Final exam
Automatic Control II
Reglerteknik II 5hp

Date: October 24, 2013

Venue: Polacksbacken, exam hall

Responsible teacher: Hans Norlander.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Please use English in your solutions when possible, that would be appreciated!

Good luck!

Problem 1

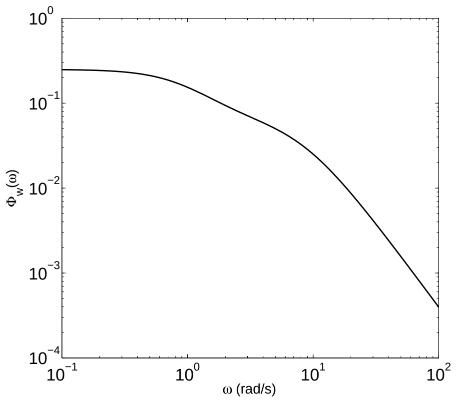
(a) A discrete-time stochastic process is described by

$$x(k+1) = 0.7x(k) + 2v(k),$$
$$w(k) = x(k),$$

where $v(k)$ is zero mean white noise, and $Ev(k)^2 = 1$. Determine the variance of w, $Ew(k)^2$. **(2p)**

(b) What is the spectrum, $\Phi_w(\omega)$, of the stochastic process w in (a)? **(2p)**

(c) The spectrum of a stationary continuous-time stochastic process, $w(t)$, is obtained on an empirical basis — see the plot below.



It is found that

$$\Phi_w(\omega) = \frac{4\omega^2 + 16}{\omega^4 + 65\omega^2 + 64}$$

is a very good approximation of the spectrum. Determine a minimum phase transfer function $G(s)$ such that $G(0) > 0$, and that the model $w(t) = G(p)v(t)$ is stationary and has the spectrum $\Phi_w(\omega)$ above. Here $v(t)$ is zero mean white noise with intensity $\Phi_v(\omega) = 1$. **(4p)**

Problem 2 A simple, linearized model of an inverted pendulum is

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t).$$

The inverted pendulum is controlled by a sampling controller, and the sampling interval is $h = \ln 1.25$ ($\Leftrightarrow e^h = 1.25$).

(a) Give the discrete-time state space model of the inverted pendulum. Use zero-order-hold sampling, i.e. let the input $u(t)$ be constant between the sampling instants. **(4p)**

(b) Assume that proportional control is used, $u(kh) = K(-y(kh))$. For which $K \in \mathbb{R}$ is the closed loop system stable? **(4p)**

Problem 3 A certain industrial process is described by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v(k),$$

where $v(k)$ is zero mean white noise, and $Ev(k)^2 = 1$. Initially both state variables were measured, but due to a malfunctioning sensor the only available measurement now is

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + e(k),$$

where $e(k)$ is zero mean white noise, uncorrelated to $v(k)$ and with intensity $Ee(k)^2 = r$.

(a) Since the value of $x_1(k)$ is required for subsequent calculations needed in the production line, the missing measurements are replaced by the estimate $\hat{x}_1(k) = Ex_1(k) = 0$ (for all k). What is the variance of the estimation error $\tilde{x}_1(k) = x_1(k) - \hat{x}_1(k)$ for this estimate? **(3p)**

(b) Is $x_1(k)$ observable from $y(k)$? **(1p)**

(c) A clever student from Uppsala suggests that a Kalman filter should be used instead for estimation of $x_1(k)$[1]. Show that

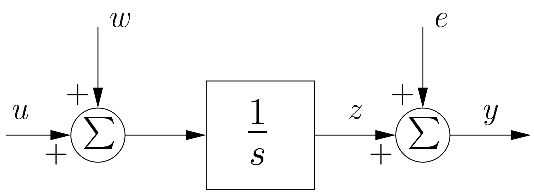
$$P = \begin{bmatrix} p_1 & 1 \\ 1 & 1 \end{bmatrix}$$

is the solution of the associated algebraic Riccati equation, and determine p_1. **(4p)**

(d) Determine the variance of the estimation error $\tilde{x}_1(k)$ when the Kalman filter in (c) is used. Is the estimate improved? **(2p)**

[1] The Kalman filter will of course provide an estimate of the full state vector, but then x_1 is part of that.

Problem 4 A simple model describing variations of the level in a water reservoir is given in the block diagram below. Here the input u is the con-



trolled net flow into the reservoir, z is the level and y is the measured level. There are two independent, zero mean noise sources: the measurement noise, e, and uncontrolled variations in the net flow, w.

It is desirable to keep the variations of the level as small as possible, at a minimal cost in terms of control power. Therefore the LQG control law $u = -L\hat{x}$, which minimizes the cost function

$$V = E\left[z^2 + \rho^2 u^2\right], \qquad \rho > 0,$$

is used. (The estimate $\hat{x}$ is obtained from a Kalman filter, which is not considered in this problem.)

