Exam in Automatic Control II Reglerteknik II 5hp

Date: October 23, 2014

Venue: Polacksbacken, exam hall

Responsible teacher: Hans Norlander.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Please use English in your solutions when possible, that would be appreciated!

Good luck!

Problem 1 A DC motor is used as a position servo, and is controlled by a sampling controller with sampling period h. Zero-order hold sampling is used, and the corresponding discrete-time model is

$$y(kh) = \frac{\beta q + \gamma}{(q-1)(q-\alpha)} u(kh), \qquad 0 < \alpha < 1, \quad 0 < \gamma < \beta.$$

The control law is

$$u(kh) = K(r(kh) - y(kh)),$$

where $r(kh)$ is the reference signal.

(a) For which $K \in \mathbb{R}$ is the closed loop system stable? **(2p)**

(b) Determine $\lim_{k\to\infty} y(kh)$ when $r(kh)$ is a unit step. **(2p)**

(c) A state space representation of the DC motor, in continuous-time, is

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t). \end{aligned}$$

When zero-order hold sampling is used the corresponding discrete-time state space model

$$\begin{aligned} x(kh+h) &= Fx(kh) + Gu(kh), \\ y(kh) &= Hx(kh), \end{aligned}$$

is obtained. Determine F, G and H. **(4p)**

Problem 2 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true" / "false" are considered!

(a) A Kalman filter is always stable.

(b) For a Kalman filter the *innovation*, $\nu = y - C\hat{x} - Du$, is white noise if the model and the noise intensities are correct.

(c) The discrete-time system $y(k) = \frac{0.25}{q+0.25} u(k)$ has unit static gain.

(d) The scalar system $x(k+1) = 0.5x(k) + u(k) + \nu(k)$, $y(k) = x(k) + \nu(k)$, ($\nu(k)$ is white noise) is on *innovations form.*

(e) The scalar system $x(k+1) = -0.5x(k) + u(k) + \nu(k)$, $y(k) = x(k) + \nu(k)$, ($\nu(k)$ is white noise) is on *innovations form.*

Each correct answer scores $+1$, each incorrect answer scores -1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(5p)**

Problem 3 The variable $x(k)$ is of importance in an industrial process, and an estimate of it, $\hat{x}(k|k-1)$, is needed. The model

$$\begin{aligned} x(k+1) &= x(k) + w(k), \\ y(k) &= x(k) + e(k), \end{aligned}$$

describes how $x(k)$ evolves over time. Here $y(k)$ is the measurement of $x(k)$. The measurement noise, $e(k)$, is zero mean white noise with covariance $Ee^2(k) = \Phi_e(\omega) = 0.75$. The process noise $w(k)$, in its turn, is described by the model

$$w(k+1) = 0.6w(k) + v(k),$$

where $v(k)$ is zero mean white noise with covariance $Ev^2(k) = \Phi_v(\omega) = 0.64$. Furthermore, $v(k)$ and $e(k)$ are uncorrelated.

(a) Show that $w(k)$ has covariance $Ew^2(k) = 1$. **(2p)**

(b) Determine the spectrum, $\Phi_w(\omega)$, of $w(k)$. **(2p)**

(c) A Kalman filter is used to get the estimate $\hat{x}(k|k-1)$, and the Kalman filter is based on the *simplified* model of $w(k)$, where it is assumed to be zero mean white noise with $Ew^2(k) = \Phi_w(\omega) = 1$. Give the Kalman filter based on this assumption. **(4p)**

(d) The P you get in (c) is the *assumed* covariance of the estimation error (that would hold if the simplified model was true). What is the *real* covariance of the estimation error, $E\tilde{x}^2(k)$, for the Kalman filter in (c) (based on the true model of $w(k)$ above)? **(4p)**

Problem 4 A system with one input and two outputs is described by

$$\begin{aligned} \dot{y}_1(t) + y_2(t) &= \dot{u}(t) + 2u(t), \\ \dot{y}_2(t) + y_2(t) + y_1(t) &= u(t). \end{aligned}$$

