Exam in Automatic Control II
Reglerteknik II 5hp

Date: August 22, 2015

Venue: Bergsbrunnagatan 15, room 2

Responsible teacher: Hans Norlander.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Good luck!

Problem 1 A discrete-time stochastic process has the state space representation

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k), \\ y(k) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(k). \end{aligned} \tag{1}$$

The input $u(k)$ is zero mean white noise with spectrum $\Phi_u(\omega) = 1$.

(a) Determine the spectrum, $\Phi_y(\omega)$, of the output for the stochastic process repesented by (1). **(2p)**

(b) Determine the covariance matrix, $\Pi_x = Ex(k)x^T(k)$, for the state vector in (1). **(2p)**

(c) A stationary continuous-time stochastic process is found to have the spectrum

$$\Phi_y(\omega) = \frac{4\omega^2 + 4}{\omega^4 - 3\omega^2 + 4}. \tag{2}$$

Find a minimum phase transfer function $G(s)$, with $G(0) > 0$, so that the model $y(t) = G(p)u(t)$ is stationary and has the spectrum (2). Here $u(t)$ is zero mean white noise with intensity $\Phi_u(\omega) = 1$. **(4p)**

Problem 2 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true"/"false" are considered!

(a) The discrete-time system $y(k) = \frac{0.8}{q+0.8}u(k)$ has unit static gain.

(b) The sampling frequency is 50% of the Nyquist frequency.

(c) In MPC the *control/input horizon* is normally much longer than the *prediction/output horizon.*

(d) In MPC the computational load increases with the *control/input horizon.*

(e) An MPC controller can only be implemented as a continuous-time controller.

(f) A commonly used model of white noise has the spectrum $\Phi(\omega) = \frac{\omega^2}{\omega^4+4}$.

(g) $\Phi(\omega) = \frac{4}{1-2\cos\omega}$ is the spectrum of a discrete-time stochastic process.

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(7p)**

Problem 3 The continuous-time system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t), \end{aligned} \tag{3}$$

is controlled by the state feedback $u(t) = -L_c x(t)$, where $L_c = \begin{bmatrix} 2 & 0 \end{bmatrix}$.

(a) Determine the poles of the closed loop system. **(1p)**

(b) The control system is modernized, and the continuous-time controller is replaced by a sampling controller, implemented in a PC. Zero-order hold (ZOH) sampling is used. In order to analyze the controlled system a discrete-time (sampled) model of the system (3) is needed. Determine the matrices and vectors F, G and H in the discrete-time state space model

$$\begin{aligned} x(kh+h) &= Fx(kh) + Gu(kh), \\ y(kh) &= Hx(kh), \end{aligned} \tag{4}$$

where $k = 0, 1, \dots$ and $h > 0$ is the sampling interval. **(2p)**

(c) Initially the same state feedback is used as for the continuous-time case, i.e. the control law $u(kh) = -L_c x(kh)$, with the same L_c as above, is implemented. Use the discrete-time model (4) and determine the closed loop poles for this case. Also determine for which $h > 0$ the closed loop system is stable. **(3p)**

(d) It turned out that the performance of the sampling controller in (c) was not satisfactory (due to a rather long sampling interval h). Redesign the controller by use of the sampled model (4). That is, find a vector L_d so that the state feedback $u(kh) = -L_d x(kh)$ gives a closed loop sampled system which behaves in the same (similar) way as the original continuous-time closed loop system in (a). (L_d will depend on h.)
Hint: Find out what the corresponding poles of the closed loop sampled system should be. **(4p)**

Problem 4 Consider the double integrator,

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_1(t), \\ z(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + v_2(t), \end{aligned}$$

where the process noise, v_1, and the measurement noise, v_2, are uncorrelated zero mean white noise processes, with intensities $R_1 = R_2 = 1$. Give the controller that minimizes the criterion

$$V = E\left\{u^2 + \gamma^2 z^2\right\}.$$

(10p)

Problem 5 Consider the discrete-time stochastic process

$$\begin{aligned} x(k+1) &= x(k) + v_1(k), \\ y(k) &= x(k) + v_2(k), \end{aligned}$$

where v_1 and v_2 are uncorrelated zero mean white noise processes, with variances $Ev_1^2 = 1$ and $Ev_2^2 = 2$. A Kalman filter is used to produce the estimate $\hat{x}(k|k-1)$.

