Exam in Automatic Control II
Reglerteknik II 5hp

Date: January 9, 2015

Venue: Polacksbacken, exam hall

Responsible teacher: Hans Norlander.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Good luck!

Problem 1 A continuous-time system is described by

$$\dot{x}(t) = w(t),$$
$$y(t) = x(t) + e(t),$$

where $e(t)$ is zero mean white noise with intensity $\Phi_e(\omega) = 1$. An observer is used for estimation of $x(t)$.

(a) The observer is designed as a Kalman filter, based on the assumption that $w(t)$ is zero mean white noise with intensity $\Phi_w(\omega) = 1$. Compute this observer/Kalman filter, and determine the *assumed* covariance of the estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$. **(3p)**

(b) The assumption in (a) turns out to be a crude approximation: the process disturbance $w(t)$ is better modelled as

$$\dot{w}(t) = -\beta w(t) + v(t), \qquad \beta > 0,$$

where $v(t)$ is zero mean white noise with intensity $\Phi_v(\omega) = 1$. Assume that the (first order) observer computed in (a) is used. What is then the covariance of the estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$? Use the improved model of $w(t)$ in your calculations. Your answer should be expressed in β.
Hint: First set up a state space model of the total system, with e.g. $\begin{bmatrix}\tilde{x} & w\end{bmatrix}^T$ as state vector. **(4p)**

(c) The process disturbance $w(t)$ has the covariance $Ew(t)^2 = 1$. Determine the value of β in the model in (b). **(2p)**

Problem 2 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true" / "false" are considered!

(a) White noise has a constant spectrum (is independent of the frequency).

(b) A Kalman filter is an observer.

(c) The weighting matrices $\{Q_1, Q_2\}$ and $\{kQ_1, kQ_2\}$ (with $k > 0$) give identical LQG controllers.

(d) LQG yields a linear time-invariant controller.

(e) MPC yields a linear time-invariant controller.

(f) A drawback with MPC is that it can only be used for linear systems with no constraints.

(g) In MPC the *control/input horizon* should always be chosen at least twice as long as the *prediction/output horizon*.

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(7p)**

Problem 3 A discrete-time system has the state space representation

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1 & 1 \\ 0 & \beta \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(k), \qquad |\beta| < 1, \\ z(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k), \end{aligned}$$

where $u(k)$ is the control input, and $v(k)$ is zero mean white noise with covariance $Ev(k)^2 = 1$. It is desired to minimize the variance of the performance variable $z(k)$, and therefor LQ control is considered, minimizing the criterion

$$V = E\left\{z(k)^2 + \rho u(k)^2\right\}, \qquad \rho \geq 0.$$

(The control law then is $u(k) = -Lx(k)$.)

(a) Let $S = \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix}$ denote the solution of the associated algebraic Riccati equation. Show that the closed loop system will *not* depend on s_2, and that the poles of the closed loop system will *only* depend on s_1. **(3p)**

(b) Determine the poles of the closed loop system for the two extreme cases; when $\rho \to 0$ and when $\rho \to \infty$. **(4p)**

(c) Determine $Ez(k)^2$ for the closed loop system when $\rho \to 0$. **(4p)**

Problem 4 Consider the continuous-time system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & \beta \end{bmatrix} x(t). \end{aligned}$$

(a) Show that the system is observable for all $\beta \in \mathbb{R}$. **(2p)**

(b) The system is to be controlled by a sampling controller, and for that reason a corresponding discrete-time model is obtained by use of zero-order hold sampling. Determine the corresponding sampled, discrete-time state space model. **(3p)**

(c) For which sampling intervals is the discrete-time state space model in (b) observable? Also show that the observability does not depend on β. **(3p)**

