Final exam
Automatic Control II
Reglerteknik II 5hp

Date: January 15, 2013

Venue: Polacksbacken, exam hall

Responsible teacher: Hans Norlander.

Aiding material: Textbooks (by Glad/Ljung), calculator, mathematical handbooks.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Your solutions can be given in Swedish or in English.

Good luck!

Problem 1 The discrete-time system

$$x(k+1) = 0.8x(k) + u(k) + v(k),$$
$$y(k) = x(k) + e(k),$$

is given. Here v and e are mutually independent white noises with intensities $\Phi_v(\omega) = Ev(k)^2 = 0.36$ and $\Phi_e(\omega) = Ee(k)^2 = 1$.

(a) The standard observer is

$$\hat{x}(k+1) = 0.8\hat{x}(k) + u(k) + K(y(k) - \hat{x}(k)).$$

For which K is the observer stable? **(1p)**

(b) The estimation error is $\tilde{x}(k) = x(k) - \hat{x}(k)$. Compute the variance $\Pi_{\tilde{x}} = E\tilde{x}(k)^2$ as a function of K, i.e. find $\Pi_{\tilde{x}}(K)$, for the standard observer in (a). **(2p)**

(c) Comment on how the values of $\Pi_{\tilde{x}}(K)$ correspond to the stabilizing values of K in (a). **(2p)**

(d) Determine the Kalman filter for the system,

$$\hat{x}(k+1|k) = 0.8\hat{x}(k|k-1) + u(k) + K(y(k) - \hat{x}(k|k-1)),$$

i.e. find the Kalman gain K. **(4p)**

(e) Show that the Kalman gain K obtained in (d) is stabilizing in accordance with your results in (a), and that K indeed minimizes $\Pi_{\tilde{x}}(K)$ in (b). **(3p)**

Problem 2 Consider the continuous-time system

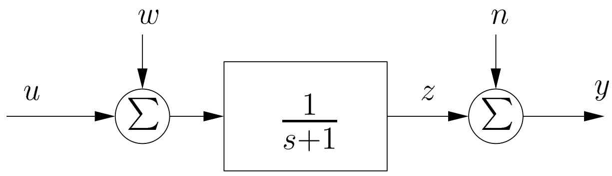
$$\dot{x}(t) = -\alpha x(t) + u(t),$$
$$y(t) = (\beta - \alpha)x(t) + u(t).$$

(a) Determine the pole and the zero of the system. Denote the pole with p and the zero with z. **(1p)**

(b) Determine the discrete-time model of the system using zero-order hold sampling (u is constant between the sampling instants) with the sampling interval h. **(2p)**

(c) Find the pole and the zero of the discrete-time system and give the mapping from p and z in (a) to these as functions of the sampling interval h. **(2p)**

Problem 3 Consider the system in the block diagram below. The system is



affected by the zero mean input disturbance w and the zero mean measurement disturbance n, modeled by

$$\begin{cases} \dot{\xi} = -2\xi + 2d, \\ w = \xi, \end{cases} \quad \Phi_d(\omega) = R_d = 1, \qquad \begin{cases} \dot{\chi} = -\chi + e, \\ n = -\chi + e, \end{cases} \quad \Phi_e(\omega) = R_e = 1.$$

Thus, d and e are both white noise, and they are mutually independent so that $\Phi_{de}(\omega) = R_{de} = 0$.

(a) Determine the variance of w, i.e. compute $Ew^2 = E\xi^2$. **(2p)**

(b) Determine the spectrum $\Phi_n(\omega)$ for the measurement disturbance n. **(2p)**

(c) The system can be represented in the standard form

$$\begin{aligned} \dot{x} &= Ax + Bu + Nv_1, \\ z &= Mx, \\ y &= Cx + v_2 \end{aligned}$$

where

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ is white noise with intensity } \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}.$$

Give A, B, C, N, M, R_1, R_2 and R_{12} for this particular case. **(5p)**

Problem 4 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true" / "false" are considered!

(a) A Kalman filter is always stable.

