

# Exam in Automatic Control II

## Reglerteknik II 5hp

**Date:** January 12, 2016

**Venue:** Bergsbrunnagatan 15, room 2

**Responsible teacher:** Hans Norlander.

**Aiding material:** Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

**Preliminary grades:** 23p for grade 3, 33p for grade 4, 43p for grade 5.

**Use separate sheets** for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

**Important:** Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

**Problem 6** is an alternative to the homework assignments. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Good luck!

**Problem 1** The velocity of a moving object is modelled as

$$Y(s) = \frac{1}{s+1}U(s). \quad (1)$$

The system is to be controlled by a zero-order-hold (ZOH) sampling controller. That is, the input is piece-wise constant between the sampling instants, so

$$u(t) = u(kh) \quad \text{for} \quad kh \leq t < kh + h,$$

where  $h$  is the sampling interval.

(a) In order to analyze the closed loop system a discrete-time model of the system is needed. Show that the ZOH sampled model of (1) is

$$y(kh) = \frac{\beta}{q - \alpha}u(kh),$$

and give the values of  $\alpha$  and  $\beta$  (expressed in  $h$ ). (2p)

(b) The proportional feedback

$$u(kh) = 3(r(kh) - y(kh))$$

is used, where  $r$  is the reference signal. Determine for which sampling intervals  $h > 0$  the closed loop system is stable. (4p)

(c) Assume that the proportional control in (b) is used, and that the sampling interval is chosen so that the closed loop system is stable. Determine the final value of the output,  $\lim_{k \rightarrow \infty} y(kh)$ , when  $r(kh)$  is a unit step. (2p)

**Problem 2** Specify for each of the following statements whether it is true or false. No motivations required — only answers “true”/“false” are considered!

(a) A Kalman filter is always stable.

(b) For a Kalman filter the *innovation*,  $\nu = y - C\hat{x} - Du$ , is white noise if the model and the noise intensities are correct.

(c) The discrete-time system  $y(k) = \frac{0.25}{q+0.25}u(k)$  has unit static gain.

(d) The scalar system  $x(k+1) = 0.5x(k) + u(k) + \nu(k)$ ,  $y(k) = x(k) + \nu(k)$ , ( $\nu(k)$  is white noise) is on *innovations form*.

(e) The scalar system  $x(k+1) = -0.5x(k) + u(k) + \nu(k)$ ,  $y(k) = x(k) + \nu(k)$ , ( $\nu(k)$  is white noise) is on *innovations form*.

Each correct answer scores +1, each incorrect answer scores -1, and omitted answers score 0 points. (Minimal total score is 0 points.) (5p)

**Problem 3** A continuous-time system has the state space representation

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_1(t), \\ y(t) &= [1 \ 0] x(t) + v_2(t),\end{aligned}$$

where  $v_1$  and  $v_2$  are uncorrelated, zero mean white noise processes with intensities  $\Phi_{v_1}(\omega) = R_1 = 48$  and  $\Phi_{v_2}(\omega) = R_2 = 1$ .

(a) In order to estimate the state vector, the observer

$$\dot{\hat{x}}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 2 \\ 5 \end{bmatrix} (y(t) - [1 \ 0] \hat{x}(t))$$

is used. What are the observer poles? (1p)

(b) Determine the covariance matrix,  $\Pi_{\tilde{x}} = E\tilde{x}(t)\tilde{x}^T(t)$ , of the estimation error,  $\tilde{x}(t) = x(t) - \hat{x}(t)$ , for the observer in (a). (3p)

(c) Determine the spectrum,  $\Phi_\nu(\omega)$ , for the output innovations,

$$\nu(t) = y(t) - [1 \ 0] \hat{x}(t) = [1 \ 0] \tilde{x}(t) + v_2(t),$$

of the observer in (a).

*Hint:* First derive the transfer operators/functions from  $v_1$  and  $v_2$  to  $\nu$ . (3p)

(d) The optimal observer is the Kalman filter. What is the covariance matrix of the estimation error  $\tilde{x}(t)$  for the Kalman filter of the system?