(a) Find the transfer function $G(s) = \begin{bmatrix} G_1(s) & G_2(s) \end{bmatrix}^T$, from the input u to the output $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$, for the system. **(3p)**

(b) Determine a state space model for the system. **(3p)**

Problem 5 A very simple model of a car, used for the design of a cruise control system, is

$$\begin{cases} \dot{z}(t) = & -z(t) + u(t) + w(t), \\ y(t) = & z(t) + e(t), \end{cases} \qquad w(t) = \frac{1}{p+0.1} v(t),$$

where z is the velocity of the car, u is the throttle opening ("gaspådraget"), which is used as input signal, and y is the measured velocity. There are disturbances in terms of a slowly varying force w, representing the shifting slope of the road, and the measurement disturbance e. Both e and v are zero mean white noise.

(a) Introduce the state vector $x = \begin{bmatrix} z & w \end{bmatrix}^T$ and give the matrices and vectors A, B, N, M and C in the state space representation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Nv(t), \\ z(t) &= Mx(t), \\ y(t) &= Cx(t) + e(t). \end{aligned}$$

(5p)

(b) The cruise control system is implemented as an observer based state feedback controller, with the control law $u(t) = -L\hat{x}(t) + L_r r(t)$. LQG is used as design technique so $\hat{x}$ is obtained from a Kalman filter, which is not considered in this problem. The cost function, which should be minimized, is

$$V = E\{1.25z^2 + u^2\}.$$

Find the optimal state feedback gain L. **(5p)**

(c) What are the poles of the closed loop system? **(2p)**

Problem 6 *The HW bonus points are exchangeable for this problem.*

(a) The stationary continuous-time stochastic process $y(t)$ is modeled as $y(t) = G(p)u(t)$, where $G(p)$ is minimum phase and $G(0) > 0$. The input $u(t)$ is zero mean white noise with intensity $\Phi_u(\omega) = 1$. The spectrum of $y(t)$ is

$$\Phi_y(\omega) = \frac{16\omega^2 + 4}{\omega^4 - 15\omega^2 + 64}.$$

Determine the transfer operator $G(p)$. **(4p)**

(b) Compute the Kalman filter estimate $\hat{x}(k|k)$ for the system

$$\begin{cases} x(k+1) &= -x(k) + v_1(k), \\ y(k) &= x(k) + v_2(k), \end{cases} \qquad R_1 = 1,\ R_2 = 6,\ R_{12} = 0,$$

when $\hat{x}(k|k-1) = -0.7$ and $y(k) = 0.2$. **(3p)**

Solutions to the exam in Automatic Control II, 2014-10-23:

1. (a) We have $y = G(q)u$ and $u = K(r - y) \Rightarrow$ the closed loop system is $y = G_c(q)r$ where $G_c = \frac{KG(q)}{1+KG(q)}$:

$$G_c(q) = \frac{K\frac{\beta q+\gamma}{(q-1)(q-\alpha)}}{1 + K\frac{\beta q+\gamma}{(q-1)(q-\alpha)}} = \frac{K(\beta q + \gamma)}{q^2 + (K\beta - 1 - \alpha)q + \alpha + K\gamma}$$

For stability all poles must lie inside the unit circle. For $z^2 + az + b$ the zeros are inside the unit circle exactly when $|a| - 1 < b < 1$. Here $b < 1$ gives $\alpha + K\gamma < 1 \Leftrightarrow K < \frac{1-\alpha}{\gamma}$.
$a - 1 < b$ gives $K\beta - 1 - \alpha - 1 < \alpha + K\gamma \Leftrightarrow K < \frac{2(1+\alpha)}{\beta-\gamma}$, which gives another possible upper bound for K.
Finally, $-a - 1 < b$ gives $-K\beta + 1 + \alpha - 1 < \alpha + K\gamma \Leftrightarrow 0 < K(\beta + \gamma)$.
Thus, stable for

$$0 < K < \min(\frac{1 - \alpha}{\gamma}, \frac{2(1 + \alpha)}{\beta - \gamma}).$$

