Final exam
Automatic Control II
Reglerteknik II 5hp

Date: January 9, 2014

Venue: Polacksbacken, exam hall

Responsible teacher: Hans Norlander.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Good luck!

Problem 1 A discrete-time system is described by

$$\begin{cases} x(k+1) &= \alpha x(k) + u(k) + v(k), \\ y(k) &= x(k), \end{cases} \quad \Leftrightarrow \quad y(k) = \frac{1}{q-\alpha}(u(k)+v(k)),$$

with $|\alpha| < 1$. The input is u, and v is a zero mean white noise with covariance $Ev^2(k) = R_v$.

(a) Determine the variance of the output, $Ey^2(k)$, when $u(k) = 0$ for all k. *Hint:* Note that $Ey^2(k) = Ex^2(k)$. **(2p)**

(b) Assume that the proportional feedback $u(k) = -Ky(k)$ is applied. Give the variance $Ey^2(k)$ for the closed loop system, expressed in K. **(2p)**

(c) What is the minimal variance $Ey^2(k)$ that can be achieved with the proportional feedback in (b), and for which K is that obtained? **(3p)**

Problem 2 A continuous-time system has the state space representation

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -3 & 0 \\ 0 & -8 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 4 \\ 0 \end{bmatrix} v_1(t), \\ z(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t), \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + v_2(t), \end{aligned}$$

where u is the input, z is the performance variable and y is the measured output. Here v_1 and v_2 are Gaussian, zero mean white noise processes, with intensities $R_1 = 1$ and $R_2 = 1$ respectively, and they are uncorrelated so $R_{12} = 0$. Determine the controller that minimizes the criterion

$$V = E[u^2 + \gamma^2 z^2]$$

Specifically, give the controllers for $\gamma = 6$ and $\gamma = 15$ respectively. **(8p)**

Problem 3 Consider the continuous-time system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & -2 & 0.5 \end{bmatrix} x(t).$$

(a) Determine the poles and zeros of the system. Is the system minimum phase? **(2p)**

(b) The system is controlled by a sampling controller, using zero-order hold sampling (i.e. the input $u(t)$ is kept constant between the sampling instants). The sampling period is $h = 0.2$ seconds. Determine the corresponding sampled, discrete-time state space model for the system. **(3p)**

(c) Determine the poles and zeros of the discrete-time model in (b). Is the system minimum phase? **(3p)**

Problem 4 Consider the system

$$y(t) = \frac{4}{p+4}\left(\frac{1}{p}u(t) + w(t)\right).$$

Here $w(t)$ is a stochastic process that can be modeled as

$$w(t) = \frac{p+b}{p^2 + a_1 p + a_2} v(t), \qquad b, a_1, a_2 > 0,$$

where $v(t)$ is zero mean white noise with intensity $\Phi_v(\omega) = R_v$.

(a) Give a state space representation for the system with y as output, u as input and v as system noise. Let $x_1 = y$.
Hint: First determine one state space representation with y as output and u and w as inputs, and another with w as output and v as input. Then combine these two into one total state space representation. **(4p)**

(b) The spectral density of $w(t)$ is

$$\Phi_w(\omega) = \frac{\omega^2 + 4}{\omega^4 + 4}.$$

Determine the spectral density of the output, $\Phi_y(\omega)$, when $u(t) \equiv 0$. **(2p)**

(c) Given $\Phi_w(\omega)$ in (b), determine the parameters b, a_1, a_2 and R_v in the model of $w(t)$ above. **(4p)**

Problem 5 In a water tower drinking water is stored at a height sufficient to pressurize the water distribution system. The water level should be constant. The water supply is monitored by measurements of the water level. This system can be modeled as (the discrete-time system)

$$\begin{aligned} x(k+1) &= x(k) + v_1(k), \\ y(k) &= x(k) + v_2(k), \end{aligned}$$

where $v_1(k)$ describes the fluctuations in the net flow into the water container, and $v_2(k)$ is the measurement noise. Both v_1 and v_2 are white noise, and have the intensities $R_1 = 1$, $R_2 = 0.5$ and $R_{12} = 0.5$. In order to estimate the water level a Kalman filter is used:

$$\hat{x}(k+1|k) = \hat{x}(k|k-1) + K(y(k) - \hat{x}(k|k-1)).$$

(a) Determine the covariance of the estimation error $P = E\tilde{x}^2(k|k-1)$. **(3p)**

