Exam in Automatic Control II
Reglerteknik II 5hp (1RT495)

Date: October 24, 2018

Venue: Bergsbrunnagatan 15, sal 2

Responsible teacher: Hans Rosth.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments from the autumn semester 2018. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Please use English in your solutions when possible, that would be appreciated!

Good luck!

Please use English in your solutions!

Problem 1

(a) Consider the continuous-time system with state space representation

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t). \end{cases} \tag{1}$$

Give the corresponding discrete-time state space model for (1), obtained with zero-order-hold (ZOH) sampling. The model should be expressed in terms of the sampling period h. **(3p)**

(b) Here (1) represents the harmonic oscillator, with the transfer function $G(s) = \frac{2}{s^2+1}$. Show that it is *not* possible to stabilize (1) with (continuous-time) proportional feedback control, $u(t) = K(r(t) - y(t))$. **(1p)**

(c) In contrast to the case in (b) it *is* possible to stabilize (1) with proportional feedback control when using a sampling controller. For a certain choice of sampling period the ZOH sampled, discrete-time model of (1) becomes

$$y(kh) = \frac{q+1}{q^2 - q + 1} u(kh).$$

For which gains $K \in \mathbb{R}$ is the closed loop system stable, when the proportional feedback control $u(kh) = K(r(kh) - y(kh))$ is used? **(2p)**

(d) Consider a general continuous-time state space system and the corresponding discrete-time model, obtained with ZOH sampling:

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \text{and} \qquad x(kT+T) = F_T x(kT) + G_T u(kT).$$

Here the sampling period is T, and the index in F_T and G_T indicates the dependence on T for the sampled model. Now, assume that $T = h$ gives a sampled model with system matrices F_h and G_h, and in the same way (if doubling the sampling period) $T = 2h$ gives a sampled model with system matrices F_{2h} and G_{2h}. Express F_{2h} and G_{2h} in terms of F_h and G_h. **(2p)**

Please use English in your solutions!

Problem 2 A continuous-time system has the state space representation

$$\begin{cases} \dot{x}(t) = Ax(t) + Nv(t), \\ y(t) = Cx(t) + e(t), \end{cases} \quad \text{where} \quad \begin{array}{ll} Ev(t) = 0, & \Phi_v(\omega) = R_1 = 5, \\ Ee(t) = 0, & \Phi_e(\omega) = R_2 = 1. \end{array} \tag{2}$$

The process noise, v, and the measurement noise, e, are correlated. The cross-spectrum/-intensity, and the model parameters are

$$\Phi_{ve}(\omega) = R_{12} = -1, \qquad A = -2, \quad N = 1, \quad C = 2. \tag{3}$$

(a) Determine the covariance (= variance here) of the state, that is, compute $Ex^2(t)$ in (2). **(2p)**

(b) What is the spectral density, $\Phi_y(\omega)$, of the output y in (2)?
Hint: The system (2) can be represented as $y(t) = \begin{bmatrix} G(p) & 1 \end{bmatrix} \eta(t)$, with $\eta(t) = \begin{bmatrix} v(t) & e(t) \end{bmatrix}^T$. What is then $\Phi_\eta(\omega)$? **(3p)**

(c) Determine the Kalman filter for the system given by (2) and (3). **(4p)**

(d) Assume that due to some kind of malfunction the state has no longer an impact on the measured output. This is manifested in that $C = 0$, but apart from that (2) and (3) still hold. What is the smallest possible covariance, $E\tilde{x}^2(t)$, of the estimation error, $\tilde{x} = x - \hat{x}$, that can be obtained (by use of a Kalman filter) in this case? **(3p)**

Problem 3 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true" / "false" are considered!

(a) A Kalman filter is always stable.

(b) The *output innovations* are white noise for a Kalman filter based on a correct model and correct noise intensities.