Problem 5 The function of a hot air balloon is based on the fact that warm air has lower density than cold air. A simple model of the vertical motion of a hot air balloon, i.e. the change of altitude, is as follows[1]. The rate of the change in altitude, i.e. the vertical speed, denoted with $r(t)$, is modelled as

$$\frac{dr}{dt} = \frac{1}{\tau_1}(w(t) - r(t)) + k\theta(t),$$

where $w(t)$ is the vertical wind speed, and $\theta(t)$ is the temperature of the air in the balloon. The temperature is modelled as

$$\frac{d\theta}{dt} = -\frac{1}{\tau_2}\theta(t) + u(t),$$

where $u(t)$ is the amount of heat produced by the burner. τ_1 and τ_2 are two time constants. We assume that the vertical wind speed is slowly varying and described by

$$w(t) = \frac{1}{p+\delta}v(t), \qquad \delta > 0,$$

where $v(t)$ is zero mean white noise with intensity $\Phi_v(\omega) = R_v$. Finally, the altitude $h(t)$ is governed by $\frac{dh}{dt} = r(t)$.

(a) Introduce the state vector $x(t) = \begin{bmatrix}\theta(t) & r(t) & h(t) & w(t)\end{bmatrix}^T$ and give the state space model in the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Nv(t),\\ z(t) &= Mx(t),\end{aligned}$$

i.e. determine the matrices and vectors A, B, N and M. Let $z(t) = h(t)$ be the output/performance variable. **(5p)**

(b) Assume that $u(t) = 0$. Determine the spectrum, $\Phi_r(\omega)$, of $r(t)$. **(3p)**

Problem 6 *The HW bonus points are exchangeable for this problem.*
A stationary discrete-time stochastic process $w(k)$ has the spectrum

$$\Phi_w(\omega) = \frac{5 + 4\cos\omega}{2.25 - 2(\cos\omega)^2} = \frac{5 + 4\cos\omega}{1.25 - \cos 2\omega}.$$

Find a transfer operator $G(q)$ that is stable, minimum phase and has positive static gain, and is such that

$$w(k) = G(q)v(k), \qquad \Phi_v(\omega) = 1,$$

is a model of $w(k)$.
Hint: Note that $(q - z_1)(q - z_2) = q^2 - (z_1 + z_2)q + z_1z_2 = q^2 + a_1q + a_2$, and that $|z_1| < 1$ and $|z_2| < 1 \Rightarrow |a_2| = |z_1z_2| = |z_1| \cdot |z_2| < 1$. Thus $|a_2| < 1$ is a *necessary* condition for the polynomial $q^2 + a_1q + a_2$ to have both its zeros inside the unit circle. **(7p)**

[1] All signals represent deviations from an equilibrium point.

Solutions to the exam in Automatic Control II, 2015-01-09:

1. (a) The Kalman filter is $\dot{\hat{x}} = A\hat{x} + K(y - C\hat{x})$, where $K = PC^T R_2^{-1}$ and $P = P^T \geq 0$ is the solution of the CARE

$$0 = AP + PA^T + NR_1N^T - PC^T R_2^{-1}CP.$$

Here $A = 0$, $N = 1$, $C = 1$ and $R_1 = R_2 = 1$, so the CARE becomes

$$0 = 1 - P^2 \qquad \Leftrightarrow \qquad P = 1 \qquad \Rightarrow \qquad K = 1.$$

(b) The estimation error, $\tilde{x} = x - \hat{x}$, is governed

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Nw - A\hat{x} - K(Cx + e - C\hat{x}) = (A - KC)\tilde{x} + Nw - Ke = -\tilde{x} + w - e.$$

Combined with the model $\dot{w} = -\beta w + v$, with the state vector $\bar{x} = \begin{bmatrix}\tilde{x} & w\end{bmatrix}^T$, this gives

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{N}\bar{v} = \begin{bmatrix}-1 & 1\\ 0 & -\beta\end{bmatrix}\bar{x} + \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}v\\ e\end{bmatrix}, \\ \tilde{x} &= \bar{C}\bar{x} = \begin{bmatrix}1 & 0\end{bmatrix}\bar{x}. \end{aligned}$$

