Uppsala University
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Systems and Control
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# Final exam: Automatic Control II (Reglerteknik II, 1TT495)

*Date*: October 26, 2012

*Responsible examiner*: Torsten Söderström

*Preliminary grades*: 3 = 23–32p, 4 = 33–42p, 5 = 43–50p.

**Instructions**

The solutions to the problems can be given in Swedish or in English, except for Problem 2 that **must** be solved in English.

Problem 1 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 1 you will be accounted for the best performance of the homework assignments and Problem 1.)

Solve each problem on a separate page.

Write your code on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

*Aiding material*: Textbooks in automatic control (such as 'Reglerteori – flervariabla och olinjära metoder', 'Reglerteknik – Grundläggande teori', and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators, copies of OH transparencies. Note that the following is **not allowed**: Exempelsamling med lösningar.

# Good luck!

**Problem 1** (*This problem is an alternative to the Homework assignments*)

Consider the first order system

$$\begin{aligned} x(t+1) &= x(t) + bu(t) \\ y(t) &= x(t) \end{aligned}$$

(a) Determine the LQ regulator (the state variable is assumed to be exactly known), when the criterion to be minimized is

$$V = \sum_{t=0}^{\infty} \left[ y^2(t) + \rho u^2(t) \right], \quad \rho \geq 0$$

Here, $\rho$ is an arbitrary weighting parameter. **3 points**

(b) Determine the closed loop pole(s), and express them as functions of $\rho$. **2 points**

(c) How to the optimal feedback vector $L$ and the closed loop poles vary when $\rho$ is increased from 0 to $\infty$? It is sufficient to determine the values of $L$ and the poles at the extreme values of $\rho$. **2 points**

**Problem 2** (*The solution to this problem must be given in English*)

Consider a two-input, one-output system with the transfer function

$$G(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{s+4}{s^2+3s+2} \end{pmatrix} = \begin{pmatrix} G_1(s) & G_2(s) \end{pmatrix}$$

(a) Represent the system in state space form using observable canonical form. **2 points**

(b) Represent the scalar system $G_1(s)$ in controllable form, using $u_1$ as input, and denote the output as $y_1$. Further, represent also the scalar system $G_2(2)$ in controllable form, using $u_2$ as input, and denote the output as $y_2$. **2 points**

(c) Describe the total system on state space form using the results of part (b). First show that $y = y_1 + y_2$. Then use all the state variables introduced in part (b). **2 points**

(d) Examine observability of the state space model derived in part (c). Give an interpretation of the result. **2 points**

**Problem 3**

Consider pole placement design for the double integrator. Assume the system to be given in continuous-time is

$$\mathcal{S}_c : \quad \dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

a) Determine the state feedback for the system $\mathcal{S}_c$ so that the poles are located in $s = -a \pm ia$. **2 points**

(b) Sample the system, with the sampling interval $h$. Present the sampled system in state-space form. **2 points**

(c) Apply the state feedback computed in part (a) to the sampled system derived in part (b). Determine the characteristic equation for the closed-loop system. **3 points**

(d) For the closed-loop system in part (c), determine the pole locations for the case of small $h$. (Make a series expansion and drop all terms that are $O(h^2)$.) Also, show that the closed-loop system becomes unstable if the sampling interval is chosen as $h = 2/a$. **5 points**

**Problem 4**

Consider the second order process

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x + v \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + e \end{aligned}$$

where $v(t)$ is a white noise process with intensity

$$R_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

and $e(t)$ is another white noise process, with intensity $R_2 = r$.

(a) Determine the Kalman filter for estimating the state vector $x(t)$ from the output measurements $y$. (Let $r$ be a parameter.) Where are the eigenvalues of $A - KC$ located? **5 points**

(b) If instead for the Kalman filter, consider an estimate of $x_2$ as $\hat{x}_2 = 0$, what would the variance of the estimation error $x_2 - \hat{x}_2$ be? **3 points**

(c) Assume that the sensor for measuring the output is changed to become very accurate. How does this influence the quality of the estimates? More specifically, what is the limit of the solution $P$ to the Riccati equation when $r \to 0$? **2 points**

