Exam in Automatic Control II
Reglerteknik II 5hp (1RT495)

Date: August 24, 2018

Venue: Bergsbrunnagatan 15 sal 2

Responsible teacher: Hans Rosth.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments from the spring semester 2018. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Good luck!

Problem 1

(a) A continuous-time system has the state space representation

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} a \\ b \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t). \end{cases} \tag{1}$$

Show that (1) is a minimal realisation for all values of a and b, except for $a = b = 0$. **(2p)**

(b) For which sampling periods $h > 0$ is the *zero-order-hold* sampled, discrete-time version of (1) observable? **(2p)**

(c) A discrete-time system is controlled by proportional feedback:

$$y(k) = \frac{q+2}{q(q-1)}u(k), \qquad u(k) = K(r(k) - y(k)).$$

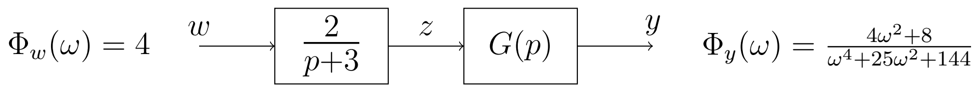
For which $K \in \mathbb{R}$ is the closed loop system stable? **(2p)**

(d) When designing a sampling controller for a continuous-time system you can either perform the design in continuous time or in discrete time. In the former case you obtain a continuous-time controller, which then must be discretized so that it can be implemented as a sampling controller. There are several possible ways to perform this discretization. One way is to use Tustin's approximation, in which the time derivative is approximated in the following way:

$$p \approx \frac{2}{h}\frac{q-1}{q+1}. \tag{2}$$

Here $h > 0$ is the sampling period, p is the differentiation operator (i.e. $py(t) = \dot{y}(t)$) and q is the forward shift operator (i.e. $qy(kh) = y(kh + h)$). Show that Tustin's approximation (2) is equivalent to the trapezoidal rule for numerical integration. **(2p)**

Problem 2 The block diagram below represents a stationary continuous-time stochastic process. The transfer operator $G(p)$ is minimum phase (with $G(0) \geq 0$), and w is zero mean white noise.



(a) Determine the spectrum for z. **(2p)**

(b) Determine the variance, Ez^2, of z. **(2p)**

(c) Determine the transfer operator $G(p)$. **(4p)**

Problem 3 A stock market analyst has the following model for the daily variations of the price for a certain share:

$$\begin{cases} x(k+1) = x(k) + w(k), & Ew(k) = 0, & \Phi_w(\omega) = 2, \\ y(k) = x(k) + e(k), & Ee(k) = 0, & \Phi_e(\omega) = 4. \end{cases} \tag{3}$$

(a) Initially it is assumed that w and e are uncorrelated. Determine the Kalman filter that gives the optimal prediction $\hat{x}(k+1|k)$ for the model (3) under this assumption. **(3p)**

(b) What is the covariance of the estimation error $\tilde{x} = x - \hat{x}$ for the Kalman filter of (3) under the assumption in (a)? That is, determine $E\tilde{x}^2$. **(1p)**

(c) A deeper analysis of the stock market reveals that w and e are *not* uncorrelated, but that

$$w(k) = v(k) - 0.5e(k), \qquad \text{where} \quad Ev(k) = 0, \quad \Phi_v(\omega) = 1, \tag{4}$$

and $v(k)$ and $e(k)$ are uncorrelated. (For $e(k)$ (3) still holds.) Assume that the Kalman filter in (a) still would be used (as an observer), what would $E\tilde{x}^2$ then be for the model (3)? **(3p)**

(d) What is the smallest possible value of $E\tilde{x}^2$ given the model (3), and under the circumstances given in (c)? **(3p)**

Problem 4 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true"/"false" are considered!

(a) A Kalman filter is an observer.

(b) For a Kalman filter, based on a correct model, the *output innovations* are white noise.

(c) MPC is typically implemented as PID controllers.

(d) In MPC the *prediction/output horizon* is typically longer than the *control/input horizon*.

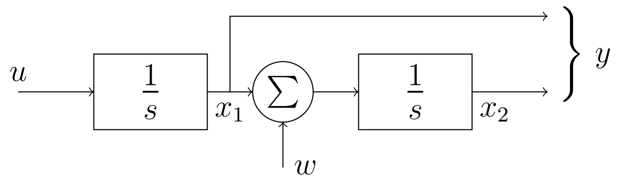
(e) An advantage with MPC is that it can account for bounds and constraints, for example of the type $|u| \leq U_{max}$.