(b) For a Kalman filter the *innovation*, $\nu = y - C\hat{x} - Du$, is white noise if the model and the noise intensities are correct.

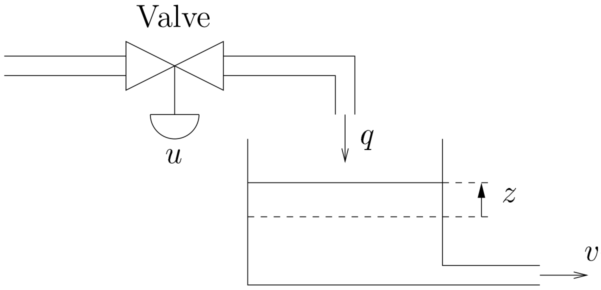
(c) The discrete-time system $y(k) = \frac{0.25}{q+0.25}u(k)$ has unit static gain.

(d) The sampling frequency is 50% of the Nyquist frequency.

(e) White noise is always periodic.

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(5p)**

Problem 5 The level in the tank in the figure below should be controlled. The objective is to keep the level as close to the operating point as possible, corresponding to a half filled tank. This is in order to avoid two situations: that the tank runs empty and overflow. All variables indicated in the figure



are deviations from the operating point: z is the level, q is the inflow which is controlled by a valve with input u, and v_1 is the deviation in the outflow. The tank is described by the state space model

$$\begin{aligned}\dot{x}(t) &= \frac{1}{A}(q(t) + v_1(t)),\\ z(t) &= x(t),\\ y(t) &= x(t) + v_2(t).\end{aligned}$$

Here $A = 1$ is the cross-section area of the tank, y is the measured level and v_2 is measurement noise. Both v_1 and v_2 are assumed to be zero mean white noise with intensities $R_1 = 1$, $R_2 = r$ and $R_{12} = 0$.

The level is controlled using an LQG controller, $u(t) = -L\hat{x}(t)$, minimizing the cost function $V = Ez(t)^2 + \rho Eu(t)^2$, $\rho > 0$.

(a) As a simplification it is assumed that $q(t) = K_q u(t)$, and $K_q = 1$. The control law can be expressed as $u(t) = -F_y(p)y(t)$. Determine the transfer operator $F_y(p)$ expressed in r and ρ for this case. **(3p)**

(b) Determine the poles of the closed loop system with the controller in (a) (under the assumptions in (a)). Discuss how ρ and r influence the properties of the closed loop system. **(2p)**

(c) It turns out that the dynamics in the valve are of significance, and can be modeled as $Q(s) = \frac{K_q}{s\tau+1}U(s)$, $\tau > 0$. Give a new state space model for the system, incorporating the valve dynamics. Also, state the equations needed for the computation of the LQG controller in this case. (You do not need to solve the LQG problem here.) **(3p)**

(d) Due to the valve the input is bounded, $|u| \leq C_u$, and the avoidance of the empty tank and overflow situations can be expressed as $|z| \leq C_z$. A possibility to account for these constraints, and still have optimal control, is to use MPC. Explain briefly how to proceed to obtain an MPC controller, e.g. how should the model be altered, which user parameters need to be chosen, what is the effect of these user parameters, etc. **(4p)**

Problem 6 *The HW bonus points are exchangeable for this problem.*

(a) Give a state space representation for the TISO (two inputs single output) discrete-time system

$$y(k) = \begin{bmatrix} \frac{1}{q-1} & \frac{1}{q^2-1} \end{bmatrix} u(k), \quad \text{where} \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

(3p)

(b) Show that the double integrator

$$Y(s) = \frac{1}{s^2} U(s)$$

is stabilized by the feedback

$$U(s) = F(s)(Y_{ref}(s) - Y(s))$$

when the controller is the PD-controller

$$F(s) = 2(s+1).$$

(1p)

(c) In practice the control in (b) must be performed by a sampling, discrete-time controller. The sampled (zero-order hold) model of the double integrator is

$$y(kh) = \frac{h^2}{2} \frac{q+1}{(q-1)^2} u(kh),$$

where h is the sampling interval. Assume that the PD-controller in (b) is implemented by using Tustin's approximation,

$$s = \frac{2}{h} \frac{q-1}{q+1}.$$

Determine for which sampling intervals h the closed loop system will be stable when using the sampling PD-controller.