(b) The static gain is $G_c(1) = 1$, and thus $\lim_{k\to\infty} y(kh) = 1$.
(c) We have that $F = e^{Ah}$, $G = \int_0^h e^{At}Bdt$ and $H = C$. Here

$$e^{At} = \mathcal{L}^{-1}[(sI-A)^{-1}] = \mathcal{L}^{-1}\left[\begin{bmatrix} s+1 & -1 \\ 0 & s \end{bmatrix}^{-1}\right] = \mathcal{L}^{-1}\left[\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s(s+1)} \\ 0 & \frac{1}{s} \end{bmatrix}\right] = \begin{bmatrix} e^{-t} & 1 - e^{-t} \\ 0 & 1 \end{bmatrix},$$

and thus

$$F = \begin{bmatrix} e^{-h} & 1 - e^{-h} \\ 0 & 1 \end{bmatrix}, \quad G = \int_0^h \begin{bmatrix} 1 - e^{-t} \\ 1 \end{bmatrix} dt = \begin{bmatrix} h - 1 + e^{-h} \\ h \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

2. (a) True (Lemma 5.1); **(b)** True (Theorem 5.5); **(c)** False (static gain is obtained for $q = 1 \Rightarrow$ the static gain is here 0.2); **(d)** True (innovations form if $v_1 = v_2 = \nu$ and $F - NH$ stable, here $F - NH = -0.5$) ; **(e)** False (here $F - NH = -1.5$, i.e. unstable);

3. (a) Use the discrete-time Lyapunov equation, $\Pi_w = F\Pi_w F^T + NR_v N^T$. Here $F = 0.6$, $N = 1$ and $R_v = 0.64$.

$$\Pi_w = 0.6^2\Pi_w + 0.64 \qquad \Leftrightarrow \qquad \Pi_w = \frac{0.64}{1 - 0.6^2} = 1.$$

(b) With $w(k) = G(q)v(k)$ we have $\Phi_w(\omega) = |G(e^{i\omega}|^2\Phi_w(\omega)$. Here $G(q) = \frac{1}{q-0.6}$, so

$$\Phi_w(\omega) = \frac{1}{e^{i\omega} - 0.6}\cdot\frac{1}{e^{-i\omega} - 0.6}R_v = \frac{0.64}{1 + 0.6^2 - 0.6(e^{i\omega} + e^{-i\omega})} = \frac{0.64}{1.36 - 1.2\cos\omega}.$$

(c) Theorem 5.6 $\Rightarrow K = FPH^T(HPH^T + R_2)^{-1}$ where $P = P^T \geq 0$ solves the DARE $P = FPF^T + NR_1N^T - FPH^T(HPH^T + R_2)^{-1}HPF^T$ (for $R_{12} =$

0). Here $F = 1$, $N = 1$, $H = 1$, $R_1 = Ew^2(k) = 1$ and $R_2 = R_e = 0.75$, so the DARE becomes

$$P = P + 1 - \frac{P^2}{P + 0.75} \qquad \Leftrightarrow \qquad P^2 - P - 0.75 = 0 \qquad \Leftrightarrow \qquad P = 0.5 \pm 1,$$

and since $P \geq 0$ we get $P = 1.5$ and $K = \frac{P}{P+0.75} = \frac{2}{3}$.

(d) The Kalman filter is

$$\hat{x}(k+1|k) = \hat{x}(k|k-1) + K(y(k) - \hat{x}(k|k-1)) = (1-K)\hat{x}(k|k-1) + Ky(k),$$

and the estimation error $\tilde{x}(k|k-1) = x(k) - \hat{x}(k|k-1)$ will then evolve over time according to

$$\begin{aligned} \tilde{x}(k+1|k) &= x(k+1) - \hat{x}(k+1|k) = x(k) + w(k) - (1-K)\hat{x}(k|k-1) - K(x(k) + e(k)) \\ &= (1-K)\tilde{x}(k|k-1) + w(k) - Ke(k) = \frac{1}{3}\tilde{x}(k|k-1) + w(k) - \frac{2}{3}e(k). \end{aligned}$$

