Final exam
Automatic Control II
Reglerteknik II 5hp

Date: August 23, 2013

Venue: Polacksbacken, exam hall

Responsible teacher: Hans Norlander.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Your solutions can be given in Swedish or in English.

Good luck!

Problem 1 A simple model of a DC-motor is given by the state space representation

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t).$$

The DC-motor should be controlled so that the cost function

$$V = \int_0^\infty [y(t)^2 + \gamma^2 u(t)^2] dt, \qquad \gamma > 0,$$

is minimized. The full state vector is measured so state feedback can be used, $u(t) = -Lx(t) + L_r r(t)$.

(a) Give the optimal state feedback gain vector, L, expressed in γ. **(5p)**

(b) Determine the transfer function, from the reference signal r to the output y, for the closed loop system. **(1p)**

(c) What are the poles of the closed loop system? Specifically, determine the closed loop poles for $\gamma = 2$, $\frac{2}{3}$, $\frac{1}{4}$ and $\frac{1}{40}$, respectively. Then mark these poles in the complex plane, and sketch the root locus (sv. rotorten) with respect to γ. Comment on how the speed and the damping of the closed loop system depends on γ (for example, in terms of rise time and overshoot of the step response). **(3p)**

(d) Determine the damping ratio ζ (sv. relativa dämpningen) for the closed loop poles as $\gamma \to 0$. **(1p)**

Problem 2 The continuous-time system

$$Y(s) = G_c(s)U(s), \quad G_c(s) = \frac{1}{(s+1)(s+2)}$$

is to be controlled by a sampling controller, and for the sake of analysis of the closed loop system a discrete-time model of the (open loop) system is needed. The sampling interval is $h = 0.1$ seconds.

(a) Assume that the discrete-time model is obtained by approximating the derivative by Euler's method (forward difference), $s \approx (z-1)/h$, so that the discrete-time transfer function becomes $G_d(z) = G_c((z-1)/h)$. Determine the discrete-time transfer function $G_d(z)$, and its poles and zeros. **(2p)**

(b) Assume that the discrete-time model is obtained by approximating the derivative by Tustin's approximation, $s \approx \frac{2(z-1)}{h(z+1)}$, so that the discrete-time transfer function becomes $G_d(z) = G_c(2(z-1)/h(z+1))$. Determine the discrete-time transfer function $G_d(z)$, and its poles and zeros. **(2p)**

(c) Now use exact sampling (zero-order hold) to get a discrete-time model. Determine the discrete-time transfer function $G_d(z)$, and its poles and zeros. *Hint:* Use e.g. a diagonal state space representation of $G_c(s)$. **(4p)**

Problem 3 A discrete-time system is modeled as

$$y(k) = u(k-1) + w(k),$$

with $w(k)$ given by

$$\begin{aligned} \nu(k+1) &= 0.8\nu(k) + 3e(k), \\ w(k) &= \nu(k) + e(k). \end{aligned}$$

Here $e(k)$ is zero mean white Gaussian noise, and $E[e(k)^2] = 1$.

(a) Determine the spectrum, $\Phi_w(\omega)$, of $w(k)$. **(3p)**

(b) The system can be represented in state space form, as

$$\begin{aligned} x(k+1) &= Fx(k) + Gu(k) + Ne(k), \\ y(k) &= Hx(k) + e(k). \end{aligned}$$

Determine the matrix F, and the vectors G, N and H. Use the state vector $x(k) = \begin{bmatrix} z(k) & \nu(k) \end{bmatrix}^T$, with $z(k) = u(k-1)$ and $\nu(k)$ as above. **(4p)**

(c) Proportional feedback, $u(k) = -K_p y(k)$, is used, and the intention is to minimize the variance of the output, $E[y(k)^2]$. Use F, G, N, H and K_p, and the state space representation in (b), to describe how $E[y(k)^2]$ (for the closed loop system) can be computed.
Hint: First describe how $E[x(k)x^T(k)] = \Pi_x$ is computed. **(3p)**

Problem 4 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true" / "false" are considered!

(a) LQG yields a linear time-invariant controller.

(b) MPC yields a linear time-invariant controller.