Hint: The zeros of the polynomial $z^2 + az + b$ lies inside the unit circle exactly when $|a| - 1 < b < 1$. **(3p)**

Solutions to the exam in Automatic Control II, 2013-01-15:

1. (a) The observer pole is $0.8 - K$, and for stability it must be inside the unit circle, i.e.

$$|0.8 - K| < 1 \quad \Leftrightarrow \quad -0.2 < K < 1.8.$$

(b) The estimation error is governed by

$$\begin{aligned}\tilde{x}(k+1) &= x(k+1) - \hat{x}(k+1) \\ &= 0.8x(k) + u(k) + v(k) - (0.8\hat{x}(k) + u(k) + K(x(k) + e(k) - \hat{x}(k))) \\ &= \underbrace{(0.8 - K)}_{=F}\tilde{x}(k) + \underbrace{\begin{bmatrix}1 & -K\end{bmatrix}}_{=G}\underbrace{\begin{bmatrix}v(k) \\ e(k)\end{bmatrix}}_{=\nu},\end{aligned}$$

i.e. $\tilde{x}(k+1) = F\tilde{x}(k) + G\nu(k)$. The (co-)variance of $\tilde{x}$ is the solution to the discrete-time Lyapunov equation $\Pi_{\tilde{x}} = F\Pi_{\tilde{x}}F^T + GR_\nu G^T$. Here we have

$$\begin{aligned}\Pi_{\tilde{x}} = (0.8-K)^2\Pi_{\tilde{x}} + \begin{bmatrix}1 & -K\end{bmatrix}\begin{bmatrix}0.36 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}1 \\ -K\end{bmatrix} = (0.8-K)^2\Pi_{\tilde{x}} + 0.36 + K^2 \\ \Leftrightarrow \quad \Pi_{\tilde{x}}(K) = \frac{K^2 + 0.36}{1 - (0.8 - K)^2} = \frac{K^2 + 0.36}{0.36 + 1.6K - K^2}.\end{aligned}$$

(c) The numerator of $\Pi_{\tilde{x}}(K)$ is always positive. The denominator is positive for $|0.8 - K| < 1$, is zero for $|0.8 - K| = 1$ and is negative elsewhere. This means that $\Pi_{\tilde{x}}(K)$ is positive exactly when K stabilizes the observer, and goes to infinity as K approaches the borders of the stabilizing interval. Outside that interval $\Pi_{\tilde{x}}(K)$ is negative and thus it is no longer a (co-)variance.

(d) Use Theorem 5.6: The associated DARE is

$$\begin{aligned}P = 0.8^2P + 0.36 - \frac{0.8^2P^2}{P+1} \quad &\Leftrightarrow \quad P(P+1) = (P+1)(0.8^2P + 0.36) - 0.8^2P^2 \\ \Leftrightarrow \quad P^2 + P &= 0.64P^2 + 0.36P + 0.64P + 0.36 - 0.64P^2 \Leftrightarrow \\ P^2 &= 0.36 \quad \Leftrightarrow \quad P = \pm 0.6,\end{aligned}$$

where the negative root is rejected. Thus $P = 0.6$, and

$$K = \frac{0.8P}{P+1} = \frac{0.8 \cdot 0.6}{1.6} = 0.3.$$

(e) Find the extremes within the stabilizing interval, i.e. find K such that $\frac{d\Pi_{\tilde{x}}}{dK} = 0$:

$$\begin{aligned}\frac{d\Pi_{\tilde{x}}}{dK} &= \frac{2K(0.36 + 1.6 - K^2) - (K^2 + 0.36)(1.6 - 2K)}{(0.36 + 1.6K - K^2)^2} \\ &= \frac{1.6(K^2 + 0.9K - 0.36)}{(0.36 + 1.6K - K^2)^2},\end{aligned}$$