(a) Determine the covariance, $E\tilde{x}(k|k-1)\tilde{x}^T(k|k-1)$, of the estimation error $\tilde{x}(k|k-1) = x(k) - \hat{x}(k|k-1)$. **(2p)**

(b) The Kalman filter is typically represented in state space form. However, it also possible to represent it with transfer functions. The Kalman filter above can be represented as

$$\hat{x}(k|k-1) = F(q)y(k).$$

Determine the transfer function $F(q)$ for the Kalman filter above. **(4p)**

(c) Now consider the general discrete-time system

$$\begin{aligned} x(k+1) &= Fx(k) + Gu(k) + Nv_1(k), \\ y(k) &= Hx(k) + v_2(k), \end{aligned} \qquad E\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}\right) = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}, \tag{5}$$

and the corresponding Kalman filter. The *output innovations*, $\nu(k) = y(k) - H\hat{x}(k|k-1)$, represent the new information fed into the Kalman filter — the Kalman filter can be represented as

$$\hat{x}(k+1|k) = F\hat{x}(k|k-1) + Gu(k) + K\nu(k).$$

For the Kalman filter the output innovations are white noise. Derive an expression for the covariance of the innovations, $E\nu(k)\nu^T(k)$, and show that this is *greater than or equal to* R_2.[1] **(2p)**

Problem 6 *The HW bonus points are exchangeable for this problem.*

(a) Give a state space representation for the system

$$y(k) = \begin{bmatrix} \frac{q+1}{q(q-1)} & \frac{q}{(q-0.5)(q-1)} \end{bmatrix} u(k), \quad u(k) = \begin{bmatrix} u_1(k) & u_2(k) \end{bmatrix}^T.$$

(4p)

(b) Is your state space representation in (a) a minimal realization? **(3p)**

[1]For the continuous-time case $\Phi_\nu(\omega) = R_2$ holds.

Solutions to the exam in Automatic Control II, 2015-08-22:

1. (a) The spectrum is $\Phi_y(\omega) = G(e^{i\omega})\Phi_u(\omega)G^*(e^{i\omega}) = |G(e^{i\omega})|^2$, where the last identity follows from that it is a scalar process, and that $\Phi_u(\omega) = 1$. The transfer function is given by $G(q) = H(qI - F)^{-1}G$, so

$$G(q) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} q - \sqrt{0.5} & 0 \\ 0 & q + \sqrt{0.5} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{q - \sqrt{0.5}} + \frac{1}{q + \sqrt{0.5}} = \frac{2q}{q^2 - 0.5}.$$

Thus

$$\Phi_y(\omega) = \frac{2e^{i\omega}}{e^{i2\omega} - 0.5} \cdot \frac{2e^{-i\omega}}{e^{-i2\omega} - 0.5} = \frac{4}{1 + 0.5^2 - 2 \cdot 0.5 \cos 2\omega} = \frac{4}{1.25 - \cos 2\omega}.$$

(b) The covariance matrix is the solution Π_x of the discrete-time Lyapunov equation $\Pi_x = F\Pi_x F^T + NRN^T$. By setting

$$\Pi_x = \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix}$$

we get the Lyapunov equation spelled out as

$$\begin{aligned} \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} &= \begin{bmatrix} \sqrt{0.5} & 0 \\ 0 & -\sqrt{0.5} \end{bmatrix} \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} \begin{bmatrix} \sqrt{0.5} & 0 \\ 0 & -\sqrt{0.5} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.5\pi_1 & -0.5\pi_{12} \\ -0.5\pi_{12} & 0.5\pi_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

Elementwise this gives the equation system

$$\begin{cases} \pi_1 = 0.5\pi_1 + 1, \\ \pi_{12} = -0.5\pi_{12} + 1, \\ \pi_2 = 0.5\pi_2 + 1, \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} \pi_1 = \dfrac{1}{1 - 0.5} = 2, \\ \pi_{12} = \dfrac{1}{1 + 0.5} = \dfrac{2}{3}, \\ \pi_2 = \dfrac{1}{1 - 0.5} = 2, \end{cases}$$

so $\Pi_x = \begin{bmatrix} 2 & 2/3 \\ 2/3 & 2 \end{bmatrix}$.

