Exam in Automatic Control II
Reglerteknik II 5hp

Date: October 21, 2015

Venue: Bergsbrunnagatan 15, room 2

Responsible teacher: Hans Norlander.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Please use English in your solutions when possible, that would be appreciated!

Good luck!

Problem 1 A discrete-time system has the state space representation

$$\begin{cases} x(k+1) = 0.8x(k) + u(k) + v_1(k), \\ z(k) = x(k), \\ y(k) = x(k) + v_2(k), \end{cases}$$

where $v_1(k)$ and $v_2(k)$ are two uncorrelated, zero mean, white noise processes with variances $Ev_1^2(k) = 0.36$ and $Ev_2^2(k) = 1$. The following cost function should be minimized:

$$V = Ez^2(k). \tag{1}$$

In a first attempt to minimize V the proportional control

$$u(k) = -\alpha y(k) \tag{2}$$

is used.

(a) For which $\alpha \in \mathbb{R}$ is the closed loop system stable, when the feedback (2) is used? **(1p)**

(b) Determine V in (1) when the proportional control (2) is used. Your answer should be an expression in α. **(2p)**

(c) Show that $V = 0.6$ is the smallest possible value of the cost function (1) when (2) is used. **(3p)**

(d) Determine the optimal controller[1], that minimizes (1). **(5p)**

Problem 2 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true" / "false" are considered!

(a) White noise is typically modelled with the spectrum $\Phi(\omega) = \frac{\omega^2}{\omega^4+4}$.

(b) $\Phi(\omega) = \frac{3.2}{1.64-2\cos\omega}$ is the spectrum of a discrete-time stochastic process.

(c) An MPC controller can only be implemented as a continuous-time LTI controller.

(d) In MPC the *prediction/output horizon* is normally much longer than the *control/input horizon.*

(e) In MPC the computational load increases with the *control/input horizon.*

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(5p)**

[1] Not restricted to proportional controllers, but the optimal of *all* (linear) controllers.

Problem 3

(a) A double tank system (like in the MPC lab) has the state space representation

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \end{cases}$$

where $u(t)$ represents the inflow into the upper tank, $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$ represents the water levels in the upper (x_1) and the lower (x_2) tanks. In order to design a sampling controller a discrete-time model,

$$\begin{cases} x(kT+T) = Fx(kT) + Gu(kT), \\ y(kT) = Hx(kT), \end{cases}$$

is needed. The discrete-time model is obtained by use of zero-order-hold (ZOH) sampling, i.e. $u(t) = u(kT)$ for $kT \le t < kT + T$. Determine F, G and H for the ZOH sampled model of the double tank system when the sampling interval is $T = 0.5$ time units. **(4p)**

(b) A continuous-time system has the state equation $\dot{x}(t) = Ax(t) + Bu(t)$. Two discrete-time models of the system are obtained by ZOH sampling. The first sampled model is $x(kT+T) = F_h x(kT) + G_h u(kT)$, where the sampling interval is $T = h > 0$. The second sampled model is $x(kT+T) = F_{2h} x(kT) + G_{2h} u(kT)$, where the sampling period is $T = 2h$. Show that the following relations hold:

$$F_{2h} = F_h^2, \qquad G_{2h} = (I + F_h)G_h.$$

(2p)

(c) Determine the transfer operator, $G_{zoh}(q)$, for the ZOH sampled version of the system $y(t) = \frac{1}{p+1} u(t - 0.15)$ when the sampling interval is $T = 0.2$ time units. **(2p)**

Problem 4

(a) Give a state space representation for the system

$$y(k) = \begin{bmatrix} \frac{q+1}{q(q-1)} & \frac{q}{(q-0.5)(q-1)} \end{bmatrix} u(k), \quad u(k) = \begin{bmatrix} u_1(k) & u_2(k) \end{bmatrix}^T.$$

(4p)

(b) Is your state space representation in (a) a minimal realization? **(2p)**

Problem 5 A continuous-time system has the state space representation

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t), \\ z(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t), \\ y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) + e(t). \end{cases} \tag{3}$$

The process noise, $v(t)$, and the measurement noise, $e(t)$, are uncorrelated, zero mean white noise processes with intensities $\Phi_v(\omega) = 9$ and $\Phi_e(\omega) = 4$.