(b) What is the Kalman gain K? **(2p)**

(c) Find the transfer function from the measured output $y(k)$ to the estimated level $\hat{x}(k|k-1)$. What is the static gain? **(2p)**

(d) A Kalman filter is designed for a third order system. One of the following matrices,

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix},$$

is the covariance matrix P of the estimation error $\tilde{x}$. Which of the matrices is P? A full motivation is required! **(3p)**

Problem 6 *The HW bonus points are exchangeable for this problem.*

(a) Give a state space representation for the TISO (two inputs single output) discrete-time system

$$y(k) = \begin{bmatrix} \frac{1}{q-1} & \frac{1}{q^2-1} \end{bmatrix} u(k), \quad \text{where} \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

(3p)

(b) Show that the double integrator

$$Y(s) = \frac{1}{s^2} U(s)$$

is stabilized by the feedback

$$U(s) = F(s)(Y_{ref}(s) - Y(s))$$

when the controller is the PD-controller

$$F(s) = 2(s+1).$$

(1p)

(c) In practice the control in (b) must be performed by a sampling, discrete-time controller. The sampled (zero-order hold) model of the double integrator is

$$y(kh) = \frac{h^2}{2} \frac{q+1}{(q-1)^2} u(kh),$$

where h is the sampling interval. Assume that the PD-controller in (b) is implemented by using Tustin's approximation,

$$s = \frac{2}{h} \frac{q-1}{q+1}.$$

Determine for which sampling intervals h the closed loop system will be stable when using the sampling PD-controller.
Hint: The zeros of the polynomial $z^2 + az + b$ lies inside the unit circle exactly when $|a| - 1 < b < 1$. **(3p)**

Solutions to the exam in Automatic Control II, 2014-01-09:

1. (a) Use that $x(k) = y(k)$ and compute the covariance of $x(k)$ by use of the discrete-time Lyapunov equation:

$$\Pi_x = F\Pi_x F^T + NR_vN^T, \qquad \Pi_x = Ex(k)x^T(k).$$

Here $F = \alpha$ and $N = 1$, so the Lyapunov equation is

$$\Pi_x = \alpha^2\Pi_x + R_v \qquad \Leftrightarrow \qquad \Pi_x = \frac{R_v}{1-\alpha^2}.$$

(b) The closed loop system is $x(k+1) = (\alpha - K)x(k) + v(k)$, $y(k) = x(k)$, so now $F = \alpha - K$, leading to the Lyapunov equation

$$\Pi_x = (\alpha - K)^2\Pi_x + R_v \qquad \Leftrightarrow \qquad \Pi_x = \frac{R_v}{1-(\alpha-K)^2}.$$

(c) To minimize Π_x, set $\frac{d\Pi_x}{dK} = 0$:

$$0 = -\frac{R_v}{[1-(\alpha-K)^2]^2}\frac{d}{dK}\left\{1-(\alpha-K)^2\right\} = \frac{-2R_v(\alpha-K)}{[1-(\alpha-K)^2]^2}.$$

Thus, $K = \alpha$ (which indeed is a minimum) and $\min \Pi_x = R_v$.

2. (a) The optimal controller is the LQG control law $u = -L\hat{x}$, where $\hat{x}$ is obtained from the corresponding Kalman filter (Theorem 9.1). The Kalman filter is (Theorem 5.4)

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}),$$

where the Kalman gain is $K = PC^TR_2^{-1}$, and $P = P^T \geq 0$ solves the CARE $0 = AP + PA^T + NR_1N^T - PC^TR_2^{-1}CP$. Here the CARE is

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}-3p_1 & -3p_{12}\\-8p_{12} & -8p_2\end{bmatrix} + \begin{bmatrix}-3p_1 & -8p_{12}\\-3p_{12} & -8p_2\end{bmatrix} + \begin{bmatrix}16 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}p_1^2 & p_1p_{12}\\p_1p_{12} & p_{12}^2\end{bmatrix},$$

resulting in the equation system

$$\begin{cases}0 &= -6p_1 + 16 - p_1^2,\\0 &= -11p_{12} - p_1p_{12},\\0 &= -16p_2 - p_{12}^2,\end{cases} \qquad \Leftrightarrow \qquad \begin{cases}0 &= p_1^2 + 6p_1 - 16,\\0 &= (11+p_1)p_{12},\\16p_2 &= p_{12}^2.\end{cases}$$