(c) For a continuous-time system, $y(t) = G(p)u(t)$, we have that

$$\Phi_y(\omega) = G(i\omega)\Phi_u(\omega)G^*(i\omega) = G(i\omega)G(-i\omega),$$

where the latter equality follows since $G(s)$ is scalar and $\Phi_u(\omega) = 1$. Since $\Phi_y(\omega)$ is rational in ω^2 it is possible to find a rational, stable and minimum phase $G(s)$ such that the relation above holds (Theorem 5.1). Since the numerator of $\Phi_y(\omega)$ has degree 2, and the denominator has degree 4, a clever guess is that

$$G(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}.$$

Then

$$G(i\omega)G(-i\omega) = \frac{ib_1\omega + b_2}{-\omega^2 + ia_1\omega + a_2} \cdot \frac{-ib_1\omega + b_2}{-\omega^2 - ia_1\omega + a_2}$$
$$= \frac{b_1^2\omega^2 + b_2^2}{(a_2 - \omega^2)^2 + a_1^2\omega^2} = \frac{b_1^2\omega^2 + b_2^2}{\omega^4 + (a_1^2 - 2a_2)\omega^2 + a_2^2}.$$

By comparing numerator and denominator, per power of ω^2, the following equation systems are obtained:

$$\begin{cases} 4 = b_1^2, \\ 4 = b_2^2, \end{cases} \qquad \text{and} \qquad \begin{cases} -3 = a_1^2 - 2a_2, \\ 4 = a_2^2. \end{cases}$$

Stationarity of $y \Rightarrow G(s)$ must be stable $\Rightarrow a_1, a_2 > 0$. This, together with $G(0) > 0$ implies that $b_2 > 0$, which together with the minimum phase requirement (no zeros in the RHP) implies that $b_1 \geq 0$. Given these conditions, the solution is

$$\begin{cases} b_1 = 2, \\ b_2 = 2, \end{cases} \qquad \begin{cases} a_1 = 1, \\ a_2 = 2 \end{cases} \qquad \Leftrightarrow \qquad G(s) = \frac{2s + 2}{s^2 + s + 2}.$$

2. **(a)** False (Here $G(1) = 0.8/1.8 \neq 1$); **(b)** False (The Nyquist frequency is per definition 50% of the sampling frequency); **(c)** False (The other way around); **(d)** True (The control horizon decides the dimension of the optimization problem); **(e)** False (The opposite); **(f)** False (White noise has flat/constant spectrum); **(g)** False (A spectrum is always non-negative — here $\Phi(0) = -4$!)

3. **(a)** The closed loop poles are given by the characteristic equation $0 = \det(sI - A + BL_c)$, which here is

$$0 = \det \begin{bmatrix} s + 1 + l_1 & l_2 \\ l_1 & s + 2 + l_2 \end{bmatrix} = s^2 + (3 + l_1 + l_2)s + 2 + 2l_1 + l_2 = s^2 + 5s + 6.$$

The poles are -2 and -3.
(b) For a ZOH sampled system we have $F = e^{Ah}$, $G = \int_0^h e^{At}B dt$ and $H = C$. Thus

$$F = \begin{bmatrix} e^{-h} & 0 \\ 0 & e^{-2h} \end{bmatrix}, \qquad G = \int_0^h \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix} dt = \begin{bmatrix} 1 - e^{-h} \\ 0.5(1 - e^{-2h}) \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

(c) Set $\alpha = e^{-h}$ and $L = \begin{bmatrix} l_1 & 0 \end{bmatrix}$. The poles of the closed loop sampled system are given by

$$0 = \det(qI - F + GL) = \det \begin{bmatrix} q - \alpha + (1 - \alpha)l_1 & 0 \\ 0.5(1 - \alpha^2)l_1 & q - \alpha^2 \end{bmatrix}$$
$$= (q - \alpha + (1 - \alpha)l_1)(q - \alpha^2).$$

Thus, the poles are $\alpha^2 = e^{-2h}$ and $(l_1+1)\alpha - l_1 = 3e^{-h} - 2$ (since $l_1 = 2$). For stability both poles must lie inside the unit circle. Since $0 < e^{-2h} < 1$ for all $h > 0$, it is only the other pole that can cause an unstable closed loop system. The condition for stability is

$$-1 < 3e^{-h} - 2 < 1 \qquad \Leftrightarrow \qquad 1 < 3e^{-h} < 3.$$

Again, since $e^{-h} < 1$ for all $h > 0$, it is only the lower bound that can be violated. Hence, the condition for stability is

$$1 < 3e^{-h} \qquad \Leftrightarrow \qquad e^h < 3 \qquad \Leftrightarrow \qquad h < \log 3 \approx 1.0986.$$