*Hint:* In the solution of the associated Riccati equation the lower right diagonal element is  $p_2 = 16$ . (4p)

(e) What is the spectrum,  $\Phi_\nu(\omega)$ , for the output innovations of the Kalman filter? (2p)

**Problem 4** For a certain choice of sampling interval the ZOH sampled model of the system in Problem 1 is

$$\begin{cases} z(k) = \frac{0.2}{q - 0.8} (u(k) + v_1(k)), \\ y(k) = z(k) + v_2(k). \end{cases} \quad (2)$$

$v_1$  and  $v_2$  are uncorrelated, zero mean white noise processes with variances  $E v_1^2 = 1.8$  and  $E v_2^2 = 0.2$ . The system should be controlled by the control law  $u(k) = -F_y(q)y(k)$ , where  $F_y(q)$  should be *strictly proper*, meaning that  $u(k)$  depends on  $y(k-1), y(k-2), \dots$ , but not on  $y(k)$ . The controller should minimize the criterion

$$V = E [z^2(k) + \rho u^2(k)], \quad \rho = \frac{1}{9}.$$

(a) Give a state space representation for the system described by (2). (2p)

(b) Determine the controller  $F_y(q)$  that minimizes  $V$ . (5p)

(c) Determine the closed loop poles when  $F_y(q)$  in (b) is used. (3p)

**Problem 5** Consider the continuous-time system<sup>1</sup>

$$y(t) = \frac{4}{p+4} \left( \frac{1}{p} u_1(t) + u_2(t) + w(t) \right).$$

Here  $w(t)$  is a stochastic process that can be modeled as

$$w(t) = \frac{p+2}{p^2+3p+5} v(t),$$

where  $v(t)$  is zero mean white noise with intensity  $\Phi_v(\omega) = R_v$ .

**(a)** Give a state space representation for the system with  $y$  as output,  $u = [u_1 \ u_2]^T$  as input and  $v$  as system noise. Let  $x_1 = y$ .

*Hint:* First determine one state space representation with  $y$  as output and  $u$  and  $w$  as inputs, and another with  $w$  as output and  $v$  as input. Then combine these two into one total state space representation. **(5p)**

**(b)** Is your state space representation in (a) controllable from  $u$ ? **(2p)**

**Problem 6** *The HW bonus points are exchangeable for this problem.*

A stationary discrete-time stochastic process  $w(k)$  has the spectrum

$$\Phi_w(\omega) = \frac{5 + 4 \cos \omega}{2.25 - 2(\cos \omega)^2} = \frac{5 + 4 \cos \omega}{1.25 - \cos 2\omega}.$$

Find a transfer operator  $G(q)$  that is stable, minimum phase and has positive static gain, and is such that

$$w(k) = G(q)v(k), \quad \Phi_v(\omega) = 1,$$

is a model of  $w(k)$ .

*Hint:* Note that  $(q - z_1)(q - z_2) = q^2 - (z_1 + z_2)q + z_1 z_2 = q^2 + a_1 q + a_2$ , and that  $|z_1| < 1$  and  $|z_2| < 1 \Rightarrow |a_2| = |z_1 z_2| = |z_1| \cdot |z_2| < 1$ . Thus  $|a_2| < 1$  is a *necessary* condition for the polynomial  $q^2 + a_1 q + a_2$  to have both its zeros inside the unit circle. **(7p)**

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<sup>1</sup>Here  $p$  denotes the differentiation operator, i.e.  $p = \frac{d}{dt}$ .

**Solutions to the exam in Automatic Control II, 2016-01-12:**

1. (a) Set up a state space representation and use Theorem 4.1:

$$\begin{cases} \dot{x} = -x + u, \\ y = x, \end{cases} \Rightarrow \begin{cases} x(kh + h) = e^{-h}x(kh) + (1 - e^{-h})u(kh), \\ y(kh) = x(kh). \end{cases}$$

Thus,

$$y(kh) = H(qI - F)^{-1}Gu(kh) = \frac{1 - e^{-h}}{q - e^{-h}}u(kh) \Rightarrow \alpha = e^{-h}, \quad \beta = 1 - e^{-h}.$$

(b) The closed loop system is

$$y(kh) = \frac{G_o(q)}{1 + G_o(q)}r(kh) = \frac{3\frac{\beta}{q-\alpha}}{1 + 3\frac{\beta}{q-\alpha}}r(kh) = \frac{3\beta}{q - \alpha + 3\beta}r(kh),$$

whose pole is  $\alpha - 3\beta = 4e^{-h} - 3$ . For stability the pole must be inside the unit circle:

$$-1 < 4e^{-h} - 3 < 1 \Leftrightarrow 2 < 4e^{-h} < 4 \Leftrightarrow 0.5 < e^{-h} < 1.$$