Combining this with the model of $w(k)$ gives the state space representation

$$\begin{bmatrix} \tilde{x}(k+1|k) \\ w(k+1) \end{bmatrix} = \begin{bmatrix} 1/3 & 1 \\ 0 & 0.6 \end{bmatrix} \begin{bmatrix} \tilde{x}(k|k-1) \\ w(k) \end{bmatrix} + \begin{bmatrix} 0 & -2/3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v(k) \\ e(k) \end{bmatrix}.$$

The covariance matrix of the state vector is obtained as the solution of the corresponding discrete-time Lyapunov equation:

$$\begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} = \begin{bmatrix} 1/3 & 1 \\ 0 & 0.6 \end{bmatrix} \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 1 & 0.6 \end{bmatrix} + \begin{bmatrix} 0 & -2/3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.64 & 0 \\ 0 & 0.75 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2/3 & 0 \end{bmatrix}$$

resulting in the equation system

$$\begin{cases} \pi_1 &= (1/3)^2\pi_1 + 2 \cdot 1/3\pi_{12} + \pi_2 + (2/3)^2 \cdot 0.75 \\ \pi_{12} &= 0.6 \cdot (1/3)\pi_{12} + 0.6\pi_2 \\ \pi_2 &= 0.36\pi_2 + 0.64 \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} \pi_1 &= 33/16 = 2.0625 \\ \pi_{12} &= 3/4 = 0.75 \\ \pi_2 &= 1 \end{cases}$$

Then $E\tilde{x}^2 = \pi_1 = 2.0625$ (which is worse than $P = 1.5$ in (c)).

4. (a) Apply the Laplace transform $\Rightarrow$

$$\begin{cases} sY_1(s) + Y(s) &= sU(s) + 2U(s), \\ sY_2(s) + Y(s) + Y_1(s) &= U(s) \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} s & 1 \\ 1 & s+1 \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} s+2 \\ 1 \end{bmatrix} U(s),$$

and thus

$$\begin{aligned} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} &= \begin{bmatrix} s & 1 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} s+2 \\ 1 \end{bmatrix} U(s) \\ &= \frac{1}{s(s+1) - 1} \begin{bmatrix} s+1 & -1 \\ -1 & s \end{bmatrix} \begin{bmatrix} s+2 \\ 1 \end{bmatrix} U(s) = \begin{bmatrix} \frac{s^2+3s+1}{s^2+s-1} \\ \frac{-2}{s^2+s-1} \end{bmatrix} U(s). \end{aligned}$$

(b) For systems with one input the controller canonical form can be used. First, though, we notice that $G_1(s)$ has a direct term — re-write it as $G_1(s) = 1 + \frac{2s+2}{s^2+s-1}$. The controller canonical form then is

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u,$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u.$$

5. **(a)** We have $(p+0.1)w = v$, and thus $\dot{w} = -0.1w + v$. Hence

$$\begin{cases} \dot{z} = & -z + w + u, \\ \dot{w} = & -0.1w + v, \\ y = & z + e, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dot{x} = \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & -0.1 \end{bmatrix}}_{=A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{=B} u + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{=N} v, \\ z = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{=M} x, \\ y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{=C} x + e. \end{cases}$$

(b) Theorem 9.1 $\Rightarrow L = Q_2^{-1} B^T S$, where $S = S^T \geq 0$ solves the CARE $0 = A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S$. Here A, B and M are determined in (a), and $Q_1 = 1.25$ and $Q_2 = 1$. The CARE is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -s_1 & -s_{12} \\ s_1 - 0.1s_{12} & s_{12} - 0.1s_2 \end{bmatrix} + \begin{bmatrix} -s_1 & s_1 - 0.1s_{12} \\ -s_{12} & s_{12} - 0.1s_2 \end{bmatrix} + \begin{bmatrix} 1.25 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} s_1^2 & s_1 s_{12} \\ s_1 s_{12} & s_{12}^2 \end{bmatrix},$$

which, element by element, gives the equation system

$$0 = -2s_1 + 1.25 - s_1^2,$$
$$0 = s_1 - 1.1s_{12} - s_1 s_{12},$$
$$0 = 2s_{12} - 0.2s_2 - s_{12}^2.$$