(a) Determine the spectral density, $\Phi_y(\omega)$, of the measured output $y(t)$ when the input is $u(t) = 0$. **(3p)**

(b) Determine the variance, $Ez^2(t)$, of the performance variable $z(t)$ when $u(t) = 0$. **(3p)**

(c) A Kalman filter is used in order to estimate the state vector in (3). Show that the solution of the associated Riccati equation has the form

$$P = \begin{bmatrix} 0 & 0 \\ 0 & p_2 \end{bmatrix}.$$

Then solve for p_2 and determine the Kalman filter. **(5p)**

(d) The *output innovations*, given as $\nu(t) = y(t) - C\hat{x}(t)$, are inputs to the Kalman filter. Determine the spectral density, $\Phi_\nu(\omega)$, of the output innovations for the Kalman filter in (c). **(2p)**

Problem 6 *The HW bonus points are exchangeable for this problem.*
Consider again the continuous-time system (3), in Problem 5. Assume that the input $u(t)$ in (3) is obtained as

$$u(t) = G_u(p)w(t), \qquad \text{with} \quad Ew(t) = 0 \quad \text{and} \quad \Phi_w(\omega) = 1,$$

where $G_u(p)$ is stable and minimum phase, and $G_u(0) > 0$. The spectral density of $u(t)$ is

$$\Phi_u(\omega) = \frac{4\omega^2 + 2.25}{\omega^4 - 2\omega^2 + 9}.$$

(a) Determine $G_u(p)$. **(3p)**

(b) Give a state space representation on "standard form" for the system (3) when $u(t)$ is generated as in (a). That is, determine the matrices and vectors $\bar{A}$, $\bar{N}$, $\bar{M}$ and $\bar{C}$ in an augmented state space model

$$\begin{cases} \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{N}v_1(t), \\ z(t) = \bar{M}\bar{x}(t), \\ y(t) = \bar{C}\bar{x}(t) + v_2(t), \end{cases} \qquad v_1(t) = \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}, \; v_2(t) = e(t),$$

where $\bar{x}(t)$ is an augmented state vector. **(4p)**

Solutions to the exam in Automatic Control II, 2015-10-21:

1. (a) The closed loop system is

$$x(k+1) = 0.8x(k) - \alpha(x(k)+v_2(k)) + v_1(k) = (0.8-\alpha)x(k) + v_1(k) - \alpha v_2(k).$$

The pole is in $0.8 - \alpha$, and for stability $|0.8 - \alpha| < 1$ must hold, that is

$$-1 < \alpha - 0.8 < 1 \quad \Leftrightarrow \quad -0.2 < \alpha < 1.8.$$

(b) We note that $V = Ex^2(k) = \Pi_x$ solves the discrete-time Lyapunov equation

$$\Pi_x = (0.8-\alpha)^2\Pi_x + 0.36 + \alpha^2 \quad \Leftrightarrow \quad \Pi_x = \frac{0.36+\alpha^2}{1-(0.8-\alpha)^2} = \frac{0.36+\alpha^2}{0.36+1.6\alpha-\alpha^2}.$$

(c) Note that the denominator in Π_x above is

$$-(\alpha^2 - 1.6\alpha - 0.36) = -(\alpha+0.2)(\alpha-1.8).$$

This means that $\Pi_x > 0$ exactly for the stabilizing α in (a), $\Pi_x < 0$ for α outside that interval, and $\Pi_x \to +\infty$ when α approaches the boundaries from inside the interval. To find the minimum, set $\frac{d\Pi_x}{d\alpha} = 0$. We have

$$\frac{d\Pi_x}{d\alpha} = \frac{2\alpha(0.36+1.6\alpha-\alpha^2) - (0.36+\alpha^2)(-2\alpha+1.6)}{(0.36+1.6-\alpha^2)^2},$$

and

$$0 = 2\alpha(0.36+1.6\alpha-\alpha^2) - (0.36+\alpha^2)(-2\alpha+1.6) = 1.6(\alpha^2+0.9\alpha-0.36).$$

