Final exam
Automatic Control II
Reglerteknik II 5hp

Date: October 24, 2011

Venue: Gimogatan 4, room 1

Responsible teacher: Hans Norlander.

Aiding material: Textbooks (by Glad/Ljung), calculator, mathematical handbooks.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Your solutions can be given in Swedish or in English, with one exception: Please use English in your solution for Problem 1.

Good luck!

Problem 1 *Please use English in your solution for Problem 1.*

(a) Give a state space representation for the continuous-time system

$$Y(s) = \begin{bmatrix} \frac{1}{s+3} & \frac{2s+3}{(s+3)(s+1)} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}.$$

(4p)

(b) The continuous-time system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$

is sampled (zero-order-hold) with sampling period h, and the discrete-time model

$$\begin{cases} x(kh+h) &= Fx(kh) + Gu(kh), \\ y(kh) &= Hx(kh), \end{cases} \quad \text{with} \quad F = \begin{bmatrix} \cos h & \sin h \\ -\sin h & \cos h \end{bmatrix}$$

is obtained. Determine the vector G. **(3p)**

(c) Consider again the sampled system in (b). For some sampling periods h the sampled system is unobservable. Determine the vector H, and specify for which sampling periods h the sampled system is *not* observable. **(2p)**

Problem 2 The unstable, scalar continuous-time system

$$\dot{x}(t) = \alpha x(t) + u(t), \quad \alpha > 0$$

is to be controlled by a LQ-controller, so that the criterion

$$V = \int_0^\infty Qx^2(t) + u^2(t)dt, \quad Q \geq 0,$$

is minimized.

(a) Show that the system is unstable. **(1p)**

(b) Find the feedback gain L in the optimal control law $u(t) = -Lx(t)$. **(4p)**

(c) What will the pole of the closed loop system be as $Q \to 0$? **(3p)**

Problem 3 Consider the (scalar) continuous-time system

$$\dot{\chi}(t) = u(t) + w(t),$$
$$y(t) = \chi(t) + e(t),$$

where $e(t)$ is white noise with intensity $\Phi_e(\omega) = 0.1$, and $w(t)$ is a disturbance described by

$$\dot{\xi}(t) = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \xi(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \nu(t),$$
$$w(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \xi(t),$$

where $\nu(t)$ is white noise with intensity $\Phi_\nu(\omega) = 1$.

(a) Determine the spectral density for $w(t)$, i.e. $\Phi_w(\omega)$. In particular, what is $\Phi_w(0)$, $\Phi_w(1)$ and $\lim_{\omega\to\infty} \Phi_w(\omega)$? **(2p)**

(b) Determine the covariance matrix $\Pi_\xi = E\xi(t)\xi^T(t)$ for the state vector in the disturbance model. **(3p)**

(c) The system model and the disturbance model can be combined into an augmented state space model (the "standard" form)

$$\dot{x}(t) = Ax(t) + Bu(t) + Nv_1(t),$$
$$y(t) = Cx(t) + v_2(t),$$

where $v_1(t)$ and $v_2(t)$ are white noise with intensities $\Phi_{v_1}(\omega) = R_1$ and $\Phi_{v_2}(\omega) = R_2$. Find such an augmented state space model, that is define the state variables in $x(t)$, determine the matrix A and the vectors B, N and C accordingly, and give the values of the noise intensities R_1 and R_2. **(4p)**

Problem 4 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true"/"false" are considered!

(a) The scalar system $\dot{x} = 2x + u + 4\nu$, $y = x + \nu$, (ν is white noise) is on *innovations form*.

(b) The scalar system $\dot{x} = 4x + u + 2\nu$, $y = x + \nu$, (ν is white noise) is on *innovations form*.

(c) In MPC the *control horizon* is normally much longer than the *prediction horizon*.

(d) In MPC the computational load increases with the *control horizon*.

(e) MPC yields a linear time-invariant controller.

(f) LQG yields a linear time-invariant controller.

(g) The sampling frequency should be chosen less than the Nyquist frequency.