(f) Stability is always preserved under zero-order-hold sampling.

(g) Observability is always preserved under zero-order-hold sampling.

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(7p)**

Problem 5 The block diagram below shows a version of a continuous-time double integrator.



A state space model of the system, with state vector $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, is

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t), \\ y(t) = x(t). \end{cases} \tag{5}$$

Here w is a process disturbance. The double integrator should be stabilized, and since the full state vector is measured (with no measurement noise) pure state feedback can be used, ie. the control law $u(t) = -Lx(t)$ can be applied.

(a) Initially it was assumed that the process disturbance is negligible, and it was set to $w \equiv 0$ so that the model (5) became purely deterministic. The feeback gain L was then computed by solving the LQ problem where the criterion function

$$V_a = \int_0^\infty \left\{ y^T(t) Q_1 y(t) + u^2(t) \right\} dt, \quad Q_1 \in \mathbb{R}^{2\times 2},$$

is minimized. It turned out that the solution of the associated Riccati equation is

$$S = \begin{bmatrix} 8 & 25 \\ 25 & 199 \end{bmatrix}.$$

How was the weighting matrix Q_1 chosen? Determine the value of Q_1 (a 2×2 matrix) such that S above solves the associated Riccati equation. **(3p)**

(b) Determine the poles of the closed loop system when the LQ controller in (a) is used. **(2p)**

(c) Later it turned out that the process disturbance is *not* negligible, but that it can be modeled as

$$w(t) = \frac{2}{p + 0.9} v(t), \quad Ev(t) = 0, \quad \Phi_v(\omega) = 1. \tag{6}$$

Combine (5) and (6) into a state space model on "standard form", with u as input, y as output and $\bar{x} = \begin{bmatrix} x_1 & x_2 & w \end{bmatrix}^T$ as state vector. **(3p)**

(d) It also turned out that w can be measured (without measurement noise), so that the pure state feedback $u(t) = -\bar{L}\bar{x}(t)$ is applicable. State the *equations* needed to find the feedback gain $\bar{L} \in \mathbb{R}^{1\times 3}$ that minimizes

$$V_d = E\left\{ y^T(t) Q_1 y(t) + u^2(t) \right\}, \quad \text{with the same } Q_1 \text{ as in (a).}$$

(You need/should *not* solve this LQ problem.) **(2p)**

Problem 6 *The HW bonus points (from the spring 2018) are exchangeable for this problem.*
A discrete-time system with one input and two outputs is described by the difference equations

$$\begin{cases} y_1(k+1) + 0.2y_1(k) - y_2(k) = 2u(k+1) - u(k), \\ y_2(k+1) - 0.7y_2(k) + y_1(k) = 2u(k). \end{cases}$$

(a) Give the transfer operator $G(q)$ in the following model of the system:

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = G(q)u(k).$$

(3p)

(b) Give a state space representation for the system. **(3p)**

(c) Is your state space model in (b) observable from y_2? **(1p)**

Solutions to the exam in Automatic Control II, 2018-08-24:

1. (a) We notice that (1) is on oberver canonical form, and hence observable. Check for controllability:

$$\mathcal{S} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} a & -2a+b \\ b & -2a \end{bmatrix}, \quad \det \mathcal{S} = -2a^2+2ab-b^2 = -(a-b)^2-a^2 \neq 0$$

for all a and b except for $a = b = 0$. Both controllable and observable $\Leftrightarrow$ minimal realisation.

(b) Theorem 4.1 $\Rightarrow$ the ZOH sampled system is $qx = Fx + Gu$, $y = Cx$, where $F = e^{Ah}$ and $G = \int_0^h e^{At} B dt$. To check for observability we need F and C. Use the Laplace transform to compute e^{At}: $e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$. We get

$$(sI - A)^{-1} = \begin{bmatrix} s+2 & -1 \\ 2 & s \end{bmatrix} = \begin{bmatrix} \frac{(s+1)-1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-2}{(s+1)^2+1} & \frac{(s+1)+1}{(s+1)^2+1} \end{bmatrix},$$

and by inverse Laplace transformation

$$e^{At} = \begin{bmatrix} e^{-t}(\cos t - \sin t) & e^{-t}\sin t \\ -2e^{-t}\sin t & e^{-t}(\cos t + \sin t) \end{bmatrix}.$$

Set $t = h$ to get F, and plug it into the observability matrix :

$$\mathcal{O} = \begin{bmatrix} C \\ CF \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ e^{-h}(\cos h - \sin h) & e^{-h}\sin h \end{bmatrix} \quad \Rightarrow \quad \det \mathcal{O} = e^{-h}\sin h.$$

Thus, the ZOH sampled system is *not* observable for $\sin h = 0$, i.e. it is observable for $h \neq n\pi$, $n = 1, 2, \ldots$.