Thus, $\frac{d\Pi_x}{d\alpha} = 0$ for $\alpha = -0.45 \pm 0.75$, and since stability is required $\alpha = 0.3$. Then

$$\Pi_x = \frac{0.36+0.3^2}{1-(0.8-0.3)^2} = \frac{0.45}{0.75} = 0.6.$$

(d) The optimal controller is the LQG controller, with $u(k) = -L\hat{x}(k|k)$ (i.e. with direct/feedthrough term). The feedback gain is $L = (G^TSG + Q_2)^{-1}G^TSF$, where $S = S^T \geq 0$ solves the DARE $S = F^TSF + M^TQ_1M - F^TSG(G^TSG+Q_2)^{-1}G^TSF$ Here

$$F = 0.8, \quad G = 1, \quad M = 1, \quad Q_1 = 1 \quad \text{and} \quad Q_2 = 0,$$

and the DARE becomes

$$S = 0.8^2S + 1 - \frac{0.8^2S}{S} = 1 \quad \Rightarrow \quad L = \frac{0.8S}{S} = 0.8.$$

(Hence, it is not necessary to solve the DARE in this case.) The Kalman filter is

$$\begin{aligned}\hat{x}(k+1|k) &= F\hat{x}(k|k-1) + Gu(k) + K[y(k) - H\hat{x}(k|k-1)],\\ \hat{x}(k|k) &= \hat{x}(k|k-1) + \tilde{K}[y(k) - H\hat{x}(k|k-1)],\end{aligned}$$

where $K = FPH^T(HPH^T + R_2)^{-1}$ and $\tilde{K} = PH^T(HPH^T + R_2)^{-1}$, and where $P = P^T \geq 0$ solves the DARE

$$P = FPF^T + NR_1N^T - FPH^T(HPH^T + R_2)^{-1}HPF^T.$$

Here

$$F = 0.8, \quad N = 1, \quad H = 1, \quad R_1 = 0.36 \quad \text{and} \quad R_2 = 1,$$

and the DARE becomes

$$P = 0.8^2P + 0.36 - \frac{0.8^2P^2}{P+1} \quad \Leftrightarrow \quad P^2 = 0.36.$$

Thus, $P = 0.6$, $K = \frac{0.8P}{P+1} = \frac{0.8 \cdot 0.6}{1.6} = 0.3$ and $\tilde{K} = \frac{P}{P+1} = \frac{0.6}{1.6} = 0.375$. (Here the optimal controller turns out to be $u(k) = -0.3y(k)$, i.e. the P-controller in (c).)

2. **(a)** False (White noise has flat/constant spectrum); **(b)** False (A spectrum is always non-negative — here $\Phi(0) < 0!$); **(c)** False; **(d)** True; **(e)** True (The control horizon decides the dimension of the optimization problem)

3. **(a)** Theorem 4.1 $\Rightarrow$ $F = e^{AT}$, $G = \int_0^T e^{At}Bdt$ and $H = C$. To compute e^{At}, exploit the Laplace transform: $\mathcal{L}\left[e^{At}\right] = (sI - A)^{-1}$. Here we have

$$(sI-A)^{-1} = \begin{bmatrix} s+1 & 0 \\ -1 & s+1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{(s+1)^2} & \frac{1}{s+1} \end{bmatrix} \quad \Leftrightarrow \quad e^{At} = \begin{bmatrix} e^{-t} & 0 \\ te^{-t} & e^{-t} \end{bmatrix}.$$