(d) The continuous-time poles are -2 and -3, and the corresponding sampled poles are $e^{-2h} = \alpha^2$ and $e^{-3h} = \alpha^3$ (see Sec 4.3 in Glad/Ljung). The desired pole polynomial for the sampled system is then $(q-\alpha^2)(q-\alpha^3)$. Compare with the obtained pole polynomial obtained in (c), $(q - \alpha + (1-\alpha)l_1)(q-\alpha^2)$. Here we see that the desired poles can be obtained with $l_2 = 0$ (as in (c)) and $l_1 = \frac{\alpha-\alpha^3}{1-\alpha} = \frac{e^{-h}-e^{-3h}}{1-e^{-h}}$. Then $L_d = \begin{bmatrix} l_1 & 0 \end{bmatrix}$.

4. (a) The controller that minimizes V is the LQG-controller:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t)), \\ u(t) &= -L\hat{x}(t), \end{aligned}$$

i.e. a Kalman filter combined with the optimal state feedback — see Theorem 9.1 in Glad/Ljung. The optimal state feedback gain is $L = Q_2^{-1}B^T S$, where $S = S^T \geq 0$ is the solution to the continuous-time algebraic Riccati (CARE) equation

$$0 = SA + A^T S + M^T Q_1 M - SBQ_2^{-1}B^T S.$$

Set $S = \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix}$

$$\Rightarrow \quad SA = \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} s_{12} & 0 \\ s_2 & 0 \end{bmatrix}, \quad SB = \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_{12} \end{bmatrix}.$$

Since $M = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $Q_1 = \gamma^2$ and $Q_2 = 1$, the CARE spells out as

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s_{12} & 0 \\ s_2 & 0 \end{bmatrix} + \begin{bmatrix} s_{12} & s_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \gamma^2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} s_1^2 & s_1 s_{12} \\ s_1 s_{12} & s_{12}^2 \end{bmatrix}.$$

Elementwise this gives the equations system

$$\begin{cases} 0 = 2s_{12} + \gamma^2 - s_1^2, \\ 0 = s_2 - s_1 s_{12}, \\ 0 = -s_{12}^2, \end{cases} \quad \Leftrightarrow \quad \begin{cases} s_1 = \gamma, \\ s_{12} = 0, \\ s_2 = 0, \end{cases} \quad \Leftrightarrow \quad S = \begin{bmatrix} \gamma & 0 \\ 0 & 0 \end{bmatrix}.$$

Now $L = Q_2^{-1}B^T S = \begin{bmatrix} 1 & 0 \end{bmatrix} S = \begin{bmatrix} s_1 & s_{12} \end{bmatrix} = \begin{bmatrix} \gamma & 0 \end{bmatrix}$.

In the Kalman filter the Kalman gain is $K = PC^T R_2^{-1}$, where $P = P^T \geq 0$ is the solution to the CARE

$$0 = AP + PA^T + NR_1N^T - PC^T R_2^{-1} CP.$$

Set $P = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$

$$\Rightarrow \quad AP = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ p_1 & p_{12} \end{bmatrix}, \quad PC^T = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_{12} \\ p_2 \end{bmatrix}.$$

We have that $N = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and $R_1 = R_2 = 1$, so the CARE is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ p_1 & p_{12} \end{bmatrix} + \begin{bmatrix} 0 & p_1 \\ 0 & p_{12} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12}p_2 \\ p_{12}p_2 & p_2^2 \end{bmatrix}.$$

Element by element this constitutes the equation system

$$\begin{cases} 0 = -p_{12}^2, \\ 0 = p_1 - p_{12}p_2, \\ 0 = 2p_{12} + 1 - p_2^2, \end{cases} \quad \Leftrightarrow \quad \begin{cases} p_1 = 0, \\ p_{12} = 0, \\ p_2 = 1, \end{cases} \quad \Leftrightarrow \quad P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The Kalman gain is then $K = PC^T R_2^{-1} = P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_{12} \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The controller is

$$\begin{aligned} \dot{\hat{x}} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(y - \begin{bmatrix} 0 & 1 \end{bmatrix} \hat{x} \right) = \begin{bmatrix} -\gamma & 0 \\ 1 & -1 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y, \\ u &= -\begin{bmatrix} \gamma & 0 \end{bmatrix} \hat{x}. \end{aligned}$$