The covariance of $\bar{x}$ is $\Pi_{\bar{x}} = E\bar{x}\bar{x}^T$, and is given as the solution of the continuous-time Lyapunov equation, here $0 = \bar{A}\Pi_{\bar{x}} + \Pi_{\bar{x}}\bar{A}^T + \bar{N}R_{\bar{v}}\bar{N}^T$. We have that

$$\begin{aligned} \bar{A}\Pi_{\bar{x}} &= \begin{bmatrix}-1 & 1\\ 0 & -\beta\end{bmatrix}\begin{bmatrix}\pi_1 & \pi_{12}\\ \pi_{12} & \pi_2\end{bmatrix} = \begin{bmatrix}\pi_{12} - \pi_1 & \pi_2 - \pi 12\\ -\beta\pi_{12} & -\beta\pi_2\end{bmatrix} \\ \bar{N}R_{\bar{x}}\bar{N}^T &= \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \end{aligned}$$

so spelled out the Lyapunov equation is

$$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}\pi_{12} - \pi_1 & \pi_2 - \pi 12\\ -\beta\pi_{12} & -\beta\pi_2\end{bmatrix} + \begin{bmatrix}\pi_{12} - \pi_1 & -\beta\pi_{12}\\ \pi_2 - \pi 12 & -\beta\pi_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Element by element this gives

$$\begin{cases} \text{1–1:} \quad 0 = 2\pi_{12} - 2\pi_1 + 1 \quad \Leftrightarrow \quad \pi_1 = \pi_{12} + 0.5, \\ \text{1–2:} \quad 0 = \pi_2 - (1+\beta)\pi_{12} \quad \Leftrightarrow \quad \pi_{12} = \frac{1}{1+\beta}\pi_2, \\ \text{2–2:} \quad 0 = -2\beta\pi_2 + 1 \quad \Leftrightarrow \quad \pi_2 = \frac{1}{2\beta}. \end{cases}$$

Thus $\pi_{12} = \frac{1}{2\beta(1+\beta)}$ and $\pi_1 = 0.5 + \frac{1}{2\beta(1+\beta)}$. Then

$$E\tilde{x}\tilde{x}^T = \bar{C}\Pi_{\bar{x}}\bar{C}^T = \pi_1 = 0.5 + \frac{1}{2\beta(1+\beta)}.$$

(c) Again, the covariance of w is the solution of a continuous-time Lyapunov equation, here $0 = (-\beta)\Pi_w + \Pi_w(-\beta) + R_v = -2\beta\Pi_w + 1$. We know that

$\Pi_w = Ew^2 = 1$, so $\beta = 0.5$.

2. **(a)** True (Def. 5.2); **(b)** True; **(c)** True (minimizing V is equivalent to minimizing kV); **(d)** True; **(e)** False; **(f)** False (the opposite, constraints are no problem for MPC); **(g)** False (typically the control horizon is much shorter than the prediction horizon.)

3. **(a)** In accordance with Theorem 9.4 in Glad/Ljung, the optimal controller is the state feedback $u(k) = -Lx(k)$, with $L = (G^TSG + Q_2)^{-1}G^TSF$, and where $S = S^T \geq 0$ is the unique solution of the DARE

$$S = F^TSF + M^TQ_1M - F^TSG(G^TSG + Q_2)^{-1}G^TSF.$$

Here

$$F = \begin{bmatrix} 1 & 1 \\ 0 & \beta \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad Q_1 = 1, \quad Q_2 = \rho.$$

The closed loop system will be

$$x(k+1) = Fx(k) + G(-Lx(k)) + Nv(k) = (F - GL)x(k) + Nv(k).$$

We note that

$$G^TSG = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = s_1, \quad SF = \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} s_1 & s_1 + \beta s_{12} \\ s_{12} & s_{12} + \beta s_2 \end{bmatrix}$$

$$\Rightarrow \quad G^TSF = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s_1 & s_1 + \beta s_{12} \\ s_{12} & s_{12} + \beta s_2 \end{bmatrix} = \begin{bmatrix} s_1 & s_1 + \beta s_{12} \end{bmatrix}.$$