(c) In MPC the *control horizon* is normally much longer than the *prediction horizon.*

(d) In MPC the computational load increases with the *control horizon.*

(e) The scalar system $x(k+1) = 0.5x(k) + u(k) + \nu(k)$, $y(k) = x(k) + \nu(k)$, ($\nu(k)$ is white noise) is on *innovations form.*

(f) The scalar system $x(k+1) = -0.5x(k) + u(k) + \nu(k)$, $y(k) = x(k) + \nu(k)$, ($\nu(k)$ is white noise) is on *innovations form.*

(g) The Nyquist frequency is always 50% of the sampling frequency.

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(7p)**

Problem 5 A system with one input and two outputs has the transfer function representation

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{2s+2}{(s+2)(s+3)} \\ \frac{2s+4}{(s+3)(s+4)} \end{bmatrix} U(s).$$

(a) Give a state space representation for the system. **(5p)**

(b) Examine whether or not your state space representation in (a) is a *minimal realization.* **(3p)**

Problem 6 *The HW bonus points are exchangeable for this problem.* Consider the scalar discrete-time system

$$x(k+1) = 0.5x(k) + 2.5v(k),$$
$$y(k) = x(k) + v(k),$$

where $v(k)$ is zero mean, Gaussian white noise, and $E[v(k)^2] = 1$.

(a) Determine the *one-step predictor* version of the Kalman filter (the "standard observer" form), for producing the estimate $\hat{x}(k|k-1)$. **(4p)**

(b) Assume that $\hat{x}(k|k-1) = -0.5$, and that $y(k) = 2$. Compute the optimal prediction $\hat{x}(k+1|k)$. **(1p)**

(c) Assume again that $\hat{x}(k|k-1) = -0.5$, and that $y(k) = 2$. Compute the optimal estimate $\hat{x}(k|k)$. **(2p)**

Solutions to the exam in Automatic Control II, 2013-08-23:

1. (a) Use Theorem 9.1: $L = Q_2^{-1}B^TS$ where $S = S^T \geq 0$ is a solution to the CARE $0 = A^TS + SA + M^TQ_1M - SBQ_2^{-1}B^TS$. Here $M = C$, $Q_1 = 1$ and $Q_2 = \gamma^2$, so spelled out the CARE becomes

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}s_1 & s_{12}\\s_{12} & s_2\end{bmatrix} + \begin{bmatrix}s_1 & s_{12}\\s_{12} & s_2\end{bmatrix}\begin{bmatrix}-1 & 0\\1 & 0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}$$
$$- \frac{1}{\gamma^2}\begin{bmatrix}s_1 & s_{12}\\s_{12} & s_2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}s_1 & s_{12}\\s_{12} & s_2\end{bmatrix}$$
$$= \begin{bmatrix}-s_1 + s_{12} & -s_{12} + s_2\\0 & 0\end{bmatrix} + \begin{bmatrix}-s_1 + s_{12} & 0\\-s_{12} + s_2 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} - \frac{1}{\gamma^2}\begin{bmatrix}s_1^2 & s_1s_{12}\\s_1s_{12} & s_{12}^2\end{bmatrix}.$$

This is equivalent to the equation system

$$0 = -2s_1 + 2s_{12} - \frac{1}{\gamma^2}s_1^2, \tag{1}$$
$$0 = -s_{12} + s_2 - \frac{1}{\gamma^2}s_1s_{12}, \tag{2}$$
$$0 = 1 - \frac{1}{\gamma^2}s_{12}^2. \tag{3}$$

Noting that $L = \frac{1}{\gamma^2}\begin{bmatrix}1 & 0\end{bmatrix}S = \frac{1}{\gamma^2}\begin{bmatrix}s_1 & s_{12}\end{bmatrix}$, we only need to solve for s_1 and s_{12}. From (3) we get $s_{12} = \pm\gamma$, which when put in (1) gives

$$0 = \frac{1}{\gamma^2}s_1^2 + 2s_1 \mp 2\gamma \quad \Leftrightarrow \quad s_1 = \gamma^2(-1 \pm \sqrt{1 \pm 2/\gamma}) = \gamma^2(-1 + \sqrt{1 + 2/\gamma}),$$

where the last identity follows from that $s_1 \geq 0$ (since $S \geq 0$). Thus $s_{12} = \gamma$, and $L = \begin{bmatrix}-1 + \sqrt{1 + 2/\gamma} & 1/\gamma\end{bmatrix}$.