(a) In a first approximation it is assumed that both w and e are white noise. Find the optimal feedback gain L, expressed in ρ. (Use $x = z$ as state variable.) **(3p)**

(b) It turns out that

$$w(t) = \frac{1}{p+1} v(t),$$

where v is white noise, is a much better description of w. The measurement noise, e, is still assumed to be white. Give a state space representation of the total model, where the dynamics of w are incorporated. Use the "standard" form, i.e. determine the matrices and vectors A, B, N, M and C in

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Nv_1(t),\\ z(t) &= Mx(t),\\ y(t) &= Cx(t) + v_2(t).\end{aligned}$$

Also, relate v_1 and v_2 to v and e. **(4p)**

(c) Find the optimal state feedback gain L for the total model in (b). **(3p)**

Problem 5 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true" / "false" are considered!

(a) White noise has a constant spectrum (is independent of the frequency).

(b) A Kalman filter is an observer.

(c) The weighting matrices $\{Q_1, Q_2\}$ and $\{kQ_1, kQ_2\}$ (with $k > 0$) give identical LQG controllers.

(d) LQG yields a linear time-invariant controller.

(e) MPC yields a linear time-invariant controller.

(f) A drawback with MPC is that it can only be used for linear systems with no constraints.

(g) In MPC the *control/input horizon* should always be chosen at least twice as long as the *prediction/output horizon.*

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(7p)**

Problem 6 *The HW bonus points are exchangeable for this problem.*
The system with transfer function

$$G(s) = \begin{bmatrix} \frac{2s+1}{s(s+1)} & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \quad \text{has} \quad \begin{cases} \dot{x}(t) & = Ax(t) + Bu(t), \\ y(t) & = Cx(t) \end{cases}$$

as state space representation, where the matrices A, B and C is one combination of the following matrices:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) Which combination of the matrices A_1–A_3, B_1–B_4 and C_1–C_3 constitutes a state space representation of $G(s)$? A full motivation is required. **(4p)**

(b) What is the impulse response of the system $G(s)$? **(1p)**

(c) Is the state space representation associated with A_2, B_2 and C_2 above a minimal realisation? **(2p)**

Solutions to the exam in Automatic Control II, 2013-10-24:

1. (a) Use that $Ew^2 = Ex^2 = \Pi_x$, and that Π_x is the solution of the discrete-time Lyapunov equation, $\Pi_x = F\Pi_x F^T + GRG^T$. Here $F = 0.7$, $G = 2$ and $R = 1 \Rightarrow$

$$\Pi_x = 0.7^2\Pi_x + 2^2 \qquad \Leftrightarrow \qquad \Pi_x = \frac{4}{1 - 0.49} \approx 7.84.$$

Thus, $Ew^2 = 7.84$.

(b) The spectrum of $w(k) = G(q)v(k)$ is $\Phi_w(\omega) = G(e^{i\omega})\Phi_v(\omega)G^*(e^{i\omega})$. Here $G(q) = H(qI - F)^{-1}G = \frac{2}{q-0.7}$ and $\Phi_v(\omega) = 1$, so

$$\begin{aligned}\Phi_w(\omega) = G(e^{i\omega})G(e^{-i\omega}) &= \frac{2}{e^{i\omega} - 0.7}\frac{2}{e^{-i\omega} - 0.7} \\ &= \frac{4}{1 + 0.7^2 - 2 \cdot 0.7(e^{i\omega} + e^{-i\omega})/2} = \frac{4}{1.49 + 1.40\cos\omega}.\end{aligned}$$

(c) For a continuous-time system, $w(t) = G(p)v(t)$, we have that

$$\Phi_w(\omega) = G(i\omega)\Phi_v(\omega)G^*(i\omega) = G(i\omega)G(-i\omega),$$

where the latter equality follows since $G(s)$ is scalar and $\Phi_v(\omega) = 1$. Since $\Phi_w(\omega)$ is rational in ω^2 it is possible to find a rational, stable and minimum phase $G(s)$ such that the relation above holds (Theorem 5.1). Since the numerator of $\Phi_w(\omega)$ has degree 2, and the denominator has degree 4, a clever guess is that

$$G(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}.$$

Then

$$\begin{aligned}G(i\omega)G(-i\omega) &= \frac{ib_1\omega + b_2}{-\omega^2 + ia_1\omega + a_2}\frac{-ib_1\omega + b_2}{-\omega^2 - ia_1\omega + a_2} \\ &= \frac{b_1^2\omega^2 + b_2^2}{(a_2 - \omega^2)^2 + a_1^2\omega^2} = \frac{b_1^2\omega^2 + b_2^2}{\omega^4 + (a_1^2 - 2a_2)\omega^2 + a_2^2}.\end{aligned}$$