Since  $e^{-h} < 1$  for all  $h > 0$  it is  $0.5 < e^{-h}$  that will restrict  $h$ :

$$0.5 < e^{-h} \Leftrightarrow \log 0.5 < -h \Leftrightarrow h < \log 2 \approx 0.693.$$

(c) Use the final value theorem:

$$\lim_{k \rightarrow \infty} y(kh) = \lim_{z \rightarrow 1} (z - 1)Y(z) = \lim_{z \rightarrow 1} (z - 1)G_c(z)R(z) = G_c(1),$$

since  $R(z) = \frac{z}{z-1}$  (a unit step). From (a) we have that

$$G_c(z) = \frac{3\beta}{z - \alpha + 3\beta} \Rightarrow G_c(1) = \frac{3\beta}{1 - \alpha + 3\beta} = \frac{3 - 3e^{-h}}{4 - 4e^{-h}} = \frac{3}{4}.$$

2. (a) True (Lemma 5.1); (b) True (Theorem 5.5); (c) False (static gain is obtained for  $q = 1 \Rightarrow$  the static gain is here 0.2); (d) True (innovations form if  $v_1 = v_2 = \nu$  and  $F - NH$  stable, here  $F - NH = -0.5$ ); (e) False (here  $F - NH = -1.5$ , i.e. unstable);

3. (a) The observer poles are given by

$$\begin{aligned} 0 = \det(sI - A + KC) &= \det \left( \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} s + 3 & -1 \\ 5 & s + 1 \end{bmatrix} = s^2 + 4s + 8. \end{aligned}$$

Hence, the observer poles are  $-2 \pm i2$ .

(b) The estimation error is governed by the state equation

$$\dot{\tilde{x}} = (A - KC)\tilde{x} + Nv_1 - Kv_2, \quad (3)$$

and  $\Pi_{\tilde{x}}$  solves the continuous-time Lyapunov equation

$$0 = (A - KC)\Pi_{\tilde{x}} + \Pi_{\tilde{x}}(A - KC)^T + NR_1N^T + KR_2K^T.$$

By noting that

$$\begin{aligned} A - KC &= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -5 & -1 \end{bmatrix} \\ \Rightarrow (A - KC)\Pi_{\tilde{x}} &= \begin{bmatrix} -3 & 1 \\ -5 & -1 \end{bmatrix} \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} = \begin{bmatrix} -3\pi_1 + \pi_{12} & -3\pi_{12} + \pi_2 \\ -5\pi_1 - \pi_{12} & -5\pi_{12} - \pi_2 \end{bmatrix} \end{aligned}$$

the Lyapunov equation can be spelled out as

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -3\pi_1 + \pi_{12} & -3\pi_{12} + \pi_2 \\ -5\pi_1 - \pi_{12} & -5\pi_{12} - \pi_2 \end{bmatrix} + \begin{bmatrix} -3\pi_1 + \pi_{12} & -5\pi_1 - \pi_{12} \\ -3\pi_{12} + \pi_2 & -5\pi_{12} - \pi_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 48 \end{bmatrix} + \begin{bmatrix} 4 & 10 \\ 10 & 25 \end{bmatrix}.$$

Element by element this gives the linear equation system

$$\begin{cases} 0 = -6\pi_1 + 2\pi_{12} + 4, \\ 0 = -5\pi_1 - 4\pi_{12} + \pi_2 + 10, \\ 0 = -10\pi_{12} - 2\pi_2 + 73, \end{cases} \Leftrightarrow \begin{cases} \pi_1 = \frac{129}{64} = 2.015625, \\ \pi_{12} = \frac{259}{64} = 4.046875, \\ \pi_2 = \frac{1041}{64} = 16.265625. \end{cases}$$