(b) The closed loop system becomes

$$\dot{x} = \begin{bmatrix}-1 & 0\\1 & 0\end{bmatrix}x + \begin{bmatrix}1\\0\end{bmatrix}\left(-\begin{bmatrix}-1 + \sqrt{1 + 2/\gamma} & 1/\gamma\end{bmatrix}x + L_rr\right)$$
$$= \begin{bmatrix}-\sqrt{1 + 2/\gamma} & -1/\gamma\\1 & 0\end{bmatrix}x + \begin{bmatrix}1\\0\end{bmatrix}L_rr,$$
$$y = \begin{bmatrix}0 & 1\end{bmatrix}x,$$

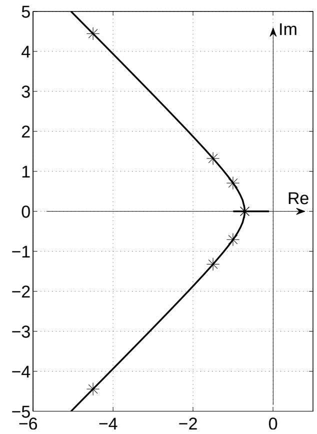
which is on controller canonical form. The transfer function is then readily obtained as

$$G(s) = \frac{L_r}{s^2 + \sqrt{1 + 2/\gamma}\cdot s + 1/\gamma}.$$

(c) The poles are given by $0 = s^2 + \sqrt{1 + 2/\gamma} \cdot s + 1/\gamma$ so with

$$
\begin{aligned}
\gamma = 2: &\quad 0 = s^2 + \sqrt{2}s + 0.4 &\Leftrightarrow&\quad s = \sqrt{0.5} \quad \text{(double)} \\
\gamma = \frac{2}{3}: &\quad 0 = s^2 + 2s + 1.5 &\Leftrightarrow&\quad s = -1 \pm i\sqrt{0.5} \\
\gamma = \frac{1}{4}: &\quad 0 = s^2 + 3s + 4 &\Leftrightarrow&\quad s = -1.5 \pm i\sqrt{1.75} \\
\gamma = \frac{1}{40}: &\quad 0 = s^2 + 9s + 40 &\Leftrightarrow&\quad s = -4.5 \pm i\sqrt{19.75}
\end{aligned}
$$

The root locus is shown below. The poles are real valued for $\gamma \geq 2$ and



complex for $\gamma < 2$, and the smaller γ gets, the longer from the origin the poles move, and the imaginary parts do not grow faster than the real part. Thus, the closed loop gets faster when γ decreases, and the damping goes to a constant value, that is, the rise time decreases while the overshoot is constant.

(d) Compare the pole polynomial $s^2 + \sqrt{1 + 2/\gamma} \cdot s + 1/\gamma$ with the standard form $s^2 + 2\zeta\omega_0 s + \omega_0^2 \Rightarrow \omega_0^2 = 1/\gamma$ and $2\zeta\omega_0 = \sqrt{1 + 2/\gamma}$. Thus

$$
\zeta = \frac{2\zeta\omega_0}{2\omega_0} = \frac{\sqrt{1 + 2/\gamma}}{2\sqrt{1/\gamma}} = \frac{\sqrt{\gamma + 2}}{2} \quad \Rightarrow \quad \lim_{\gamma \to 0} \zeta = \frac{1}{\sqrt{2}}.
$$

2. (a) Here, with $h = 0.1$,

$$
G_d(z) = \frac{1}{(10(z-1)+1)(10(z-1)+2)} = \frac{1}{(10z-9)(10z-8)}.
$$

No zeros, and poles in $+0.9$ and $+0.8$.

(b) Now

$$
G_d(z) = \frac{1}{\left(\frac{20(z-1)}{z+1} + 1\right)\left(\frac{20(z-1)}{z+1} + 2\right)} = \frac{(z+1)^2}{(21z-19)(22z-18)},
$$

with a double zero in -1, and poles in $+19/21 = +0.9048$ and $+18/22 = +0.8182$.