Thus

$$L = \begin{bmatrix} l_1 & l_2 \end{bmatrix} = \begin{bmatrix} \frac{s_1}{s_1+\rho} & \frac{s_1+\beta s_{12}}{s_1+\rho} \end{bmatrix} \qquad \text{and} \qquad F - GL = \begin{bmatrix} 1 - l_1 & 1 - l_2 \\ 0 & \beta \end{bmatrix}.$$

Since $F - GL$ is triangular the poles/eigenvalues are on the diagonal, i.e. they are β and $1 - l_1 = 1 - \frac{s_1}{s_1+\rho} = \frac{\rho}{s_1+\rho}$.

(b) Here we need to solve the DARE, at least for s_1. Using the results from (a) we get that

$$F^TSF = \begin{bmatrix} 1 & 0 \\ 1 & \beta \end{bmatrix} \begin{bmatrix} s_1 & s_1 + \beta s_{12} \\ s_{12} & s_{12} + \beta s_2 \end{bmatrix} = \begin{bmatrix} s_1 & s_1 + \beta s_{12} \\ s_1 + \beta s_{12} & s_1 + 2\beta s_{12} + \beta^2 s_2 \end{bmatrix},$$

$$M^TQ_1M = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad F^TSG(G^TSG + Q_2)^{-1}G^TSF =$$

$$= \frac{1}{s_1 + \rho} \begin{bmatrix} s_1 \\ s_1 + \beta s_{12} \end{bmatrix} \begin{bmatrix} s_1 & s_1 + \beta s_{12} \end{bmatrix} = \frac{1}{s_1 + \rho} \begin{bmatrix} s_1^2 & s_1(s_1 + \beta s_{12}) \\ s_1(s_1 + \beta s_{12}) & (s_1 + \beta s_{12})^2 \end{bmatrix}.$$

The DARE then is

$$\begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix} = \begin{bmatrix} s_1 & s_1 + \beta s_{12} \\ s_1 + \beta s_{12} & s_1 + 2\beta s_{12} + \beta^2 s_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \frac{1}{s_1 + \rho} \begin{bmatrix} s_1^2 & s_1(s_1 + \beta s_{12}) \\ s_1(s_1 + \beta s_{12}) & (s_1 + \beta s_{12})^2 \end{bmatrix}.$$

The 1–1 element gives

$$s_1 = s_1 + 1 - \frac{s_1^2}{s_1 + \rho} \quad \Leftrightarrow \quad s_1^2 - s_1 - \rho = 0 \quad \Leftrightarrow \quad s_1 = 0.5 \pm \sqrt{0.25 + \rho},$$

where the negative root is omitted. The poles are $\lambda_1 = \beta$ (regardless of ρ) and $\lambda_2(\rho) = \frac{\rho}{0.5+\rho+\sqrt{0.25+\rho}}$. We see that

$$\lim_{\rho \to 0} \lambda_2(\rho) = 0 \qquad \text{and} \qquad \lim_{\rho \to \infty} \lambda_2(\rho) = 1.$$

(c) Setting $\rho = 0$ in the DARE above gives for the 1–1 and 1–2 elements

$$\begin{cases} s_1 & = s_1 + 1 - \frac{s_1^2}{s_1} = 1, \\ s_{12} & = s_1 + \beta s_{12} - \frac{s_1(s_1+\beta s_{12})}{s_1} = 0, \end{cases}$$

which gives $L = \begin{bmatrix} \frac{s_1}{s_1+\rho} & \frac{s_1+\beta s_{12}}{s_1+\rho} \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$. The closed loop system then becomes

$$\begin{aligned} x(k+1) &= (F - GL)x(k) + Nv(k) = \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(k), \\ z(k) &= Mx(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k). \end{aligned}$$