From this follows that $p_{12} = p_2 = 0$, and $p_1 = -3 \pm 5$. Since $P \geq 0$ requires $p_1 \geq 0$, the negative root is omitted. Thus

$$P = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}, \quad \text{and} \quad K = PC^T = \begin{bmatrix}p_1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}.$$

The optimal state feedback gain is $L = Q_2^{-1}B^TS$, where $S = S^T \geq 0$ is the solution to the CARE $0 = A^TS + SA + M^TQ_1M - SBQ_2^{-1}B^TS$. Here $Q_1 = \gamma^2$ and $Q_2 = 1$, so the CARE becomes

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}-3s_1 & -3s_{12}\\-8s_{12} & -8s_2\end{bmatrix} + \begin{bmatrix}-3s_1 & -8s_{12}\\-3s_{12} & -8s_2\end{bmatrix} + \begin{bmatrix}\gamma^2 & \gamma^2\\\gamma^2 & \gamma^2\end{bmatrix} - \begin{bmatrix}s_{12}^2 & s_{12}s_2\\s_{12}s_2 & s_2^2\end{bmatrix},$$

resulting in the equation system

$$\begin{cases} 0 &= -6s_1 + \gamma^2 - s_{12}^2, \\ 0 &= -11s_{12} + \gamma^2 - s_{12}s_2, \\ 0 &= -16s_2 + \gamma^2 - s_2^2, \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} s_1 &= (\gamma^2 - s_{12}^2)/6, \\ s_{12} &= \gamma^2/(11 + s_2), \\ 0 &= s_2^2 + 16s_2 - \gamma^2. \end{cases}$$

To have $S \geq 0$, we must have $s_2 \geq 0$. Thus $s_2 = -8 + \sqrt{64 + \gamma^2}$ (the negative root is omitted), and $s_{12} = \gamma^2/(3 + \sqrt{64 + \gamma^2})$. Since

$$L = B^T S = \begin{bmatrix} 0 & 1 \end{bmatrix} S = \begin{bmatrix} s_{12} & s_2 \end{bmatrix},$$

s_1 is not really needed explicitely. With $\gamma = 6$ we have $L = \begin{bmatrix} 36/13 & 2 \end{bmatrix}$, and with $\gamma = 15$ we get $L = \begin{bmatrix} 225/20 & 9 \end{bmatrix}$.

3. (a) The transfer function is $G(s) = C(sI - A)^{-1}B$, so

$$G(s) = \begin{bmatrix} 1 & -2 & 0.5 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & 0 & 0 \\ 0 & \frac{1}{s+1} & 0 \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2} = \frac{2}{s(s+1)(s+2)}.$$

The system has no zeros, and poles in the origin, -1 and -2. No poles or zeros in the right half plane $\Rightarrow$ minimum phase.

(b) The discrete-time model is $x(kh + h) = Fx(kh) + Gu(kh)$, $y(kh) = Cx(kh)$, where $F = e^{Ah}$ and $G = \int_0^h e^{At} B dt$. Here

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-h} & 0 \\ 0 & 0 & e^{-2h} \end{bmatrix} \quad \text{and} \quad G = \int_0^h \begin{bmatrix} 1 \\ e^{-t} \\ 2e^{-2t} \end{bmatrix} dt = \begin{bmatrix} h \\ 1 - e^{-h} \\ 1 - e^{-2h} \end{bmatrix}.$$

With $h = 0.2 \Rightarrow e^{-h} = 0.819$ the state-space model becomes

$$\begin{aligned} x(kh + h) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.819 & 0 \\ 0 & 0 & 0.670 \end{bmatrix} x(kh) + \begin{bmatrix} 0.2 \\ 0.181 \\ 0.330 \end{bmatrix} u(kh), \\ y(kh) &= \begin{bmatrix} 1 & -2 & 0.5 \end{bmatrix} x(kh). \end{aligned}$$