which is zero for $0 = K^2 + 0.9K - 0.36 \Leftrightarrow K = -0.45 \pm \sqrt{0.45^2 + 0.36} = -0.45 \pm 0.75$, of which only the positive root, $K = 0.3$, lies in the stabilizing interval. Also, $\frac{d\Pi_{\tilde{x}}}{dK} < 0$ for $K < 0.3$ and $\frac{d\Pi_{\tilde{x}}}{dK} > 0$ for $K > 0.3$, so $K = 0.3$ is indeed a minimum of $\Pi_{\tilde{x}}(K)$. Furthermore,

$$\Pi_{\tilde{x}}(0.3) = \frac{0.3^2 + 0.36}{1 - 0.5^2} = \frac{0.45}{0.75} = 0.6.$$

2. (a) The transfer function is

$$G(s) = 1 + \frac{\beta - \alpha}{s + \alpha} = \frac{s + \beta}{s + \alpha}$$

The pole is $p = -\alpha$ and the zero is $z = -\beta$.
(b) The sampled system is $qx = Fx + Gu$, $y = (\beta - \alpha)x + u$, where $F = e^{-\alpha h}$ and $G = \int_0^h e^{\alpha t} dt = (1 - e^{-\alpha h})/\alpha$. That is,

$$\begin{aligned} x(kh + h) &= e^{-\alpha h} x(kh) + \frac{1 - e^{-\alpha h}}{\alpha} u(kh), \\ y(kh) &= (\beta - \alpha)x(kh) + u(kh). \end{aligned}$$

(c) The transfer function for the sampled system is

$$\begin{aligned} G(q) &= 1 + \frac{(\beta - \alpha)(1 - e^{-\alpha h})/\alpha}{q - e^{-\alpha h}} \\ &= \frac{q - e^{-\alpha h} + (\beta - \alpha)(1 - e^{-\alpha h})/\alpha}{q - e^{-\alpha h}} = \frac{q - 1 + \frac{\beta}{\alpha}(1 - e^{-\alpha h})}{q - e^{-\alpha h}}. \end{aligned}$$

The pole is $e^{-\alpha h} = e^{ph}$, and the zero is $1 - \frac{\beta}{\alpha}(1 - e^{-\alpha h}) = 1 - \frac{z}{p}(1 - e^{ph})$.

3. (a) For a continuous-time system $\dot{x} = Ax + Nd$, where d is white noise with intensity R_d, the covariance of x is $Exx^T = \Pi_x$ given as the solution of the continuous-time Lyapunov equation $A\Pi_x + \Pi_x A^T + NR_dN^T = 0$ (Thm. 5.3). This means that $Ew^2 = \Pi_\xi$, and

$$-4\Pi_\xi + 4 = 0 \qquad \Leftrightarrow \qquad Ew^2 = \Pi_\xi = 1.$$

(b) We have that $n(t) = \left(-\frac{1}{p+1} + 1\right) e(t) = \frac{p}{p+1} e(t)$, and since $\Phi_n(\omega) = |G(i\omega)|^2 \Phi_e(\omega)$ and $\Phi_e(\omega) = R_e = 1$ we get

$$\Phi_n(\omega) = \frac{i\omega(-i\omega)}{(i\omega + 1)(-i\omega + 1)} = \frac{\omega^2}{\omega^2 + 1}.$$

(c) Note that $\dot{z} = -z + u + w$ (from the block diagram). By setting $x = \begin{bmatrix} z & \xi & \chi \end{bmatrix}^T$, and observing (from the block diagram) that

$$\begin{aligned} \dot{z} &= -z + \xi + u, \\ \dot{\xi} &= -2\xi + 2d, \\ \dot{\chi} &= -\chi + e, \\ z &= z, \\ y &= z - \chi + e, \end{aligned}$$

we get

$$\dot{x} = \overbrace{\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}}^{=A} x + \overbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}^{=B} u + \overbrace{\begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}}^{=N} \overbrace{\begin{bmatrix} d \\ e \end{bmatrix}}^{=v_1},$$

$$z = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{=M} x,$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}}_{=C} x + \underbrace{e}_{=v_2}.$$