The discrete-time Lyapunov equation, here $\Pi_x = (F - GL)\Pi_x(F - GL)^T + NR_vV^T$, gives the covariance of $x(k)$: $Ex(k)x^T(k) = \Pi_x$. Then $Ez(k)^2 = M[Ex(k)x^T(k)]M^T = M\Pi_x M^T$. Here

$$\Pi_x = \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \beta^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $\Pi_x = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{1-\beta^2} \end{bmatrix}$, and $Ez(k)^2 = M\Pi_x M^T = 0$.

4. (a) The observability matrix is

$$\mathcal{O}_c = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & \beta \\ \beta & -1 \end{bmatrix} \quad \Rightarrow \quad \det \mathcal{O}_c = -(1 + \beta^2) \neq 0 \quad \forall \beta \in \mathbb{R}.$$

(b) Let h denote the sampling interval. The sampled system is $x(kh + h) = Fx(kh) + Gu(kh)$, $y(kh) = Hx(kh)$, where

$$F = e^{Ah}, \quad G = \int_0^h e^{At} B dt, \quad H = C = \begin{bmatrix} 1 & \beta \end{bmatrix}.$$

Use the Laplace transform, $e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$:

$$(sI - A)^{-1} = \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{-1}{s^2+1} \\ \frac{1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} \quad \Leftrightarrow \quad e^{At} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Thus

$$F = \begin{bmatrix} \cos h & -\sin h \\ \sin h & \cos h \end{bmatrix}, \quad \text{and} \quad G = \int_0^h \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} dt = \begin{bmatrix} \sin h \\ 1 - \cos h \end{bmatrix}.$$

(c) The observability matrix of the sampled system is

$$\mathcal{O}_d = \begin{bmatrix} C \\ CF \end{bmatrix} = \begin{bmatrix} 1 & \beta \\ \cos h + \beta \sin h & -\sin h + \beta \cos h \end{bmatrix} \quad \Rightarrow \quad \det \mathcal{O}_d = -(1+\beta^2) \sin h.$$

Since $1+\beta^2 \neq 0$ for all $\beta \in \mathbb{R}$ the rank of $\mathcal{O}_d$ does not depend on β. However, $\det \mathcal{O}_d = 0$ when $\sin h = 0$. Thus the sampled system is observable for all $h \neq l\pi$, $l = 1, 2, \ldots$.

5. (a) First note that

$$(p + \delta)w = v \qquad \Leftrightarrow \qquad pw = \dot{w} = -\delta w + v.$$

All together we then have

$$\begin{aligned} \dot{\theta} &= -\frac{1}{\tau_2}\theta + u, \\ \dot{r} &= k\theta - \frac{1}{\tau_1} r + \frac{1}{\tau_1} w, \\ \dot{h} &= r, \\ \dot{w} &= -\delta w + v, \\ z &= h, \end{aligned}$$

which with $x = \begin{bmatrix} \theta & r & h & w \end{bmatrix}^T$ can be written as

$$\begin{aligned} \dot{x} &= \overbrace{\begin{bmatrix} -1/\tau_2 & 0 & 0 & 0 \\ k & -1/\tau_1 & 0 & 1/\tau_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\delta \end{bmatrix}}^{=A} x + \overbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}^{=B} u + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}^{=N} v, \\ z &= \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}}_{=M} x. \end{aligned}$$

(b) We that

$$\left(p + \frac{1}{\tau_1}\right) r = k\theta + \frac{1}{\tau_1} w, \qquad \left(p + \frac{1}{\tau_2}\right) \theta = u, \qquad w = \frac{1}{p + \delta} v.$$

Since $u = 0$, $\theta = 0$ also holds, and we get

$$r = \frac{1/\tau_1}{(p + 1/\tau_1)(p + \delta)} v = \frac{1}{(\tau_1 p + 1)(p + \delta)} v = G_{rv}(p) v.$$