(c) The transfer function is $G(q) = C(qI - F)^{-1}G$, so

$$\begin{aligned} G(q) &= \begin{bmatrix} 1 & -2 & 0.5 \end{bmatrix} \begin{bmatrix} \frac{1}{q-1} & 0 & 0 \\ 0 & \frac{1}{q-e^{-h}} & 0 \\ 0 & 0 & \frac{1}{q-e^{-2h}} \end{bmatrix} \begin{bmatrix} h \\ 1 - e^{-h} \\ 1 - e^{-2h} \end{bmatrix} \\ &= \frac{h}{q-1} - \frac{2(1 - e^{-h})}{q - e^{-h}} + \frac{0.5(1 - e^{-2h})}{q - e^{-2h}} = \frac{b_1 q^2 + b_2 q + b_3}{(q-1)(q - e^{-h})(q - e^{-2h})}, \end{aligned}$$

where
$b_1 = h - 2(1 - e^{-h}) + 0.5(1 - e^{-2h}) = 2.3015 \cdot 10^{-3}$,

$b_2 = -h(e^{-h}+e^{-2h})+2(1-e^{-h})(1+e^{-2h})-0.5(1-e^{-2h})(1+e^{-h}) = 7.9456\cdot 10^{-3}$ and
$b_3 = he^{-3h} - 2(1-e^{-h})e^{-2h} + 0.5(1-e^{-2h})e^{-h} = 1.7051 \cdot 10^{-3}$.
The poles are 1, $e^{-h} = 0.819$ and $e^{-2h} = 0.670$. The zeros are given by $0 = b_1q^2 + b_2q + b_3$, i.e.

$$z = -\frac{b_2}{2b_1} \pm \sqrt{\left[\frac{b_2}{2b_1}\right]^2 - \frac{b_3}{b_1}} = -1.7262 \pm 1.4963.$$

Thus, the zeros are -0.2299 and -3.2225. One zero is outside the unit circle $\Rightarrow$ non-minimum phase.

4. (a) With $x_1 = y$ we have $(p+4)x_1 = 4(\frac{1}{p}u + w) \Leftrightarrow \dot{x}_1 = -4x_1 + 4\frac{1}{p}u + 4w$. Now set (for example) $x_2 = \frac{1}{p}u \Rightarrow \dot{x}_2 = u$. Using e.g. the controller canonical form for describing w we get

$$\begin{aligned} \dot{x}_3 &= -a_1x_3 - a_2x_4 + v, \\ \dot{x}_4 &= x_3, \\ w &= x_3 + bx_4, \end{aligned}$$

and in total we get

$$\begin{aligned} \dot{x}_1 &= -4x_1 + 4x_2 + 4x_3 + 4bx_4, \\ \dot{x}_2 &= u, \\ \dot{x}_3 &= -a_1x_3 - a_2x_4 + v, \\ \dot{x}_4 &= x_3, \\ y &= x_1. \end{aligned}$$

In vector form this is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -4 & 4 & 4 & 4b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & -a_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} v, \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x \end{aligned}$$

(b) With $u = 0$ we have $y = \frac{4}{p+4}w$, and since $\Phi_y(\omega) = |G(i\omega)|^2\Phi_w(\omega)$ we get

$$\Phi_y(\omega) = \frac{4^2}{|i\omega + 4|^2}\Phi_w(\omega) = \frac{16(\omega^2+4)}{(\omega^2+16)(\omega^4+4)}.$$

(c) Use that $\Phi_w(\omega) = |G_w(i\omega)|^2\Phi_v(s)$:

$$\begin{aligned} |G_w(i\omega)|^2\Phi_v(s) &= \frac{|i\omega + b|^2}{|(i\omega)^2 + ia_1\omega + a_2|^2}R_v \\ &= \frac{\omega^2 + b^2}{(a_2 - \omega^2)^2 + a_1^2\omega^2}R_v = \frac{\omega^2 + b^2}{\omega^4 + (a_1^2 - 2a_2)\omega^2 + a_2^2}R_v \end{aligned}$$

Comparing coefficients with $\Phi_w(\omega) = \frac{\omega^2+4}{\omega^4+4}$ gives

$$\begin{cases} b^2 &= 4 \\ a_1^2 - 2a_2 &= 0 \\ a_2^2 &= 4 \\ R_v &= 1 \end{cases} \quad \Rightarrow \quad \begin{cases} b &= 2 \\ a_1 &= 2 \\ a_2 &= 2 \\ R_v &= 1 \end{cases}$$