Furthermore

$$R_1 = \begin{bmatrix} R_d & 0 \\ 0 & R_e \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_2 = R_e = 1 \quad \text{and} \quad R_{12} = \begin{bmatrix} R_{de} \\ R_e \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

4. (a) True (Lemma 5.1); **(b)** True (Theorem 5.5); **(c)** False (static gain is obtained for $q = 1 \Rightarrow$ the static gain is here 0.2); **(d)** False (the other way around); **(e)** False (for white noise $Ee(t)e(s) = 0$, $\forall t \neq s$, which does not hold for periodic signals).

5. (a) With $A = 1$ and $K_q = 1$ the system becomes

$$\dot{x} = u + v_1, \quad z = x, \quad y = x + v_2.$$

The Kalman filter is $\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) = u + K(y - \hat{x})$, where $K = PC^T R_2^{-1} = P/r$ and $P > 0$ is the solution to the CARE

$$0 = AP + PA^T + NR_1N^T - PC^T R_2^{-1} CP = 1 - P^2/r.$$

Thus $P = \sqrt{r}$ and $K = 1/\sqrt{r}$. The state feedback gain is $L = Q_2^{-1} B^T S = S/\rho$, where $S > 0$ is the solution to the CARE

$$0 = A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S = 1 - S^2/\rho,$$

and thus $S = \sqrt{\rho}$ and $L = 1/\sqrt{\rho}$. The controller is

$$\begin{cases} \dot{\hat{x}} & = u - K(y - \hat{x}), \\ u & = -L\hat{x}, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dot{\hat{x}} & = -(K + L)\hat{x} + Ky, \\ u & = -L\hat{x}, \end{cases}$$

so the control law is $u = -\frac{KL}{p+K+L} y$. Hence (with $K = 1/\sqrt{r}$ and $L = 1/\sqrt{\rho}$)

$$F_y(p) = \frac{1/\sqrt{r\rho}}{p + 1/\sqrt{r} + 1/\sqrt{\rho}}.$$

(b) The closed loop system: $z = \frac{1}{p}(u + v_1)$, $u = -F_y(p)(z + v_2) \Rightarrow$

$$z = \frac{\frac{1}{p}}{1 + \frac{1}{p} F_y(p)} (v_1 - F_y(p) v_2) = \frac{1}{p + \frac{KL}{p+K+L}} \left(v_1 - \frac{KL}{p + K + L} v_2 \right)$$
$$= \frac{p + K + L}{(p + K)(p + L)} v_1 - \frac{KL}{(p + K)(p + L)} v_2$$

The closed loop system has two poles, in $-L = -1/\sqrt{\rho}$ and $-K = -1/\sqrt{r}$. The speed in the response depends on the distance to the origin for the poles. Thus, the response will be faster if the greatest one of ρ and r is decreased (corresponding to the dominating pole).

(c) We note that $sQ(s) = -\frac{1}{\tau}Q(s) + \frac{K_q}{\tau}U(s)$, and $\dot{z} = q + v_1$, so

$$\begin{aligned}\dot{z} &= q + v_1,\\ \dot{q} &= -\frac{1}{\tau}q + \frac{K_q}{\tau}u,\\ y &= z + v_2.\end{aligned}$$

With $x = \begin{bmatrix} z & q \end{bmatrix}^T$ the standard form state space model is $\dot{x} = Ax + Bu + Nv_1$, $z = Mx$, $y = Cx + v_2$, with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{\tau} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{K_q}{\tau} \end{bmatrix}, \quad N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = M = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The controller will be $u = -L\hat{x}$, with $\hat{x}$ from the Kalman filter, $L = \frac{1}{\rho}B^T S$, $K = \frac{1}{r}PC^T$ with S and P from the corresponding CAREs as in (a) and $Q_1 = 1$, $Q_2 = \rho$, $R_1 = 1$ and $R_2 = r$, but with the A, B, C, M and N above $\Rightarrow$ S and P are 2×2-matrices.