(c) We note that  $\nu = y - C + \hat{x} = C\tilde{x} + v_2$ , which together with (3) gives

$$\begin{cases} \dot{\tilde{x}} = (A - KC)\tilde{x} + Nv_1 - Kv_2, \\ \nu = C\tilde{x} + v_2, \end{cases}$$

as state space representation for  $\nu$ . Thus,

$$\begin{aligned} \nu(t) &= \underbrace{C(pI - A + KC)^{-1}N}_{=G_1(p)} v_1(t) + \underbrace{[1 - C(pI - A + KC)^{-1}K]}_{=G_2(p)} v_2(t) \\ &= [G_1(p) \quad G_2(p)] \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}. \end{aligned}$$

The spectrum of  $\nu$  then is (in accordance with Eq. (5.15))

$$\Phi_\nu(\omega) = [G_1(i\omega) \quad G_2(i\omega)] \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} G_1(-i\omega) \\ G_2(-i\omega) \end{bmatrix} = |G_1(i\omega)|^2 R_1 + |G_2(i\omega)|^2 R_2.$$

Since  $C(pI - A + KC)^{-1}$  is part of both  $G_1(p)$  and  $G_2(p)$  we compute that first:

$$C(pI - A + KC)^{-1} = [1 \quad 0] \begin{bmatrix} p+3 & -1 \\ 5 & p+1 \end{bmatrix}^{-1} = \frac{1}{p^2 + 4p + 8} [p+1 \quad 1].$$

Then

$$G_1(p) = \frac{1}{p^2 + 4p + 8} \begin{bmatrix} p+1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{p^2 + 4p + 8}$$

$$\Rightarrow |G_1(i\omega)|^2 = \frac{1}{(8 - \omega^2)^2 + (4\omega)^2} = \frac{1}{\omega^4 + 64},$$

and

$$G_2(p) = 1 - \frac{1}{p^2 + 4p + 8} \begin{bmatrix} p+1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 1 - \frac{2p+7}{p^2 + 4p + 8}$$

$$= \frac{p^2 + 2p + 1}{p^2 + 4p + 8} = \frac{(p+1)^2}{p^2 + 4p + 8} \quad \Rightarrow \quad |G_2(i\omega)|^2 = \frac{(\omega^2 + 1)^2}{\omega^4 + 64}.$$

The spectrum is then

$$\Phi_\nu(\omega) = 48|G_1(i\omega)|^2 + |G_2(i\omega)|^2 = \frac{48 + (\omega^2 + 1)^2}{\omega^4 + 64} = 1 + \frac{2\omega^2 - 15}{\omega^4 + 64}.$$

(d) The covariance of the estimation error for the Kalman filter,  $E\tilde{x}\tilde{x}^T = P$ , is the solution of the CARE

$$0 = AP + PA^T + NR_1N^T - PC^TR_2^{-1}CP.$$

Set  $P = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} \Rightarrow$  the CARE spells out as

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} + \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} + 48 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$- \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}.$$

Element by element this gives the equation system

$$\begin{cases} 0 = -2p_1 + 2p_{12} - p_1^2, \\ 0 = -2p_{12} + p_2 - p_1p_{12}, \\ 0 = -2p_2 + 48 - p_{12}^2, \end{cases}$$

Since we know that  $p_2 = 16$  we get

$$\begin{cases} 0 = -2p_1 + 2p_{12} - p_1^2, \\ 0 = -2p_{12} + 16 - p_1p_{12} = 16 - p_{12}(2 + p_1), \\ 0 = -32 + 48 - p_{12}^2 = 16 - p_{12}^2. \end{cases}$$

The last equation gives that  $p_{12} = \pm 4$ , and since  $p_1 \geq 0$  must hold ( $\Leftarrow P > 0$ ), the second equation gives that  $p_{12} > 0$ . Thus,

$$\begin{cases} p_1 = 2, \\ p_{12} = 4, \\ p_2 = 16, \end{cases} \quad \Leftrightarrow \quad P = \begin{bmatrix} 2 & 4 \\ 4 & 16 \end{bmatrix}.$$

(e) Theorem 5.5  $\Rightarrow \Phi_\nu(\omega) = R_2 = 1$ .