(d) One might have guessed that the result in part (c) would imply

$$\lim_{r \to 0} P = 0$$

Explain why this is not the case for the given example. **2 points**

**Problem 5**

Consider a double integrator. The acceleration is due both to an input $u$ and to some white process noise, so with the state variables $x_1 = y$, $x_2 = \dot{y}$ the state space model is

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v$$

(a) Assume that the aim of control is to minimize the variance of the position $y = x_1$. Consider first the noise free case and find a full state feedback controller, $u(t) = -Lx(t)$, so that the criterion

$$V = \int_0^\infty [x_1^2(t) + \rho^4 u^2(t)]dt$$

is minimized. **4 points**

(b) Determine the closed loop poles when the state feedback from (a) is applied. **3 points**

(c) Let the feedback from part (a) be applied. Determine the covariance matrix of the state vector, that is compute

$$P = Ex(t)x^T(t)$$

**4 points**

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# Automatic control II, October 26, 2012
# Answers and brief solutions

### Problem 1

(a) The Ricatti equation

$$S = F^T SF + M^T Q_1 M - F^T SG \left[ G^T SG + Q_2 \right]^{-1} G^T SF$$

gives directly the scalar equation

$$s = s + 1 - \frac{s^2 b^2}{b^2 s + \rho}$$

leading to

$$\begin{aligned} s^2 - s - \rho/b^2 &= 0 \\ s &= \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{\rho}{b^2}} \\ s &= \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{4\rho}{b^2}} \right] \end{aligned}$$

The positive sign is chosen, as $s > 0$ must hold.

One then gets

$$\begin{aligned} L &= \left[ G^T SG + Q_2 \right]^{-1} G^T SF \\ &= \frac{bs}{b^2 s + \rho} \\ &= \frac{\frac{b}{2} \left[ 1 + \sqrt{1 + \frac{4\rho}{b^2}} \right]}{b^2 \left[ \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\rho}{b^2}} + \frac{\rho}{b^2} \right]} \\ &= \frac{1}{b} \frac{1 + \sqrt{1 + \frac{4\rho}{b^2}}}{1 + \frac{2\rho}{b^2} + \sqrt{1 + \frac{4\rho}{b^2}}} \frac{1 + \frac{2\rho}{b^2} - \sqrt{1 + \frac{4\rho}{b^2}}}{1 + \frac{2\rho}{b^2} - \sqrt{1 + \frac{4\rho}{b^2}}} \\ &= \frac{1}{b} \frac{-\frac{2\rho}{b^2} + \frac{2\rho}{b^2}\sqrt{1 + \frac{4\rho}{b^2}}}{\frac{4\rho^2}{b^4}} \\ &= \frac{b}{2\rho} \left[ -1 + \sqrt{1 + \frac{4\rho}{b^2}} \right] \end{aligned}$$

The optimal feedback is

$$u(t) = -Lx(t) = -Ly(t)$$

with $L$ as above.

(b) The closed loop system becomes

$$x(t+1) = x(t) - bLx(t)$$

It is of first order. The only pole is

$$p_d = 1 - bL = 1 - \frac{b^2}{2\rho}\left[-1 + \sqrt{1 + \frac{4\rho}{b^2}}\right]$$

(c) Both $L$ and the pole $p_d$ vary with the weighting parameter $\rho$.

The extreme cases are:

$$\begin{aligned}
\rho = 0 \quad \Rightarrow \quad & L = \frac{b}{2\rho}\left[-1 + 1 + \frac{2\rho}{b^2} + O(\rho^2)\right] = \frac{1}{b} + O(\rho) \to \frac{1}{b} \\
& p_d = 1 - b\left[\frac{1}{b} + O(\rho)\right] = O(\rho) \to 0 \\
\rho = \infty \quad \Rightarrow \quad & L \to 0 \\
& p_d \to 1
\end{aligned}$$

**Problem 2**

(a) It is straightforward to obtain

$$\begin{aligned}
G(s) &= \frac{1}{s^2 + 3s + 2}\begin{pmatrix} s+2 & s+4 \end{pmatrix} \\
\dot{x} &= \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} u \\
y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x
\end{aligned}$$

(b) The two state space forms are readily obtained as

$$\begin{aligned}
\dot{x}_1 &= -x_1 + u_1 \\
y_1 &= x_1
\end{aligned}$$

and

$$\begin{aligned}
\dot{x}_2 &= \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_2 \\
y_2 &= \begin{pmatrix} 1 & 4 \end{pmatrix} x_2
\end{aligned}$$

respectively. Note that $x_1$ is a one-dimensional state vector, while the state vector $x_2$ is of dimension 2.