By comparing numerator and denominator, per power of ω, the following equation systems are obtained:

$$\begin{cases} 4 &= b_1^2, \\ 16 &= b_2^2, \end{cases} \qquad \text{and} \qquad \begin{cases} 65 &= a_1^2 - 2a_2, \\ 64 &= a_2^2. \end{cases}$$

Stationarity of $w \Rightarrow G(s)$ must be stable $\Rightarrow a_1, a_2 > 0$. This, together with $G(0) > 0$ implies that $b_2 > 0$, which together with the minimum phase requirement (no zeros in the RHP) implies that $b_1 \geq 0$. Given these conditions, the solution is

$$\begin{cases} b_1 &= 2, \\ b_2 &= 4, \end{cases} \qquad \begin{cases} a_1 &= 9, \\ a_2 &= 8 \end{cases} \qquad \Leftrightarrow \qquad G(s) = \frac{2s + 4}{s^2 + 9s + 8} = \frac{2(s + 2)}{(s + 1)(s + 8)}.$$

2. **(a)** With zero-order-hold sampling the discrete-time system becomes

$$\begin{cases} x(kh+h) &= Fx(kh) + Gu(kh), \\ y(kh) &= Cx(kh), \end{cases} \qquad \text{with} \qquad F = e^{Ah}, \quad G = \int_0^h e^{At} B dt.$$

Here

$$F = e^{Ah} = \begin{bmatrix} e^h & 0 \\ 0 & e^{-h} \end{bmatrix} = \begin{bmatrix} 1.25 & 0 \\ 0 & 0.8 \end{bmatrix}$$

$$\text{and} \quad G = \int_0^h \begin{bmatrix} e^t \\ -e^{-t} \end{bmatrix} dt = \begin{bmatrix} e^h - 1 \\ e^{-h} - 1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ -0.20 \end{bmatrix}.$$

The sampled state space model is then

$$\begin{aligned} x(kh+h) &= \begin{bmatrix} 1.25 & 0 \\ 0 & 0.8 \end{bmatrix} x(kh) + \begin{bmatrix} 0.25 \\ -0.20 \end{bmatrix} u(kh), \\ y(kh) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(kh). \end{aligned}$$

(b) The closed loop system becomes

$$\begin{aligned} x(kh+h) &= Fx(kh) + G(-KCx(kh)) = (F - GKC)x(kh), \\ y(kh) &= Cx(kh), \end{aligned}$$

and the poles are given by $0 = \det(zI - F + GKC)$. Here the pole polynomial is

$$\begin{aligned} \det&\left\{ \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1.25 & 0 \\ 0 & 0.8 \end{bmatrix} + \begin{bmatrix} 0.25 \\ -0.20 \end{bmatrix} K \begin{bmatrix} 1 & 1 \end{bmatrix} \right\} \\ &= \det \begin{bmatrix} z - 1.25 + 0.25K & 0.25K \\ -0.2K & z - 0.8 - 0.2K \end{bmatrix} \\ &= z^2 + (-2.05 + 0.05K)z + 1 + 0.05K. \end{aligned}$$

The polynomial $z^2 + az + b$ has both zeros inside the unit circle if and only if $|a| - 1 < b < 1$. Here $a = -2.05 + 0.05K$ and $b = 1 + 0.05K$. The condition $b < 1$ rules out every $K \geq 0$. Furthermore, for $K < 0$ we have

$$|a| - 1 = |-2.05 + 0.05K| - 1 = 2.05 + 0.05|K| - 1 = 1.05 + 0.05|K| > b = 1 - 0.05|K|.$$

Thus, the stability conditions are violated for both $K \geq 0$ and for $K < 0$, so the closed loop system is never stable, for any $K \in \mathbb{R}$.

3. **(a)** Since $\hat{x}_1(k) = 0$, we have $\tilde{x}_1(k) = x_1(k)$, and thus

$$\tilde{x}_1(k+1) = 0.5\tilde{x}_1(k) + v(k).$$

The variance $E\tilde{x}_1(k)^2 = \Pi_{\tilde{x}}$ is then obtained from the discrete-time Lyapunov function $\Pi_{\tilde{x}} = F\Pi_{\tilde{x}}F^T + GR_vG^T$. Here $F = 0.5$, $G = 1$ and $R_v = 1$, so

$$\Pi_{\tilde{x}} = 0.5^2 \Pi_{\tilde{x}} + 1 \quad \Leftrightarrow \quad \Pi_{\tilde{x}} = \frac{1}{0.75} = \frac{4}{3}.$$

(b) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \mathcal{O} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = 0$$

regardless of x_1. Therefore it is *not* observable.
(c) The associated DARE is $P = FPF^T + NR_1N^T - FPH^T(HPH^T + R_2)^{-1}HPF^T$ (since $R_{12} = 0$), and here $R_1 = 1$ and $R_2 = r$, so written out the DARE becomes

$$\begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$
$$- \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + r \right)^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0.25p_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \frac{0.25p_{12}^2}{p_2+r} & 0 \\ 0 & 0 \end{bmatrix}.$$