By noting that $L = Q_2^{-1} B^T S = \begin{bmatrix} s_1 & s_{12} \end{bmatrix}$ it is realized that only s_1 and s_{12} need to be solved for. The first equation gives $0 = s_1^2 + 2s_1 - 1.25 \Leftrightarrow s_1 = -1 + \sqrt{1 + 1.25} = 0.5$. The second equation gives $s_{12} = \frac{s_1}{1.1+s_1} = \frac{0.5}{1.6} = 0.3125$. Thus, $L = \begin{bmatrix} 0.5 & 0.3125 \end{bmatrix}$.

(c) The closed loop system is $Z(s) = M(sI - A + BL)^{-1} BL_r R(s)$, and the poles are the roots of

$$0 = \det(sI - A + BL) = \det \begin{bmatrix} s + 1 + 0.5 & -1 + 0.3125 \\ 0 & s + 0.1 \end{bmatrix} = (s + 1.5)(s + 0.1).$$

Thus, the poles are -1.5 and -0.1.

6. **(a)** $G(p)$ stationary and minimum phase $\Leftrightarrow$ all poles in the left half plane and no zeros in the right half plane. The numerator is a first order polynomial, and the denominator is a second order polynomial in ω^2 $\Rightarrow$ try with $G(p) = \frac{b_1 p + b_2}{p^2 + a_1 p + a_2}$, with $b_1, b_2, a_1, a_2 > 0$ (to also have $G(0) > 0$). The spectrum then is

$$\begin{aligned}\Phi_y(\omega) = G(i\omega)G(-i\omega) &= \frac{(b_2 + ib_1\omega)(b_2 - ib_1\omega)}{(-\omega^2 + a_2 + ia_1\omega)(-\omega^2 + a_2 - ia_1\omega)} \\ &= \frac{b_2^2 + b_1^2\omega^2}{(-\omega^2 + a_2)^2 + a_1^2\omega^2} = \frac{b_2^2 + b_1^2\omega^2}{\omega^4 + (a_1^2 - 2a_2)\omega^2 + a_2^2}\end{aligned}$$

Comparing with the given spectrum, and equating equal powers of ω^2 in both numerator and denominator, gives

$$\begin{cases} b_1^2 & = 16, \\ b_2^2 & = 4, \\ a_1^2 - 2a_2 & = -15, \\ a_2^2 & = 64, \end{cases} \qquad \Rightarrow \qquad \begin{cases} b_1 & = 4, \\ b_2 & = 2, \\ a_1 & = 1, \\ a_2 & = 8. \end{cases}$$

Thus $G(p) = \frac{4p+2}{p^2+p+8}$.

(b) Theorem 5.6, Eqs. (5.100)–(5.101) $\Rightarrow$

$$\hat{x}(k|k) = \hat{x}(k|k-1) + \tilde{K}(y(k) - H\hat{x}(k|k-1)), \qquad \tilde{K} = PH^T(HPH^T + R_2)^{-1}.$$

Here $F = -1$, $N = H = 1$, $R_1 = 1$, $R_2 = 6$ and $R_{12} = 0$ so the associated DARE is

$$P = P + 1 - \frac{P^2}{P + 6} \qquad \Leftrightarrow \qquad P^2 - P - 6 = 0,$$

with solution $P = 0.5 + \sqrt{0.5^2 + 6} = 3 \Rightarrow \tilde{K} = \frac{P}{P+6} = \frac{1}{3}$. Hence

$$\hat{x}(k|k) = -0.7 + \frac{1}{3}(0.2 - (-0.7)) = -0.4.$$