(c) First represent the continuous-time system in state space form. Note that $G(s) = \frac{1}{s+1} - \frac{1}{s+2}$, so on diagonal form

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & -1 \end{bmatrix} x. \end{aligned}$$

With zero-order hold sampling the discrete-time state space model becomes

$$\begin{cases} x(kh+h) &= Fx(kh) + Gu(kh), \\ y(kh) &= Cx(kh), \end{cases} \qquad \text{where} \quad F = e^{Ah}, \quad G = \int_0^h e^{At} B dt.$$

Here

$$e^{Ah} = \begin{bmatrix} e^{-h} & 0 \\ 0 & e^{-2h} \end{bmatrix}, \quad \int_0^h e^{At} B dt = \int_0^h \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix} dt = \begin{bmatrix} 1 - e^{-h} \\ (1 - e^{-2h})/2 \end{bmatrix},$$

so

$$\begin{aligned} G_d(z) &= C(zI - F)^{-1} G = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} z - e^{-h} & 0 \\ 0 & z - e^{-2h} \end{bmatrix}^{-1} \begin{bmatrix} 1 - e^{-h} \\ (1 - e^{-2h})/2 \end{bmatrix} \\ &= \frac{1 - e^{-h}}{z - e^{-h}} - \frac{1 - e^{-2h}}{2(z - e^{-2h})} = \frac{2(z - e^{-2h})(1 - e^{-h}) - (z - e^{-h})(1 - e^{-2h})}{2(z - e^{-h})(z - e^{-2h})} \\ &= \frac{(1 - e^{-h})^2 (z + e^{-h})}{2(z - e^{-h})(z - e^{-2h})}. \end{aligned}$$

There is one zero in $-e^{-h} = -0.9048$, and poles in $e^{-h} = +0.9048$ and $e^{-2h} = +0.8187$.

3. (a) Generally, for a discrete-time system $y(k) = G(q)u(q)$, the spectrum is $G(e^{i\omega})\Phi_u(\omega)G^*(e^{i\omega})$, which for the scalar case equals $G(e^{i\omega})G(e^{-i\omega})\Phi_u(\omega) = |G(e^{i\omega})|^2 \Phi(\omega)$. Here we have $w(k) = G(q)e(k)$, with $\Phi_e(\omega) = E[e(k)^2] = 1$ and $G(q) = 1 \cdot (q - 0.8)^{-1} \cdot 3 + 1 = \frac{q+2.2}{q-0.8}$, so

$$\Phi_w(\omega) = \frac{e^{i\omega} + 2.2}{e^{i\omega} - 0.8} \cdot \frac{e^{-i\omega} + 2.2}{e^{-i\omega} - 0.8} = \frac{1 + 2.2^2 + 2.2(e^{i\omega} + e^{-i\omega})}{1 + 0.8^2 - 0.8(e^{i\omega} + e^{-i\omega})} = \frac{5.84 + 4.4\cos\omega}{1.64 - 1.6\cos\omega}$$

(b) We have

$$\begin{cases} z(k+1) &= u(k), \\ \nu(k+1) &= 0.8\nu(k) + 3e(k), \\ y(k) &= z(k) + \nu(k) + e(k), \end{cases} \qquad \Leftrightarrow$$

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0 & 0 \\ 0 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 3 \end{bmatrix} e(k) = Fx(k) + Gu(k) + Ne(k), \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(k) + e(k) = Hx(k) + e(k). \end{aligned}$$

(c) Apply the control law, $u(k) = -K_p y(k) = -K_p Hx(k) - K_p e(k)$, to the state space model to obtain the closed loop system:

$$\begin{aligned} x(k+1) &= Fx(k) + G(-K_p Hx(k) - K_p e(k)) + Ne(k) \\ &= (F - GK_p H)x(k) + (N - GK_p)e(k), \\ y(k) &= Hx(k) + e(k). \end{aligned}$$

The covariance $\Pi_x = E[x(k)x^T(k)]$ is the solution to the discrete-time Lyapunov equation

$$\Pi_x = (F - GK_p H)\Pi_x(F - GK_p H)^T + (N - GK_p)R_e(N - GK_p)^T, \quad R_e = E[e(k)^2] = 1.$$

The variance of the output is then

$$\begin{aligned} E[y(k)^2] &= E[(Hx(k) + e(k))(Hx(k) + e(k))^T] \\ &= HE[x(k)x^T(k)]H^T + 2HE[x(k)e(k)] + E[e(k)^2] = H\Pi_x H^T + 1. \end{aligned}$$

($E[x(k)e(k)] = 0$ since $e(k)$ is white noise and $x(k)$ only depend on $e(t)$ for $t \le k-1$.)