Thus

$$F = \begin{bmatrix} e^{-0.5} & 0 \\ 0.5e^{-0.5} & e^{-0.5} \end{bmatrix}, \quad G = \int_0^{0.5} \begin{bmatrix} e^{-t} \\ te^{-t} \end{bmatrix} dt = \begin{bmatrix} 1 - e^{-0.5} \\ 1 - 1.5e^{-0.5} \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

(b) First we note that $F_h = e^{Ah}$ and $G_h = \int_0^h e^{At}Bdt$. Then

$$F_{2h} = e^{A2h} = (e^{Ah})^2 = F_h^2,$$

and

$$\begin{aligned} G_{2h} = \int_0^{2h} e^{At}Bdt &= \int_0^h e^{At}Bdt + \int_h^{2h} e^{At}Bdt = G_h + \int_0^h e^{A(h+t)}Bdt \\ &= G_h + e^{Ah}\int_0^h e^{At}Bdt = G_h + F_hG_h = (I + F_h)G_h. \end{aligned}$$

(c) The solution of the state equation is

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau,$$

so from one sampling instant, $t_0 = kT$, to the next, $t = kT + T$, we have

$$x(kT+T) = e^{AT}x(kT) + \int_{kT}^{kT+T} e^{A(kT+T-\tau)}Bu(\tau)d\tau.$$

ZOH means that $u(t)$ is piecewise constant, and thus can be moved outside of the integral. Normally $u(t)$ is constant during the whole interval, $t \in [kT, kT+T)$. However, if $u(t)$ changes from $u(t) = u_1$ to $u(t) = u_2$ at time $t_1 \in [kT, kT+T)$, we get

$$x(kT+T) = e^{AT}x(kT) + \int_{kT}^{t_1} e^{A(kT+T-\tau)}Bd\tau u_1 + \int_{t_1}^{kT+T} e^{A(kT+T-\tau)}Bd\tau u_2.$$

Here we can represent the system as

$$\begin{cases} \dot{x} = -x + \bar{u}, \\ y = x, \end{cases} \qquad \text{with} \quad \bar{u}(t) = u(t - 0.015).$$

Then, with $u(t) = u(kT)$ for $t \in [kT, kT+T)$, we have that $\bar{u}(t) = u(kT-T)$ for $t < kT+0.15$ (and $t \geq kT-T+0.15$) and $\bar{u}(t) = u(kT)$ for $t \geq kT+0.15$ (and $t < kT+T+0.15$). So here we can set $t_1 = kT+0.15$, and $u_1 = u(kT-T)$ and $u_2 = u(kT)$. Thus,

$$\begin{aligned} x(kT+T) &= e^{-T}x(kT) + \int_{kT}^{kT+0.15} e^{-(kT+T-\tau)}d\tau \cdot u(kT-T) + \int_{kT+0.15}^{kT+T} e^{-(kT+T-\tau)}d\tau \cdot u(kT) \\ &= e^{-T}x(kT) + (e^{0.15-T} - e^{-T})u(kT-T) + (1 - e^{0.15-T})u(kT) \\ &= e^{-0.2}x(kT) + (e^{-0.05} - e^{-0.2})u(kT-T) + (1 - e^{-0.05})u(kT). \end{aligned}$$

With $\alpha = e^{-0.2}$, $\beta = e^{-0.05}$ and the shift operator q we now can write

$$\begin{aligned} qx(kT) &= \alpha x(kT) + (\beta - \alpha)q^{-1}u(kT) + (1-\beta)u(kT) \\ \Leftrightarrow \quad q(q-\alpha)x(kT) &= [(1-\beta)q + \beta - \alpha]u(kT) \\ \Leftrightarrow \quad y(kT) = x(kT) &= \underbrace{\frac{(1-\beta)q + \beta - \alpha}{q(q-\alpha)}}_{=G_{zoh}(q)} u(kT) \end{aligned}$$