(c) The closed loop poles are given by

$$0 = 1+KG(q) = 1+K\frac{q+2}{q(q-1)} \quad \Leftrightarrow \quad 0 = q(q-1)+K(q+2) = q^2+(K-1)q+2K.$$

The zeros of the polynomial $z^2 + \alpha z + \beta$ lies inside the unit circle if and only if $|\alpha| - 1 < \beta < 1$:

$$\begin{aligned} \beta < 1: &\quad 2K < 1 &\Leftrightarrow&\quad K < 0.5, \\ \alpha - 1 < \beta: &\quad K-1-1 < 2K &\Leftrightarrow&\quad -2 < K, \\ -\alpha - 1 < \beta: &\quad -K+1-1 < 2K &\Leftrightarrow&\quad 0 < K. \end{aligned}$$

Hence, the closed loop system is stable for $0 < K < 0.5$.

(d) Let $D(kh)$ denote the Tustin approximation the time derivative of $y(kh)$ (i.e. $D(kh) \approx \dot{y}(kh)$). Then

$$D(kh) = \frac{2(q-1)}{h(q+1)}y(kh) \quad \Leftrightarrow \quad h(D(kh+h)+D(kh)) = 2(y(kh+h)-y(kh)).$$

This is a difference equation that shows how $D(kh+h)$ can be computed (recursively) from $D(kh)$, $y(kh)$ and $y(kh+h)$, where the latter two can be regarded as inputs. Now observe that integration is the inverse operation of differentiation. Thus, if we instead regard $D(t)$ as input and $y(t)$ as output, we can see $y(t)$ as the integral of $D(t)$. Particulary, in the time interval $kh \leq t \leq kh+h$ we can see the difference $y(kh+h) - y(kh)$ as the integral $\int_{kh}^{kh+h} D(t)dt$. Numerical integration by use of the trapezoid rule then gives

$$\int_{kh}^{kh+h} D(t)dt \approx \frac{D(kh)+D(kh+h)}{2} \cdot (kh+h-kh) = \frac{h}{2}(D(kh)+D(kh+h)).$$

By equating this with the difference above we get

$$y(kh+h) - y(kh) = \frac{h}{2}(D(kh)+D(kh+h)),$$

which is exactly the same difference equation as the one for Tustin's approximation.

2. (a) We have

$$\Phi_z(\omega) = \left|\frac{2}{i\omega+3}\right|^2 \Phi_w(\omega) = \frac{2}{i\omega+3} \cdot \frac{2}{-i\omega+3} \cdot 4 = \frac{16}{\omega^2+9}.$$

(b) Set up a state space representation for z:

$$z = \frac{2}{p+3}w \quad \Leftrightarrow \quad \dot{z} = -3z + 2w.$$

Now $\Pi_z = Ez^2$ solves the continuous-time Lyapunov equation,

$$0 = A\Pi_z + \Pi_z A^T + NR_wN^T = 2\cdot(-3)\Pi_z + 2^2\cdot 4 = -6\Pi_z + 16 \quad \Leftrightarrow \quad \Pi_z = \frac{16}{6} = \frac{8}{3}.$$

(c) Again we can use that $\Phi_y(\omega) = |G(i\omega)|^2\Phi_z(\omega)$, and from (a) we have $\Phi_z(\omega)$. Thus,

$$\begin{aligned}|G(i\omega)|^2 = \frac{\Phi_y(\omega)}{\Phi_z(\omega)} = \frac{\frac{4\omega^2+8}{\omega^4+25\omega^2+144}}{\frac{16}{\omega^2+9}} &= \frac{(4\omega^2+8)(\omega^2+9)}{16(\omega^4+25\omega^2+144)} \\ &= \frac{(4\omega^2+8)(\omega^2+9)}{16(\omega^2+9)(\omega^2+16)} = \frac{0.25\omega^2+0.5}{\omega^2+16}.\end{aligned}$$

Based on the degrees of ω^2 in numerator and denominator we try with

$$G(p) = \frac{b_1p+b_2}{p+a} \quad \Rightarrow \quad |G(i\omega)|^2 = \frac{(ib_1\omega+b_2)(-ib_1\omega+b_2)}{(i\omega+a)(-i\omega+a)} = \frac{b_1^2\omega^2+b_2^2}{\omega^2+a^2}.$$