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(7p)**

Problem 5 In a water tower drinking water is stored at a height sufficient to pressurize the water distribution system. The water level should be constant. The water supply is monitored by measurements of the water level. This system can be modeled as (the discrete-time system)

$$\begin{aligned} x(k+1) &= x(k) + v_1(k), \\ y(k) &= x(k) + v_2(k), \end{aligned}$$

where $v_1(k)$ describes the fluctuations in the net flow into the water container, and $v_2(k)$ is the measurement noise. Both v_1 and v_2 are white noise, and have the intensities $R_1 = 1$, $R_2 = 0.5$ and $R_{12} = 0.5$. In order to estimate the water level a Kalman filter is used:

$$\hat{x}(k+1|k) = \hat{x}(k|k-1) + K(y(k) - \hat{x}(k|k-1)).$$

(a) Determine the covariance of the estimation error $P = E\tilde{x}^2(k|k-1)$. **(3p)**

(b) What is the Kalman gain K? **(2p)**

(c) Find the transfer function from the measured output $y(k)$ to the estimated level $\hat{x}(k|k-1)$. What is the static gain? **(2p)**

(d) A Kalman filter is designed for a third order system. One of the following matrices,

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix},$$

is the covariance matrix P of the estimation error $\tilde{x}$. Which of the matrices is P? A full motivation is required! **(3p)**

Problem 6 *The HW bonus points are exchangeable for this problem.*
A DC-motor, modeled as

$$Y(s) = \frac{1}{s(s+2)} U(s),$$

is to be controlled by a sampling controller, using proportional control, meaning that the control law is $u(t) = u(kh) = Ke(kh)$ for $kh \le t < (k+1)h$, where $e = y_{ref} - y$ is the control error and $h = 0.5$ is the sampling period (and k is an integer).

(a) Determine the transfer operator $G^d(q)$ of the sampled version of the system (the DC-motor) (assuming that u is constant between the sampling instants). **(3p)**

(b) For which values of the gain K is the (sampled) closed loop system stable when the proportional control above is used? **(4p)**

Solutions to the exam in Automatic Control II, 2011-10-24:

1. (a) Use e.g. the observer canonical form. We have

$$Y(s) = \begin{bmatrix} \frac{1}{s+3} & \frac{2s+3}{(s+3)(s+1)} \end{bmatrix} U(s) = \begin{bmatrix} \frac{s+1}{s^2+4s+3} & \frac{2s+3}{s^2+4s+3} \end{bmatrix} U(s).$$

The observer canonical form is then

$$\dot{x} = \begin{bmatrix} -4 & 1 \\ -3 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

Alternative solution: rewrite the system as

$$Y(s) = \frac{1}{s+3}\left(U_1(s) + \frac{2s+3}{s+1}U_2(s)\right) = \frac{1}{s+3}\left(U_1(s) + 2U_2(s) + \frac{1}{s+1}U_2(s)\right).$$

The choice $X_1(s) = \frac{1}{s+1}U_2(s)$ and $X_2(s) = Y(s)$ gives

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x.$$

(b) Sampling of the continuous-time system $\dot{x} = Ax + Bu$, $y = Cx$ gives the discrete-time system $qx = Fx + Gu$, $y = Hx$, where $F = e^{At}$, $G = \int_0^h e^{At} B dt$, and $H = C$. Here

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \quad \Rightarrow \quad G = \int_0^h \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} dt = \begin{bmatrix} \sin h \\ \cos h - 1 \end{bmatrix}.$$

(c) From (b) $H = C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. The observability matrix is

$$\mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos h & \sin h \end{bmatrix} \quad \text{and} \quad \det \mathcal{O} = \sin h.$$

Thus, unobservable when $\sin h = 0$, i.e. for $h = k\pi$, $k = 1, 2, \ldots$ $(h > 0)$.

2. (a) The system has its pole in $\alpha > 0$, i.e. in the right half plane, and is therefor unstable.

