-\cos h)q+1-\sin h-\cos h}{q^2-2\cos hq+1}.$$

(c) With $h = 2k\pi$ for $k = 1, 2, \ldots$ will give $G(q) = 0$. For these cases the sampling controller "sees" no system, and it will not be possible to stabilize the system.