4. **(a)** True; **(b)** False (MPC = nonlinear and time-varying); **(c)** False (the other way around); **(d)** True; **(e)** True (innovations form if $v_1 = v_2 = \nu$ and $F - NH$ stable, here $F - NH = -0.5$); **(f)** False (here $F - NH = -1.5$, i.e. unstable); **(g)** True (That is the definition of the Nyquist frequency.)

5. **(a)** For systems with one input the controller canonical form works. To be able to use this, see that both elements in the transfer function matrix have the same denominator:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{(2s+2)(s+4)}{(s+2)(s+3)(s+4)} \\ \frac{(2s+4)(s+2)}{(s+2)(s+3)(s+4)} \end{bmatrix} U(s) = \begin{bmatrix} \frac{2s^2+10s+8}{s^3+9s^2+26s+24} \\ \frac{2s^2+8s+8}{s^3+9s^2+26s+24} \end{bmatrix} U(s).$$

The controller canonical form is then

$$\begin{aligned} x(t) &= \begin{bmatrix} -9 & -26 & -24 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t), \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} 2 & 10 & 8 \\ 2 & 8 & 8 \end{bmatrix} x(t). \end{aligned}$$

(b) Minimal realization $\Leftrightarrow$ both controllable and observable. The controllability matrix,

$$\mathcal{S} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -9 & 55 \\ 0 & 1 & -9 \\ 0 & 0 & 1 \end{bmatrix},$$

clearly has full rank $\Leftrightarrow$ controllable. The observatbility matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 2 & 10 & 8 \\ 2 & 8 & 8 \\ -8 & -44 & -48 \\ -10 & -44 & -48 \\ 28 & 160 & 192 \\ 46 & 212 & 240 \end{bmatrix}.$$

The three first rows of $\mathcal{O}$ are linearily independent, so $\mathcal{O}$ has full rank $\Leftrightarrow$ observable. Thus, the state space representation is a minimal realization.

6. **(a)** For a discrete-time system, with no input, the Kalman predictor is

$$\hat{x}(k+1|k) = F\hat{x}(k|k-1) + K(y(k) - H\hat{x}(k|k-1))$$

where $K = (FPH^T + NR_{12})(HPH^T + R_2)^{-1}$, and $P = P^T \geq 0$ is a solution to the DARE

$$P = FPF^T + NR_1N^T - (FPH^T + NR_{12})(HPH + R_2)^{-1}(FPH^T + NR_{12})^T.$$

Here $F = 0.5$, $N = 2.5$, $H = 1$ and $R_1 = R_{12} = R_2 = 1$, so the DARE becomes

$$P = 0.5^2P + 2.5^2 - \frac{(0.5P + 2.5)^2}{P+1} \quad \Leftrightarrow$$
$$P(P+1) = (0.25P+6.25)(P+1) - (0.25P^2+2.5P+6.25) \quad \Leftrightarrow \quad P(P-3) = 0.$$

Thus $P = 3$ (the largest solution), and $K = \frac{0.5P+2.5}{P+1} = 1$.
(b) From (a) we have that

$$\hat{x}(k+1|k) = 0.5\hat{x}(k|k-1) + 1\cdot(y(k) - \hat{x}(k|k-1)) = 0.5(-0.5) + 2 - (-0.5) = 2.25.$$

(c) The estimate is $\hat{x}(k|k) = \hat{x}(k|k-1) + \tilde{K}(y(k) - H\hat{x}(k|k-1))$, where $\tilde{K} = PH^T(HPH^T + R_2)^{-1}$ (see Eq. (5.100) in Glad/Ljung). Here $\tilde{K} = \frac{P}{P+1} = 0.75$, so

$$\hat{x}(k|k) = -0.5 + 0.75(2 - (-0.5)) = 1.375.$$