(c) As $Y(s) = G(s)U(s) = G_1(s)U_1(s) + G_2(s)U_2(s) = Y_1(s) + Y_2(s)$, we have easily with the total state vector $x$ being

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & -2 \\ 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 1 & 4 \end{pmatrix} x\end{aligned}$$

(d) The observability matrix becomes

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 4 \\ -1 & 1 & -2 \\ 1 & -5 & -2 \end{pmatrix}$$

and it holds $\det(\mathcal{O}) = 0$. Hence the system is not observable. The reason is that the realization is not minimal, as it can be represented also as a second order system according to part (a).

**Problem 3**

(a) With the state feedback $u = -\begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix} x$, the closed-loop system becomes

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\ell_1 & -\ell_2 \end{pmatrix} x$$

and has the characteristic polynomial $p(s) = s^2 + \ell_2 s + \ell_1$. The desired characteristic polynomial is $(s+a)^2 + a^2 = s^2 + 2as + 2a^2$. Hence,

$$L = \begin{pmatrix} 2a^2 & 2a \end{pmatrix}$$

(b) The sampled system reads

$$\mathcal{S}_d : \quad x(kh+h) = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} x(kh) + \begin{pmatrix} h^2/2 \\ h \end{pmatrix} u(kh)$$

(c) The closed-loop system becomes $x(kh+h) = Fx(kh)$, with

$$F = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} h^2/2 \\ h \end{pmatrix} \begin{pmatrix} 2a^2 & 2a \end{pmatrix} = \begin{pmatrix} 1 - a^2h^2 & h - ah^2 \\ -2a^2h & 1 - 2ah \end{pmatrix}$$

It has the characterietic polynomial

$$\begin{aligned} p(z) &= z^2 + \left(-2 + 2ah + a^2h^2\right) z + \left(1 - 2ah - a^2h^2 + 2a^3h^3 + 2a^2h^2 - 2a^3h^3\right) \\ &= z^2 + \left(-2 + 2ah + a^2h^2\right) z + (1 - ah)^2 \end{aligned}$$

(d) The poles of the closed loop system can be written as

$$\begin{aligned} p &= 1 - ah - a^2h^2/2 \pm \left[ \left(1 - ah - \frac{a^2h^2}{2}\right)^2 - (1 - ah)^2 \right]^{1/2} \\ &= 1 - ah - a^2h^2/2 \pm \left[-a^2h^2 + a^3h^3 + a^4h^4/4\right]^{1/2} \\ &\approx 1 - ah \pm iah + O(h^2), \quad \text{for small } h \end{aligned}$$

Note that for small values of $h$ it thus holds that the poles in continuous and in discrete time are related as $p_d = e^{p_c h}$.

When $h = 2/a$ the characteristic polynomial becomes

$$p(z) = z^2 + 6z + 1$$

which has zeros at $z = -3 \pm \sqrt{9-1}$. One of the poles is obviously outside the unit cirle.

**Problem 4**

(a) The Kalman filter takes the form

$$\frac{d}{dt}\hat{x} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \hat{x} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \left[ y - \begin{pmatrix} 1 & 0 \end{pmatrix} \hat{x} \right]$$

To find the Kalman gain, set

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

The Riccati equation becomes

$$\begin{aligned} 0 \; = \; & \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ & - \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \frac{1}{r} \end{aligned}$$

leading to the system of equations

$$\begin{aligned} 0 &= 0 + 1 - p_{11}^2/r \\ 0 &= 0 - p_{12} + 1 - p_{11}p_{12}/r \\ 0 &= -2p_{22} + 2 - p_{12}^2/r \end{aligned}$$

and the solution is found to be

$$\begin{aligned} p_{11} &= \sqrt{r} \\ p_{12} &= \frac{1}{1 + p_{11}/r} = \frac{1}{1 + \sqrt{1/r}} = \frac{\sqrt{r}}{\sqrt{r} + 1} \\ p_{22} &= 1 - \frac{p_{12}^2}{2r} = 1 - \frac{1}{2\left(\sqrt{r} + 1\right)^2} \end{aligned}$$