The spectrum then is

$$\begin{aligned}\Phi_r(\omega) &= |G_{rv}(i\omega)|^2\Phi_v(\omega)\\ &= \frac{1}{(i\tau_1\omega+1)(i\omega+\delta)(-i\tau_1\omega+1)(-i\omega+\delta)}R_v = \frac{R_v}{(\tau_1^2\omega^2+1)(\omega^2+\delta^2)}.\end{aligned}$$

6. The modelled spectrum is $G(e^{i\omega})\Phi_v(\omega)G(e^{-i\omega}) = |G(e^{i\omega})|^2$ (since $\Phi_v(\omega) = 1$). The powers of $\cos\omega$ in the given spectrum suggest that

$$G(q) = \frac{b_1 q + b_2}{q^2 + a_1 q + a_2},$$

for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Stability and minimum phase mean that the poles and zeros must be inside the unit circle (zeros on the unit circle are allowed). This means that $|a_2| < 1$ and $|b_2| < |b_1|$. Furthermore, the static gain must be positive, meaning that $G(1) = \frac{b_1+b_2}{1+a_1+a_2} > 0$. The modelled spectrum becomes

$$\begin{aligned}G(e^{i\omega})G(e^{-i\omega}) &= \frac{(b_1e^{i\omega}+b_2)(b_1e^{-i\omega}+b_2)}{(e^{i2\omega}+a_1e^{i\omega}+a_2)(e^{-i2\omega}+a_1e^{-i\omega}+a_2)}\\ &= \frac{b_1^2+b_2^2+b_1b_2(e^{i\omega}+e^{-i\omega})}{1+a_1^2+a_2^2+a_1(1+a_2)(e^{i\omega}+e^{-i\omega})+a_2(e^{i2\omega}+e^{-i2\omega})}\\ &= \frac{b_1^2+b_2^2+2b_1b_2\cos\omega}{1+a_1^2+a_2^2+2a_1(1+a_2)\cos\omega+2a_2\cos 2\omega}\\ &= \frac{(b_1^2+b_2^2+2b_1b_2\cos\omega)/2a_2}{\frac{1+a_1^2+a_2^2}{2a_2}+\frac{2a_1(1+a_2)}{2a_2}\cos\omega+\cos 2\omega}.\end{aligned}$$

Compare with the given spectrum:

$$\Phi_w(\omega) = \frac{5+4\cos\omega}{1.25-\cos 2\omega} = \frac{-(5+4\cos\omega)}{-1.25+\cos 2\omega}.$$

The denominator:

$$\frac{1+a_1^2+a_2^2}{2a_2} = -1.25, \qquad \text{and} \qquad \frac{2a_1(1+a_2)}{2a_2} = 0$$

The second equation gives that $a_1 = 0$ or $a_2 = -1$, but the latter can be excluded since $|a_2| < 1$. Thus $a_1 = 0$, which when put into the first equation gives $a_2^2 + 2.5a_2 + 1 = 0 \Leftrightarrow a_2 = -1.25 \pm -0.75$, and again $|a_2| < 1 \Rightarrow a_2 = -0.5$. The numerator then is $-(b_1^2 + b_2^2 + 2b_1b_2\cos\omega)$, which when compared when the given spectrum gives

$$b_1^2 + b_2^2 = 5, \qquad \text{and} \qquad 2b_1b_2 = 4.$$

This is solved by $b_{1,2} = \pm 1.5 \pm 0.5$. The second equation implies that b_1 and b_2 have the same sign, and since $G(1) = \frac{b_1+b_2}{1+a_1+a_2} = \frac{b_1+b_2}{0.5} > 0$ must hold both b_1 and b_2 must be positive. Furthermore, $|b_2| < |b_1| \Rightarrow b_1 = 2$ and $b_2 = 1$. Hence $G(q) = \frac{2q+1}{q^2-0.5}$.