5. **(a)** The covariance $P = P^T \geq 0$ is the solution of the DARE

$$P = FPF^T + NR_1N^T - (FPH^T + NR_{12})(HPH^T + R_2)^{-1}(FPH^T + NR_{12})^T.$$

Here $F = 1$, $N = 1$, $H = 1$, $R_1 = 1$, $R_2 = 0.5$ and $R_{12} = 0.5 \Rightarrow$

$$P = P + 1 - \frac{(P+0.5)^2}{P+0.5} \quad \Leftrightarrow \quad (P+0.5)^2 = P + 0.5 \quad \Leftrightarrow \quad P = \pm 0.5,$$

where the negative solution is rejected. Thus $P = 0.5$.
(b) $K = (FPH^T + NR_{12})(HPH^T + R_2)^{-1} \Rightarrow K = \frac{0.5+0.5}{0.5+0.5} = 1.$
(c) We have

$$q\hat{x}(k|k-1) = \hat{x}(k|k-1) + y(k) - \hat{x}(k|k-1) = y(k) \quad \Leftrightarrow \quad \hat{x}(k|k-1) = \frac{1}{q}y(k).$$

The transfer function is $G_{\hat{x}y}(z) = \frac{1}{z}$, and the static gain is $G_{\hat{x}y}(1) = 1$.
(d) $P = P^T \geq 0$ and P is 3×3 (since third order system) $\Rightarrow M_4$ is the only possible P.

6. **(a)** For systems with only one output the observer canonical form works. See that all elements of the transfer operator $G(q)$ have identical denominators — notice that $q^2 - 1 = (q-1)(q+1)$:

$$G(q) = \begin{bmatrix} \frac{q+1}{q^2-1} & \frac{1}{q^2-1} \end{bmatrix}$$

The observer canonical form is then

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} u(k),$$
$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k).$$

(b) The loop gain is $G(s)F(s) = \frac{2(s+1)}{s^2}$, and the poles of the closed loop system are the roots of the characteristic equation

$$0 = 1 + G(s)F(s) = 1 + \frac{2(s+1)}{s^2} \quad \Leftrightarrow \quad 0 = s^2 + 2s + 2,$$

which are $s = -1 \pm i$. Both poles are in the left half plane, and thus the closed loop system is stable.
(c) With Tustin's approximation the PD-controller becomes

$$F^d(q) = F\left(\frac{2}{h}\frac{q-1}{q+1}\right) = 2 \cdot \left(\frac{2}{h}\frac{q-1}{q+1} + 1\right) = 2 \cdot \frac{(1+2/h)q + 1 - 2/h}{q+1},$$

and the loop gain is then

$$G(q)F(q) = \frac{h^2}{2}\frac{q+1}{(q-1)^2}\cdot 2\cdot\frac{(1+2/h)q+1-2/h}{q+1} = h^2\frac{(1+2/h)q+1-2/h}{(q-1)^2}.$$

The poles are given by the characteristic equation:

$$0 = 1 + G(q)F(q) = 1 + h^2\frac{(1+2/h)q+1-2/h}{(q-1)^2} \quad \Leftrightarrow$$
$$0 = (q-1)^2 + h^2((1+2/h)q+1-2/h) = q^2 + (h^2+2h-2)q + h^2 - 2h + 1$$

For stability the poles must lie inside the unit circle — use the hint. Here $a = h^2 + 2h - 2$ and $b = h^2 - 2h + 1$. The inequality in the hint can be split into *(i)* $b < 1$, *(ii)* $a + b > -1$ and *(iii)* $a - b < 1$:

$$\textit{(i)}:\quad h^2 - 2h + 1 < 1 \quad \Leftrightarrow \quad h(h-2) < 0 \quad \Leftrightarrow \quad 0 < h < 2,$$

giving the lower bound $h > 0$ (which is always fulfilled) and an upper bound.

$$\textit{(ii)}:\quad h^2 + 2h - 2 + h^2 - 2h + 1 > -1 \quad \Leftrightarrow \quad 2h^2 > 0,$$

which is always fulfilled.

$$\textit{(iii)}:\quad h^2 + 2h - 2 - (h^2 - 2h + 1) < 1 \quad \Leftrightarrow \quad 4h - 3 < 1 \quad \Leftrightarrow \quad h < 1,$$

which is a tighter upper bound than *(i)*. Thus, the closed loop system is stable for $0 < h < 1$.