This gives the equation system

$$\begin{cases} p_1 &= 0.25p_1 + 1 - \frac{0.25p_{12}^2}{p_2+r}, \\ p_{12} &= 1, \\ p_2 &= 1, \end{cases}$$

and obviously $p_{12} = p_2 = 1$, and then

$$p_1 = 0.25p_1 + 1 - \frac{0.25}{1+r} \qquad \Leftrightarrow \qquad p_1 = \frac{4}{3} - \frac{1}{3(1+r)}.$$

(d) Since $P = E\tilde{x}\tilde{x}^T$, and $\tilde{x}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \tilde{x}$, we get

$$E\tilde{x}_1^2 = \begin{bmatrix} 1 & 0 \end{bmatrix} P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = p_1 = \frac{4}{3} - \frac{1}{3(1+r)}.$$

Furthermore, $1 < p_1 < \frac{4}{3}$ for every $r > 0$, so the estimate from the Kalman filter *is* better than the estimate $\hat{x}_1 = 0$!

4. (a) A state space representation is

$$\dot{x} = u + w, \quad z = x, \quad y = x + e.$$

Theorem 9.1 $\Rightarrow$ $L = Q_2^{-1}B^TS$, where $S = S^T > 0$ solves the CARE $0 = A^TS + SA + M^TQ_1M - SBQ_2^{-1}B^TS$. Here $Q_1 = 1$, $Q_2 = \rho^2$, $A = 0$, $B = 1$ and $M = 1$, leading to

$$0 = 1 - S^2/\rho^2 \qquad \Rightarrow \qquad S = \rho,$$

and then $L = \rho^{-2}\rho = 1/\rho$.
(b) We have that $pw = -w + v$, and therefore

$$\begin{cases} \dot{x} &= w + u, \\ \dot{w} &= -w + v, \\ z &= x, \\ y &= x + e \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v, \\ z &= \begin{bmatrix} 1 & 0 \end{bmatrix} x, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x + e, \end{cases}$$

with $x = \begin{bmatrix} z & w \end{bmatrix}^T$.
(c) The CARE becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix} + \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rho^{-2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix},$$

which, taken element by element, leads to the equation system

$$\begin{cases} 0 &= 1 - s_1^2/\rho^2, \\ 0 &= s_1 - s_{12} - s_1 s_{12}/\rho^2, \\ 0 &= 2s_{12} - 2s_2 - s_{12}^2/\rho^2, \end{cases}$$

with the solution

$$\begin{cases} s_1 &= \rho, \\ s_{12} &= \frac{s_1}{1+s_1/\rho^2} = \frac{\rho^2}{\rho+1}, \\ s_2 &= (1 - s_{12}/\rho^2)\, s_{12}/2 = \frac{\rho^3}{2(\rho+1)^2}. \end{cases}$$

The state feedback gain is then $L = \rho^{-2} \begin{bmatrix} s_1 & s_{12} \end{bmatrix} = \begin{bmatrix} 1/\rho & 1/(\rho+1) \end{bmatrix}$.

5. **(a)** True (Def. 5.2); **(b)** True; **(c)** True (minimizing V is equivalent to minimizing kV); **(d)** True; **(e)** False; **(f)** False (the opposite, constraints are no problem for MPC); **(g)** False (typically the control horizon is much shorter than the prediction horizon.)

6. **(a)** The system $G(s)$ has two inputs, two outputs and poles in the origin and in -1. These fact excludes A_2, B_1, B_3, C_1 and C_2. From the B- and C-matrices it is clear that it is a third order system, why also A_3 is excluded. Thus the A-matrix must be A_1, and the C-matrix is C_3. The B-vector is either B_2 or B_4.

$$C_3(sI - A_1)^{-1}B_2 = G(s) \quad \text{while } C_3(sI - A_1)^{-1}B_4 \neq G(s),$$

and hence $\{A_1, B_2, C_3\}$ is the correct triple.
(b) Use either $g(t) = \mathcal{L}^{-1}[G(s)]$ or $g(t) = Ce^{At}B$. The latter gives

$$g(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + e^{-t} & 0 \\ e^{-t} & e^{-t} \end{bmatrix}$$

(c) The controllability and observability matrices are

$$\mathcal{S} = \begin{bmatrix} B_2 & A_2B_2 & A_2^2B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 4 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} C_2 \\ C_2A_2 \\ C_2A_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 4 \end{bmatrix}.$$

Both $\mathcal{S}$ and $\mathcal{O}$ are rank deficient, and thus the system is neither controllable, nor observable. Hence it is not a minimal realisation.