4. (a) For systems with one output the observer canonical form applies. See that the elements of $G(q)$ have the same denominator: the least common denominator is $q(q-0.5)(q-1) = q^3 - 1.5q^2 + 0.5q$. Rewrite the transfer function:

$$G(q) = \begin{bmatrix} \frac{(q+1)(q-0.5)}{q(q-0.5)(q-1)} & \frac{q^2}{q(q-0.5)(q-1)} \end{bmatrix} = \begin{bmatrix} \frac{q^2+0.5q-0.5}{q^3-1.5q^2+0.5q} & \frac{q^2}{q^3-1.5q^2+0.5q} \end{bmatrix}$$

The observer canonical form then is

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1.5 & 1 & 0 \\ -0.5 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 1 \\ 0.5 & 0 \\ -5 & 0 \end{bmatrix} u(k), \\ y(k) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(k). \end{aligned}$$

(b) A minimal realization is both controllable and observable. The suggested state space representation is on observer canonical form, and is thus observable. Controllability is checked by investigation of the rank of the controllability matrix,

$$\mathcal{S} = \begin{bmatrix} G & FG & F^2G \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 1.5 & 2 & 1.75 \\ 0.5 & 0 & -1 & -0.5 & -1 & -0.75 \\ -0.5 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly $\mathcal{S}$ has full rank (e.g. the three first columns are linearly independent). The state space representation is both observable and controllable, and is therefore a minimal realization.

5. (a) For a state space model

$$\begin{cases} \dot{x} = Ax + Bu + Nv, \\ z = Mx, \\ y = Cx + e, \end{cases}$$

with $u = 0$, we get

$$y = \underbrace{C(pI - A)^{-1}N}_{=G_v(p)} v + e = \underbrace{\begin{bmatrix} G_v(p) & 1 \end{bmatrix}}_{=\bar{G}(p)} \underbrace{\begin{bmatrix} v \\ e \end{bmatrix}}_{=\mu}.$$

Then

$$\Phi_y(\omega) = \bar{G}(i\omega)\Phi_\mu(\omega)\bar{G}^*(i\omega), \quad \text{where} \quad \Phi_\mu(\omega) = \begin{bmatrix} \Phi_v(\omega) & 0 \\ 0 & \Phi_e(\omega) \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}.$$

Since $\bar{G}^*(i\omega) = \bar{G}^T(-i\omega)$ we get

$$\Phi_y(\omega) = \begin{bmatrix} G_v(i\omega) & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} G_v(-i\omega) \\ 1 \end{bmatrix} = G_v(i\omega)G_v(-i\omega) \cdot 9 + 4.$$

Here we have

$$G_v(p) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} p+1 & 0 \\ 0 & p+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{p+2} \quad \Rightarrow$$

$$\Phi_y(\omega) = \frac{1}{i\omega + 2} \cdot \frac{1}{-i\omega + 2} \cdot 9 + 4 = \frac{9}{\omega^2 + 4} + 4 = \frac{4\omega^2 + 25}{\omega^2 + 4}.$$

(b) Note that

$$Ez^2 = E\{Mxx^TM^T\} = MExx^TM^T = M\Pi_x M^T,$$

where Π_x solves the continuous-time Lyapunov equation $0 = A\Pi_x + \Pi_x A^T + NR_1N^T$. Here we get

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\pi_1 & -\pi_{12} \\ -2\pi_{12} & -2\pi_2 \end{bmatrix} + \begin{bmatrix} -\pi_1 & -2\pi_{12} \\ -\pi_{12} & -2\pi_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix} \quad \Leftrightarrow \quad \Pi_x = \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2.25 \end{bmatrix},$$

and thus $Ez^2 = M\Pi_x M^T = \pi_1 + 2\pi_{12} + \pi_2 = 2.25$.
(c) The Kalman filter is $\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$, where $K = PC^T R_2^{-1}$, and $P = P^T \geq 0$ solves the CARE $0 = AP + PA^T + NR_1N^T - PC^T R_2^{-1} CP$. Here

$$P = \begin{bmatrix} 0 & 0 \\ 0 & p_2 \end{bmatrix} \quad \Rightarrow \quad AP = \begin{bmatrix} 0 & 0 \\ 0 & -2p_2 \end{bmatrix}, \; PC^T = \begin{bmatrix} 0 \\ p_2 \end{bmatrix}, \; NR_1N^T = \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix},$$

so the CARE becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -2p_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -2p_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \frac{p_2^2}{4} \end{bmatrix}$$