4. (a) With  $x = z$  the state space representation becomes

$$\begin{cases} x(k+1) = 0.8x(k) + 0.2u(k) + 0.2v_1(k), \\ z(k) = x(k), \\ y(k) = x(k) + v_2(k). \end{cases}$$

(b) The minimizing controller is the LQG controller  $u(k) = -L\hat{x}(k|k-1)$ , where  $\hat{x}(k|k-1)$  is given by the Kalman filter,

$$\hat{x}(k+1|k) = F\hat{x}(k|k-1) + Gu(k) + K(y(k) - H\hat{x}(k|k-1)).$$

(With  $u(k) = -L\hat{x}(k|k)$  the input depends on  $y(k)$ , and then  $F_y(q)$  is not strictly proper.) The Kalman gain  $K$  is given by

$$K = FPH^T(HPH^T + R_2)^{-1}, \quad P = FPF^T + NR_1N^T - FPH^T(HPH^T + R_2)^{-1}HPF^T.$$

Here  $F = 0.8$ ,  $N = 0.2$ ,  $R_1 = 1.8$ ,  $H = 1$ ,  $R_2 = 0.2$ , so the DARE becomes

$$P = 0.8^2P + 0.2^2 \cdot 1.8 - \frac{0.8^2P^2}{P + 0.2} \quad \Leftrightarrow \quad P^2 = 0.12^2 \quad \Rightarrow \quad P = 0.12,$$

and

$$K = \frac{0.8P}{P + 0.2} = \frac{0.8 \cdot 0.12}{0.32} = 0.3.$$

The feedback gain  $L$  is given by

$$L = (G^T SG + Q_2)^{-1} G^T SF, \quad S = F^T SF + M^T Q_1 M - F^T SG (G^T SG + Q_2)^{-1} G^T SF.$$

Here we we have  $F = 0.8$ ,  $G = 0.2$ ,  $M = 1$ ,  $Q_1 = 1$ ,  $Q_2 = \rho = \frac{1}{9}$ , so the DARE becomes

$$S = 0.8^2 S + 1 - \frac{0.8^2 \cdot 0.2^2 P^2}{0.2^2 P + 1/9} \quad \Leftrightarrow \quad S^2 = \frac{25}{9} \quad \Rightarrow \quad S = \frac{5}{3},$$

and

$$L = \frac{0.2 \cdot 0.8 S}{0.2^2 S + 1/9} = \frac{0.16 \cdot \frac{5}{3}}{0.04 \cdot \frac{5}{3} + \frac{1}{9}} = 1.5.$$

The controller then is

$$\begin{aligned} & \begin{cases} q\hat{x} = F\hat{x} + Gu + K(y - H\hat{x}) = (F - GL - KH)\hat{x} + Ky, \\ u = -L\hat{x}, \end{cases} \\ \Rightarrow & \begin{cases} q\hat{x} = (0.8 - 0.2 \cdot 1.5 - 0.3)\hat{x} + 0.3y = 0.2\hat{x} + 0.3y, \\ u = -1.5\hat{x}. \end{cases} \end{aligned}$$

Then

$$F_y(q) = L(qI - F + GL + KH)^{-1}K = \frac{1.5 \cdot 0.3}{q - 0.8 + 0.2 \cdot 1.5 + 0.3} = \frac{0.45}{q - 0.2}.$$



(c) The closed loop poles are given by

$$0 = \det(zI - F + GL) \det(zI - F + KH) = (z - 0.8 + 0.2 \cdot 1.5)(z - 0.8 + 0.3) = (z - 0.5)^2,$$

i.e., a double pole in 0.5. (The closed loop poles can also be determined by  $0 = 1 + G(z)F_y(z)$ .)

5. (a) With  $x_1 = y$  we have  $(p + 4)x_1 = 4(\frac{1}{p}u_1 + u_2 + w) \Leftrightarrow \dot{x}_1 = -4x_1 + 4\frac{1}{p}u_1 + 4u_2 + 4w$ . Now set (for example)  $x_2 = \frac{1}{p}u_1 \Rightarrow \dot{x}_2 = u_1$ . Using e.g. the controller canonical form for describing  $w$  we get

$$\begin{cases} \dot{x}_3 = -3x_3 - 5x_4 + v, \\ \dot{x}_4 = x_3, \\ w = x_3 + 2x_4, \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = -4x_1 + 4x_2 + 4x_3 + 8x_4 + 4u_2, \\ \dot{x}_2 = u_1, \\ \dot{x}_3 = -3x_3 - 5x_4 + v, \\ \dot{x}_4 = x_3, \\ y = x_1. \end{cases}$$

In vector form this is

$$\begin{cases} \dot{x} = \begin{bmatrix} -4 & 4 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 4 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} v, \\ y = [1 \ 0 \ 0 \ 0] x. \end{cases}$$

(b) To examine the controllability, compute the controllability matrix:

$$\mathcal{S} = [B \ AB \ A^2B \ A^3B] = \begin{bmatrix} 0 & 4 & 4 & -16 & -16 & 64 & 64 & -256 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\mathcal{S}$  has rank two, so the system is not controllable.