(b) Optimal state feedback gain is $L = Q_2^{-1} B^T S$, where $S = S^T \geq 0$ is the solution to the CARE $0 = A^T S + SA + M^T Q_1 M - SBQ_2^{-1}B^T S$ (see Theorem 9.1). Here $A = \alpha$, $B = 1$, $M = 1$, $Q_1 = Q$ and $Q_2 = 1$. The CARE becomes

$$0 = 2\alpha S + Q - S^2 \quad \Leftrightarrow \quad S^2 - 2\alpha S - Q = 0,$$

with solution $S = \alpha + \sqrt{\alpha^2 + Q}$ (the negative squareroot is rejected since $S \geq 0$). Thus $L = S = \alpha + \sqrt{\alpha^2 + Q}$.

(c) The closed loop system is

$$\dot{x} = \alpha x - Lx = (\alpha - L)x = -\sqrt{\alpha^2 + Q}x,$$

with the pole $-\sqrt{\alpha^2+Q} \to -\alpha$ as $Q \to 0$. (This means that the "cheapest" — only u is penalized — way to stabilize the system is to mirror the unstable pole.)

3. **(a)** We have that $\Phi_w(\omega) = |G_w(i\omega)|^2 \Phi_\nu(\omega)$ if $w(t) = G_w(p)\nu(t)$. Here $\Phi_\nu(\omega) = R_\nu = 1$, and

$$G_w(p) = \frac{p}{p^2+2p+1} = \frac{p}{(p+1)^2},$$

since the state space model is on conroller canonical form. Thus $\Phi_w(\omega) = \frac{\omega^2}{(\omega^2+1)^2} \Rightarrow \Phi_w(0) = 0$, $\Phi_w(1) = 0.25$ and $\lim_{\omega\to\infty} \Phi_w(\omega) = 0$.
(b) For $\dot{\xi} = A_\xi \xi + B_\xi \nu$ the covariance matrix Π_ξ is the solution to the Lyapunov equation $0 = A_\xi \Pi_\xi + \Pi_\xi A_\xi^T + B_\xi R_\nu B_\xi^T$. Setting

$$\Pi_\xi = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} \quad \Rightarrow \quad 0 = \begin{bmatrix} -2p_1 - p_{12} & -2p_{12} - p_2 \\ p_1 & p_{12} \end{bmatrix} + \begin{bmatrix} -2p_1 - p_{12} & p_1 \\ -2p_{12} - p_2 & p_{12} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

or

$$\begin{cases} 0 = -4p_1 - 2p_{12} + 1, \\ 0 = p_1 - 2p_{12} - p_2, \\ 0 = 2p_{12} \end{cases} \quad \Rightarrow \quad \begin{cases} p_1 &= 0.25, \\ p_{12} &= 0, \\ p_2 &= 0.25 \end{cases} \quad \Leftrightarrow \quad \Pi_\xi = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}.$$

(c) Set e.g. $x = \begin{bmatrix} \chi \\ \xi \end{bmatrix} \Rightarrow$

$$\begin{aligned} \dot{x}_1 &= \dot{\chi} = \begin{bmatrix} 1 & 0 \end{bmatrix} \xi + u = x_2 + u, \\ \dot{x}_2 &= \dot{\xi}_1 = -2\xi_1 - \xi + \nu = -2x_2 - x_3 + \nu, \\ \dot{x}_3 &= \dot{\xi}_2 = \xi_1 = x_2, \\ y &= \chi + e = x_1 + e, \end{aligned}$$

and thus

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \nu, \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + e. \end{aligned}$$

Here $v_1 = \nu$ and $v_2 = e$ so $R_1 = R_\nu = 1$ and $R_2 = R_e = 0.1$.

4. **(a)** True (innovations form if $v_1 = v_2 = \nu$ and $A - NC$ stable, here $A - NC = -2$); **(b)** False (here $A - NC = 2$, i.e. unstable); **(c)** False (the other way around); **(d)** True; **(e)** False (MPC = nonlinear and time-varying); **(f)** True; **(f)** False (the Nyquist frequency is half the sampling frequency).