There are only non-zero elements in the 2–2-element, which reads

$$0 = -4p_2 + 9 - \frac{p_2^2}{4} \quad \Leftrightarrow \quad 0 = p_2^2 + 16p_2 - 36 \quad \Leftrightarrow \quad p_2 = -8 \pm \sqrt{8^2 + 36} = 2.$$

where the negative root is omitted. The Kalman gain then is $K = PC^T R_2^{-1} = \frac{1}{4}\begin{bmatrix} 0 & p_2 \end{bmatrix}^T = \begin{bmatrix} 0 & 0.5 \end{bmatrix}^T$.
(d) Theorem 5.5 $\Rightarrow$ $\nu(t)$ is white noise with intensity R_2. Hence, $\Phi_\nu(\omega) = R_2 = 4$.

6. **(a)** The spectral density is $\Phi_u(\omega) = |G_u(i\omega)|^2 \Phi_w(\omega)$, and based on the powers of ω^2 we try with

$$G_u(p) = \frac{b_1 p + b_2}{p^2 + a_1 p + a_2} \quad \Rightarrow \quad \Phi_u(\omega) = |G_u(i\omega)|^2 \cdot 1 =$$
$$= \frac{ib_1\omega + b_2}{a_2 - \omega^2 + ia_1\omega} \cdot \frac{-ib_1\omega + b_2}{a_2 - \omega^2 - ia_1\omega} = \frac{b_1^2\omega^2 + b_2^2}{\omega^4 + (a_1^2 - 2a_2)\omega^2 + a_2^2}$$

Stability $\Leftrightarrow$ $a_1, a_2 > 0$, $G_u(0) > 0 \Rightarrow b_2 > 0$, and then minimum phase $\Rightarrow$ $b_1 > 0$. Comparing the expression above with the given spectrum gives the following equation system

$$\begin{cases} b_1^2 = 4, \\ b_2^2 = 2.25, \\ a_1^2 - 2a_2 = -2, \\ a_2^2 = 9, \end{cases} \quad \Leftrightarrow \quad \begin{cases} b_1 = 2, \\ b_2 = 1.5, \\ a_1 = 2, \\ a_2 = 3 \end{cases} \quad \Rightarrow \quad G_u(p) = \frac{2p + 1.5}{p^2 + 2p + 4}.$$

(b) By use of eg. controller canonical form $u(t) = G_u(p)w(t)$ can be represented as

$$\begin{cases} \dot{x}_u = A_u x_u + N_u w, \\ u = C_u x_u, \end{cases} \qquad A_u = \begin{bmatrix} -2 & -4 \\ 1 & 0 \end{bmatrix}, \; N_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; C_u = \begin{bmatrix} 2 & 1.5 \end{bmatrix}.$$

Together with the state space representation in Problem 5 we then have

$$\begin{cases} \dot{x}_u = A_u x_u + N_u w, \\ \dot{x} = Ax + Bu + Nv = BC_u x_u + Ax + Nv, \\ z = Mx, \\ y = Cx + e, \end{cases} \Leftrightarrow \begin{cases} \dot{\bar{x}} = \begin{bmatrix} A_u & 0 \\ BC_u & A \end{bmatrix} \bar{x} + \begin{bmatrix} N_u & 0 \\ 0 & N \end{bmatrix} v_1, \\ z = \begin{bmatrix} 0 & M \end{bmatrix} \bar{x}, \\ y = \begin{bmatrix} 0 & C \end{bmatrix} \bar{x} + v_2. \end{cases}$$

That is

$$\bar{A} = \begin{bmatrix} -2 & -4 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1.5 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \quad \bar{N} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{M} = \bar{C} = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}.$$