(d) First a discrete-time model is needed $\Rightarrow$ use zero-order hold sampling. The cost function then is (e.g.)

$$V_k = \sum_{t=k+1}^{k+M} \hat{z}^2(t|k) + \sum_{t=k}^{k+N} \rho u^2(t),$$

which, in every timestep k, is minimized under the constraints $|u(t)| \leq C_u$, $|z(t)| \leq C_z$. The user parameters are (in addition to ρ and r) the sampling period h, the prediction horizon M and the control horizon N. M should be chosen sufficiently long to capture the transient part ($\Rightarrow$ better stability properties), $N < M$ in general and the longer the smoother behavior, but also more computationally expensive. The sampling period h should be chosen sufficiently short, but the shorter h is, the greater M and N are required $\Rightarrow$ heavier computations must be performed in shorter time.

6. (a) For systems with only one output the observer canonical form works. See that all elements of the transfer operator $G(q)$ have identical denominators — notice that $q^2 - 1 = (q-1)(q+1)$:

$$G(q) = \begin{bmatrix} \frac{q+1}{q^2-1} & \frac{1}{q^2-1} \end{bmatrix}$$

The observer canonical form is then

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} u(k),\\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k).\end{aligned}$$

(b) The loop gain is $G(s)F(s) = \frac{2(s+1)}{s^2}$, and the poles of the closed loop system are the roots of the characteristic equation

$$0 = 1 + G(s)F(s) = 1 + \frac{2(s+1)}{s^2} \quad \Leftrightarrow \quad 0 = s^2 + 2s + 2,$$

which are $s = -1 \pm i$. Both poles are in the left half plane, and thus the closed loop system is stable.
(c) With Tustin's approximation the PD-controller becomes

$$F^d(q) = F\left(\frac{2}{h}\frac{q-1}{q+1}\right) = 2 \cdot \left(\frac{2}{h}\frac{q-1}{q+1} + 1\right) = 2 \cdot \frac{(1+2/h)q + 1 - 2/h}{q+1},$$

and the loop gain is then

$$G(q)F(q) = \frac{h^2}{2}\frac{q+1}{(q-1)^2} \cdot 2 \cdot \frac{(1+2/h)q + 1 - 2/h}{q+1} = h^2 \frac{(1+2/h)q + 1 - 2/h}{(q-1)^2}.$$

The poles are given by the characteristic equation:

$$0 = 1 + G(q)F(q) = 1 + h^2 \frac{(1+2/h)q + 1 - 2/h}{(q-1)^2} \quad \Leftrightarrow$$
$$0 = (q-1)^2 + h^2((1+2/h)q + 1 - 2/h) = q^2 + (h^2 + 2h - 2)q + h^2 - 2h + 1$$

For stability the poles must lie inside the unit circle — use the hint. Here $a = h^2 + 2h - 2$ and $b = h^2 - 2h + 1$. The inequality in the hint can be split into *(i)* $b < 1$, *(ii)* $a + b > -1$ and *(iii)* $a - b < 1$:

$$\textit{(i)}: \quad h^2 - 2h + 1 < 1 \quad \Leftrightarrow \quad h(h-2) < 0 \quad \Leftrightarrow \quad 0 < h < 2,$$

giving the lower bound $h > 0$ (which is always fulfilled) and an upper bound.

$$\textit{(ii)}: \quad h^2 + 2h - 2 + h^2 - 2h + 1 > -1 \quad \Leftrightarrow \quad 2h^2 > 0,$$

which is always fulfilled.

$$\textit{(iii)}: \quad h^2 + 2h - 2 - (h^2 - 2h + 1) < 1 \quad \Leftrightarrow \quad 4h - 3 < 1 \quad \Leftrightarrow \quad h < 1,$$

which is a tighter upper bound than *(i)*. Thus, the closed loop system is stable for $0 < h < 1$.