5. **(a)** The covariance $P = P^T \geq 0$ is the solution of the DARE

$P = FPF^T + NR_1N^T - (FPH^T + NR_{12})(HPH^T + R_2)^{-1}(FPH^T + NR_{12})^T.$

Here $F = 1$, $N = 1$, $H = 1$, $R_1 = 1$, $R_2 = 0.5$ and $R_{12} = 0.5 \Rightarrow$

$$P = P + 1 + \frac{(P+0.5)^2}{P+0.5} \quad \Leftrightarrow \quad (P+0.5)^2 = P + 0.5 \quad \Leftrightarrow \quad P = \pm 0.5,$$

where the negative solution is rejected. Thus $P = 0.5$.
(b) $K = (FPH^T + NR_{12})(HPH^T + R_2)^{-1} \Rightarrow K = \frac{0.5+0.5}{0.5+0.5} = 1.$
(c) We have

$$q\hat{x}(k|k-1) = \hat{x}(k|k-1) + y(k) - \hat{x}(k|k-1) = y(k) \quad \Leftrightarrow \quad \hat{x}(k|k-1) = \frac{1}{q}y(k).$$

The transfer function is $G_{\hat{x}y}(z) = \frac{1}{z}$, and the static gain is $G_{\hat{x}y}(1) = 1$.
(d) $P = P^T \geq 0$ and P is 3×3 (since third order system) $\Rightarrow M_4$ is the only possible P.

6. **(a)** In order to sample the system, represent it in state space form, e.g. in controller canonical form:

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad \Rightarrow \quad F = e^{Ah} = \begin{bmatrix} e^{-2h} & 0 \\ 0.5(1-e^{-2h}) & 1 \end{bmatrix} = \begin{bmatrix} e^{-1} & 0 \\ 0.5(1-e^{-1}) & 1 \end{bmatrix},$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x, \qquad H = C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \text{and}$$

$$G = \int_0^h e^{At} B dt = \begin{bmatrix} 0.5(1-e^{-2h}) \\ 0.5(h-0.5) + 0.25e^{-2h} \end{bmatrix} = \begin{bmatrix} 0.5(1-e^{-1}) \\ 0.25e^{-1} \end{bmatrix}.$$

The transfer operator is then

$$\begin{aligned} G^d(q) = H(qI - F)^{-1}G &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} q - e^{-1} & 0 \\ -0.5(1-e^{-1}) & q-1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5(1-e^{-1}) \\ 0.25e^{-1} \end{bmatrix} \\ &= \frac{0.25(1-e^{-1})^2 + 0.25e^{-1}(q-e^{-1})}{(q-1)(q-e^{-1})} = \frac{0.25e^{-1}(q+e-2)}{(q-1)(q-e^{-1})}. \end{aligned}$$

(b) The poles of the closed loop system are given by

$$\begin{aligned} 0 = 1 + G_o(z) = 1 + KG^d(z) \quad &\Leftrightarrow \quad 0 = (z-1)(z-e^{-1}) + 0.25e^{-1}K(z+e-2) \\ &= z^2 + (0.25e^{-1}K - 1 - e^{-1})z + e^{-1} + 0.25K(1-2e^{-1}). \end{aligned}$$

Stability for $z^2 + az + b \Leftrightarrow b < 1$, $a + b > -1$ and $a - b < 1$, so

$$b < 1 \quad \Rightarrow \quad K < 4\frac{1-e^{-1}}{1-2e^{-1}} \approx 9.57,$$

$$a + b > -1 \quad \Rightarrow \quad 0.25K(1-e^{-1}) > 0 \quad \Leftrightarrow \quad K > 0,$$

$$\left(\text{and} \quad a - b < 1 \quad \Rightarrow \quad K < 8\frac{1+e^{-1}}{3e^{-1}-1} \approx 105.6 \right).$$

Hence, the closed loop system is stable for $0 < K < 4\frac{1-e^{-1}}{1-2e^{-1}} \approx 9.57$.