The corresponding Kalman gain becomes

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = PC^T R_2^{-1} = \begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} \frac{1}{r} = \begin{pmatrix} 1/\sqrt{r} \\ \frac{1}{r + \sqrt{r}} \end{pmatrix}$$

The error dynamics $A - KC$ becomes

$$A - KC = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1/\sqrt{r} \\ \frac{1}{r+\sqrt{r}} \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{r} & 0 \\ -\frac{1}{r+\sqrt{r}} & -1 \end{pmatrix}$$

The eigenvalues of $A - KC$ are located in -1 and in $-1/\sqrt{r}$.

(b) The variance of $x_2$ is easily found: As

$$\dot{x}_2 = -x_2 + v_2$$

using the Lyapunov equation gives

$$0 = -2\text{var}(x_2) + 2 \Rightarrow \text{var}(x_2) = 1$$

This would also be the variance of the estimation error if we take $\hat{x}_2 \equiv 0$. In the current example, the components $v_1$ and $v_2$ are correlated. As the measurements of $y$ gives much information about $v_1$, we also get some information about $v_2$, As usual for a Kalman filter, the available information is exploited in an optimal way, and

$$p_{22} < 1$$

holds always (that is, for all $r$).

(c) One obtains directly

$$\lim_{r \to 0} P = \begin{pmatrix} 0 & 0 \\ 0 & 1 - \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0.5 \end{pmatrix}$$

(d) The reason why $P$ does not converge to zero in this example, is that the system is not observable.

## Problem 5

(a) The Riccati equation becomes

$$\begin{aligned} 0 &= A^T S + SA + Q_1 - SBQ_2^{-1}B^T S \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} + \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{\rho^4} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \end{aligned}$$

which leads to the following system of equations

$$\begin{aligned} 0 &= 1 - s_{12}^2/\rho^4 \\ 0 &= s_{11} - s_{12}s_{22}/\rho^4 \\ 0 &= 2s_{12} - s_{22}^2/\rho^4 \end{aligned}$$

The solution is readily found to be

$$S = \begin{pmatrix} \sqrt{2}\rho & \rho^2 \\ \rho^2 & \sqrt{2}\rho^3 \end{pmatrix}$$

The corresponding feedback vector is

$$\begin{aligned} L &= \frac{1}{\rho^4} B^T S \\ &= \frac{1}{\rho^4} \begin{pmatrix} s_{12} & s_{22} \end{pmatrix} \\ &= \begin{pmatrix} \rho^2 & \sqrt{2}\rho^3 \end{pmatrix} / \rho^4 \end{aligned}$$

(b) The characteristic equation for the closed loop system is

$$\begin{aligned} 0 &= \det(sI - A + BL) \\ &= \det\begin{pmatrix} s & -1 \\ \rho^{-2} & s + \sqrt{2}\rho^{-1} \end{pmatrix} = s^2 + \sqrt{2}\rho^{-1}s + \rho^{-2} \end{aligned}$$

which has its solutions in

$$s = \frac{\sqrt{2}}{2}(-1 \pm i)\frac{1}{\rho}$$

(c) The closed loop system is given by the state space model

$$\dot{x} = (A - BL)x + Nv$$

or more explicitly,

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\rho^{-2} & -\sqrt{2}\rho^{-1} \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v$$

Assume that the process noise have unit spectrum. The covariance matrix

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

is found from the Lyapunov equation

$$0 = \begin{pmatrix} 0 & 1 \\ -\rho^{-2} & -\sqrt{2}\rho^{-1} \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 & -\rho^{-2} \\ 1 & -\sqrt{2}\rho^{-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Evaluating the elements leads to the following system of equations

$$\begin{aligned} 0 &= 2p_{12} \\ 0 &= p_{22} - \rho^{-2}p_{11} - \sqrt{2}\rho^{-1}p_{12} \\ 0 &= -2\rho^{-2}p_{12} - 2\sqrt{2}\rho^{-1}p_{22} + 1 \end{aligned}$$

The solution to these equations are found to be

$$P = \frac{1}{2\sqrt{2}}\begin{pmatrix} \rho^3 & 0 \\ 0 & \rho \end{pmatrix}$$