This means that $u = -F_y(p)y$ with $F_y(p) = 0$, i.e. no feedback! Try to figure out why![2]

5. (a) $E\tilde{x}(k|k-1)\tilde{x}^T(k|k-1) = P$ which (for $R_{12} = 0$) solves the DARE

$$P = FPF^T + NR_1N^T - FPH^T(HPH^T + R_2)^{-1}HPF^T.$$

Here $F = N = H = R_1 = 1$ and $R_2 = 2$, so the DARE becomes

$$P = P + 1 - \frac{P^2}{P+2} \quad \Leftrightarrow \quad P^2 - P - 2 = 0 \quad P = 0.5 \pm 1.5.$$

The negative root is omitted since $P \geq 0$ must hold, and thus $P = 2$.
(b) The Kalman filter is

$$\hat{x}(k+1|k) = F\hat{x}(k|k-1) + K(y(k) - H\hat{x}(k|k-1)) = (F - KH)\hat{x}(k|k-1) + Ky(k),$$

[2]This is a pathological case — the conditions in Theorem 9.1 are not completely fulfilled.

where the Kalman gain is $K = FPH^T(HPH^T + R_2)^{-1}$. Thus

$$\hat{x}(k|k-1) = \underbrace{(qI - F + KH)^{-1}K}_{=F(q)} y(k).$$

Here $K = \frac{P}{P+2} = 0.5$, and $F - KH = 0.5$, so the filter/transfer function is $F(q) = \frac{0.5}{q-0.5}$.

(c) We have $\nu(k) = y(k) - H\hat{x}(k|k-1) = Hx(k) + v_2(k) - H\hat{x}(k|k-1) = H\tilde{x}(k|k-1) + v_2(k)$. The covariance is then

$$\begin{aligned} E\nu(k)\nu^T(k) &= E\left[(H\tilde{x}(k|k-1) + v_2(k))(H\tilde{x}(k|k-1) + v_2(k))^T\right] \\ &= HE\tilde{x}(k|k-1)\tilde{x}^T(k|k-1)H^T + HE\tilde{x}(k|k-1)v_2^T(k) + \\ &\quad + Ev_2(k)\tilde{x}^T(k|k-1)H^T + Ev_2(k)v_2^T(k) = HPH^T + R_2 \geq R_2, \end{aligned}$$

where the last equality follows from that $E\tilde{x}(k|k-1)\tilde{x}^T(k|k-1) = P$, $Ev_2(k)v_2^T(k) = R_2$, and that $E\tilde{x}(k|k-1)v_2^T(k) = 0$.[3]

6. (a) For systems with one output the observer canonical form applies. See that the elements of $G(q)$ have the same denominator: the least common denominator is $q(q-0.5)(q-1) = q^3 - 1.5q^2 + 0.5q$. Rewrite the transfer function:

$$G(q) = \begin{bmatrix} \frac{(q+1)(q-0.5)}{q(q-0.5)(q-1)} & \frac{q^2}{q(q-0.5)(q-1)} \end{bmatrix} = \begin{bmatrix} \frac{q^2+0.5q-0.5}{q^3-1.5q^2+0.5q} & \frac{q^2}{q^3-1.5q^2+0.5q} \end{bmatrix}$$

The observer canonical form then is

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1.5 & 1 & 0 \\ -0.5 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 1 \\ 0.5 & 0 \\ -5 & 0 \end{bmatrix} u(k), \\ y(k) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(k). \end{aligned}$$

(b) A minimal realization is both controllable and observable. The suggested state space representation is on observer canonical form, and is thus observable. Controllability is checked by investigation of the rank of the controllability matrix,

$$\mathcal{S} = \begin{bmatrix} G & FG & F^2G \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 1.5 & 2 & 1.75 \\ 0.5 & 0 & -1 & -0.5 & -1 & -0.75 \\ -0.5 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly $\mathcal{S}$ has full rank (e.g. the three first columns are linearly independent). The state space representation is both observable and controllable, and is therefore a minimal realization.

[3] $\tilde{x}(k|k-1)$ and $v_2(k)$ are uncorrelated.