Comparison with the expression above gives

$$\begin{cases} b_1^2 = 0.25, \\ b_2^2 = 0.5, \\ a^2 = 16, \end{cases} \quad \Rightarrow \quad \begin{cases} b_1 = 0.5, \\ b_2 = \sqrt{0.5}, \\ a = 4, \end{cases} \quad \Rightarrow \quad G(p) = \frac{0.5p+\sqrt{0.5}}{p+4}.$$

The negative roots are omitted since stationarity $\Leftrightarrow$ stability $\Leftrightarrow a > 0$, minimum phase and $G(0) \geq 0 \Leftrightarrow b_1, b_2 \geq 0$.

3. **(a)** Theorem 5.6: The Kalman filter is $q\hat{x} = F\hat{x} + Gu + K(y - H\hat{x})$ with $K = (FPH^T + NR_{12})(HPH^T + R_2)^{-1}$, where $P = P^T \geq 0$ solves the DARE $P = FPF^T + NR_1N^T - (FPH^T + NR_{12})(HPH^T + R_2)^{-1}(FPH^T + NR_{12})^T$. Here $F = N = H = 1$, $R_1 = 2$, $R_2 = 4$ and $R_{12} = 0$. The DARE then is

$$P = P + 2 - \frac{P^2}{P+4} \quad \Leftrightarrow \quad P^2 - 2P - 8 = 0 \quad \Leftrightarrow \quad P = 1 + \sqrt{1+8} = 4.$$

Thus, $K = \frac{P}{P+R_2} = \frac{4}{4+4} = 0.5$ and the Kalman filter is

$$q\hat{x} = \hat{x} + 0.5(y - \hat{x}) = 0.5\hat{x} + 0.5y.$$

(b) Theorem 5.6 $\Rightarrow E\tilde{x}\tilde{x}^T = P = 4$.
(c) For any observer $q\tilde{x} = (F - KH)\tilde{x} + Nv_1 - Kv_2$ holds. Then $E\tilde{x}\tilde{x}^T = \Pi_{\tilde{x}}$ solves the discrete-time Lyapunov equation $\Pi_{\tilde{x}} = (F-KH)\Pi_{\tilde{x}}(F-KH)^T + \begin{bmatrix} N & -K \end{bmatrix} R_\nu \begin{bmatrix} N & -K \end{bmatrix}^T$, where $R_\nu = E\nu\nu^T$, $\nu = \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}^T$. Here we have $F - KH = 0.5$, $N = 1$, $K = 0.5$ and $w = v - 0.5e$, so

$$q\tilde{x} = 0.5\tilde{x} + w - 0.5e = 0.5\tilde{x} + v - 0.5e - 0.5e = 0.5\tilde{x} + v - e,$$

and the Lyapunov equation becomes (since v and e are uncorrelated)

$$\Pi_{\tilde{x}} = 0.5^2\Pi_{\tilde{x}} + 1 + 4 \quad \Leftrightarrow \quad 0.75\Pi_{\tilde{x}} = 5 \quad \Leftrightarrow \quad \Pi_{\tilde{x}} = \frac{20}{3} = 6\frac{2}{3}.$$

(d) The minimal covariance is obtained for the Kalman filter and, as stated in (a) and (b), is the P that solves the DARE. Here the circumstances differ from those in (a): We still have $F = N = H = 1$ and $R_2 = Ee^2 = 4$. However,

$$R_1 = Ew^2 = E(v - 0.5e)^2 = Ev^2 - Eve + 0.5^2Ee^2 = 1 + 0.25 \cdot 4 = 2,$$
$$R_{12} = Ewe = E(v - 0.5e)e = Eve - 0.5Ee^2 = -0.5 \cdot 4 = -2.$$

The DARE becomes

$$P = P + 2 - \frac{(P-2)^2}{P+4} \quad \Leftrightarrow \quad P^2 - 6P - 4 = 0 \quad \Leftrightarrow \quad P = 3 + \sqrt{9+4} = 3 + \sqrt{13} \approx 6.606.$$

4. **(a)** True; **(b)** True (Theorem 5.5); **(c)** False (MPC requires numerical optimization); **(d)** True; **(e)** True (This is handled by the numerical optimization); **(f)** True (The left half plane is mapped onto the unit disc); **(g)** False (For some systems observability is lost for certain sampling intervals);