6. The modelled spectrum is  $G(e^{i\omega})\Phi_v(\omega)G(e^{-i\omega}) = |G(e^{i\omega})|^2$  (since  $\Phi_v(\omega) = 1$ ). The powers of  $\cos \omega$  in the given spectrum suggest that

$$G(q) = \frac{b_1q + b_2}{q^2 + a_1q + a_2},$$

for some  $a_1, a_2, b_1, b_2 \in \mathbb{R}$ . Stability and minimum phase mean that the poles and zeros must be inside the unit circle (zeros on the unit circle are allowed). This means that  $|a_2| < 1$  and  $|b_2| < |b_1|$ . Furthermore, the static gain must be positive, meaning that  $G(1) = \frac{b_1+b_2}{1+a_1+a_2} > 0$ . The modelled spectrum

becomes

$$\begin{aligned}
G(e^{i\omega})G(e^{-i\omega}) &= \frac{(b_1 e^{i\omega} + b_2)(b_1 e^{-i\omega} + b_2)}{(e^{i2\omega} + a_1 e^{i\omega} + a_2)(e^{-i2\omega} + a_1 e^{-i\omega} + a_2)} \\
&= \frac{b_1^2 + b_2^2 + b_1 b_2 (e^{i\omega} + e^{-i\omega})}{1 + a_1^2 + a_2^2 + a_1(1 + a_2)(e^{i\omega} + e^{-i\omega}) + a_2(e^{i2\omega} + e^{-i2\omega})} \\
&= \frac{b_1^2 + b_2^2 + 2b_1 b_2 \cos \omega}{1 + a_1^2 + a_2^2 + 2a_1(1 + a_2) \cos \omega + 2a_2 \cos 2\omega} \\
&= \frac{(b_1^2 + b_2^2 + 2b_1 b_2 \cos \omega)/2a_2}{\frac{1+a_1^2+a_2^2}{2a_2} + \frac{2a_1(1+a_2)}{2a_2} \cos \omega + \cos 2\omega}.
\end{aligned}$$

Compare with the given spectrum:

$$\Phi_w(\omega) = \frac{5 + 4 \cos \omega}{1.25 - \cos 2\omega} = \frac{-(5 + 4 \cos \omega)}{-1.25 + \cos 2\omega}.$$

The denominator:

$$\frac{1 + a_1^2 + a_2^2}{2a_2} = -1.25, \quad \text{and} \quad \frac{2a_1(1 + a_2)}{2a_2} = 0$$

The second equation gives that  $a_1 = 0$  or  $a_2 = -1$ , but the latter can be excluded since  $|a_2| < 1$ . Thus  $a_1 = 0$ , which when put into the first equation gives  $a_2^2 + 2.5a_2 + 1 = 0 \Leftrightarrow a_2 = -1.25 \pm -0.75$ , and again  $|a_2| < 1 \Rightarrow a_2 = -0.5$ . The numerator then is  $-(b_1^2 + b_2^2 + 2b_1 b_2 \cos \omega)$ , which when compared when the given spectrum gives

$$b_1^2 + b_2^2 = 5, \quad \text{and} \quad 2b_1 b_2 = 4.$$

This is solved by  $b_{1,2} = \pm 1.5 \pm 0.5$ . The second equation implies that  $b_1$  and  $b_2$  have the same sign, and since  $G(1) = \frac{b_1 + b_2}{1 + a_1 + a_2} = \frac{b_1 + b_2}{0.5} > 0$  must hold both  $b_1$  and  $b_2$  must be positive. Furthermore,  $|b_2| < |b_1| \Rightarrow b_1 = 2$  and  $b_2 = 1$ . Hence  $G(q) = \frac{2q+1}{q^2-0.5}$ .