5. **(a)** The associated Riccati equation for the LQ problem is the CARE $0 = A^TS + SA + M^TQ_1M - SBQ_2^{-1}B^TS$. Here we notice that $z = y = x \Leftrightarrow$

$M = I$, and that $B = \begin{bmatrix}1 & 0\end{bmatrix}^T$ and $Q_2 = 1$. Thus we get

$$Q_1 = -A^T S - SA + SBB^T S = -\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}8 & 25\\25 & 199\end{bmatrix} - \begin{bmatrix}8 & 25\\25 & 199\end{bmatrix}\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$
$$+\begin{bmatrix}8 & 25\\25 & 199\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}8 & 25\\25 & 199\end{bmatrix} = \begin{bmatrix}14 & 1\\1 & 625\end{bmatrix}.$$

(b) The feedback gain is $L = Q_2^{-1}B^T S = \begin{bmatrix}8 & 25\end{bmatrix}$. The poles are then given by

$$0 = \det(sI - A + BL) = \det\left(\begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + \begin{bmatrix}8 & 25\\0 & 0\end{bmatrix}\right)$$
$$= \det\begin{bmatrix}s+8 & 25\\-1 & s\end{bmatrix} = s^2 + 8s + 25.$$

The poles are $s = -4 \pm \sqrt{4^2 - 25} = -4 \pm i3$.
(c) From (6) we get $pw = -0.9w + 2v$. Combining this with (5) gives

$$\begin{cases} px_1 = u,\\ px_2 = x_1 + w,\\ pw = -0.9w + 2v,\\ y_1 = x_1,\\ y_2 = x_2, \end{cases} \Leftrightarrow \begin{cases} \dot{\bar{x}} = \underbrace{\begin{bmatrix}0 & 0 & 0\\1 & 0 & 1\\0 & 0 & -0.9\end{bmatrix}}_{=\bar{A}}\bar{x} + \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{=\bar{B}}u + \underbrace{\begin{bmatrix}0\\0\\2\end{bmatrix}}_{=\bar{N}}v,\\ y = \underbrace{\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}}_{=\bar{C}}\bar{x}. \end{cases}$$

(d) Now $z = y = \bar{C}\bar{x} \Leftrightarrow M = \bar{C}$. Then $\bar{L} = \bar{B}^T\bar{S}$, where $\bar{S} = \bar{S}^T \geq 0$ solves the CARE

$$0 = \bar{A}^T\bar{S} + \bar{S}\bar{A} + \bar{C}^T Q_1 \bar{C} - \bar{S}\bar{B}\bar{B}^T\bar{S},$$

(since $Q_2 = 1$,) with matrices given in (c). (See Theorem 9.1.)

6. (a) Rewrite the difference equations with the shift operator q:

$$\begin{cases}(q+0.2)y_1 - y_2 = (2q-1)u,\\ (q-0.7)y_2 + y_1 = 2u,\end{cases} \Leftrightarrow \begin{bmatrix}q+0.2 & -1\\1 & q-0.7\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}2q-1\\2\end{bmatrix}u$$

$$\Leftrightarrow$$

$$\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}q+0.2 & -1\\1 & q-0.7\end{bmatrix}^{-1}\begin{bmatrix}2q-1\\2\end{bmatrix}u = \begin{bmatrix}\frac{2q^2-2.4q+2.7}{q^2-0.5q+0.86}\\ \frac{1.4}{q^2-0.5q+0.86}\end{bmatrix}u = \begin{bmatrix}2 + \frac{-1.4q+0.98}{q^2-0.5q+0.86}\\ \frac{1.4}{q^2-0.5q+0.86}\end{bmatrix}u.$$

(b) One input $\Rightarrow$ controller canonical form works. Notice that y_1 requires a direct term:

$$\begin{cases} qx = \begin{bmatrix}0.5 & -0.86\\1 & 0\end{bmatrix}x + \begin{bmatrix}1\\0\end{bmatrix}u,\\ \begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}-1.4 & 0.98\\0 & 1.4\end{bmatrix}x + \begin{bmatrix}2\\0\end{bmatrix}u. \end{cases}$$

(c) Observability from $y_2 \Leftrightarrow$ check whether or not the observability matrix $\mathcal{O}_2 = \begin{bmatrix} H_2 \\ H_2 F \end{bmatrix}$ has full rank:

$$\mathcal{O}_2 = \begin{bmatrix} 0 & 1.4 \\ 1.4 & 0 \end{bmatrix} \quad \text{full rank} \quad \Leftrightarrow \quad \text{observable from } y_2.$$