(c) The scalar system $x(k+1) = x(k) + u(k) + \nu(k)$, $y(k) = x(k) + \nu(k)$ is on *innovations form.*

(d) The scalar system $x(k+1) = -x(k) + u(k) + \nu(k)$, $y(k) = x(k) + \nu(k)$ is on *innovations form.*

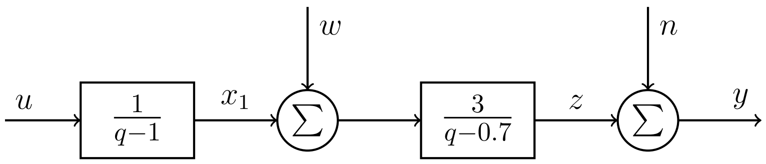
(e) Observability is always preserved under zero-order-hold sampling.

(f) Stability is always preserved under zero-order-hold sampling.

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(6p)**

Please use English in your solutions!

Problem 4 Consider the discrete-time system in the block diagram below:



The process disturbance, w, is described by the model

$$w(k) = \frac{4}{q+0.6}v(k), \qquad Ev(k) = 0, \quad \Phi_v(\omega) = 4,$$

and the measurement disturbance, n, is modeled as

$$\begin{cases} x_n(k+1) = 0.2x_n(k) + 0.3e(k), \\ n(k) = x_n(k) + e(k), \end{cases} \qquad Ee(k) = 0, \quad \Phi_e(\omega) = 1.$$

The white noise processes v and e are uncorrelated.

(a) Give the state space representation, on the "standard form", of the system above with the state vector $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$, where $x_2 = z$, $x_3 = w$ and $x_4 = x_n$. That is, find matrices F, G, N, M and H such that the system is described by

$$\begin{cases} x(k+1) = Fx(k) + Gu(k) + Nv_1(k), \\ z(k) = Mx(k), \\ y(k) = Hx(k) + v_2(k), \end{cases}$$

where v_1 and v_2 are white noise processes. **(4p)**

(b) What are the noise intensities,

$$R_1 = Ev_1(k)v_1^T(k), \quad R_2 = Ev_2(k)v_2^T(k) \quad \text{and} \quad R_{12} = Ev_1(k)v_2^T(k),$$

for v_1 and v_2 in (a)? **(3p)**

Please use English in your solutions!

Problem 5 The discrete-time system with state space representation

$$\begin{cases} x(k+1) = x(k) + u(k) + v(k), & Ev(k) = 0, \quad Ev^2(k) = R_1 = 12, \\ \quad z(k) = x(k), & \\ \quad y(k) = x(k) + e(k), & Ee(k) = 0, \quad Ee^2(k) = R_2 = 45, \end{cases} \tag{4}$$

should be stabilized by feedback control, $u(k) = -F_y(q)y(k)$, so that the criterion

$$V = E\left[Q_1 z^2(k) + u^2(k)\right], \qquad Q_1 > 0, \tag{5}$$

is minimized. The controller, $F_y(q)$, should be strictly proper[1]. Process noise, v, and measurement noise, e, are uncorrelated white noise processes.

(a) Find the controller that minimizes (5). **(5p)**

(b) What are the poles of the closed loop system? **(2p)**

(c) For $Q_1 = 0.5$ the controller is $F_y(q) = \frac{0.2}{q-0.1}$. What is then the value of V in (5)? **(3p)**

Problem 6 *The HW bonus points (from the autumn 2018) are exchangeable for this problem.*

(a) An old, continuous-time control system needs to be replaced (due to malfunction), and the new control system is a sampling controller. The old controller was tuned so that the closed loop pole polynomial was

$$p_c(s) = (s+1)(s^2 + 2s + 2).$$

When tuning the new, sampling controller, pole placement is used for a zero-order-hold sampled model of the system. What *discrete-time* closed loop pole polynomial,

$$p_d(q) = q^3 + aq^2 + bq + c,$$

should one aim for in order to have an equivalent performance with the sampling controller? **(3p)**

(b) The stationary continuous-time stochastic process $y(t)$ is modeled as $y(t) = G(p)u(t)$, where $G(p)$ is minimum phase and $G(0) > 0$. The input $u(t)$ is zero mean white noise with intensity $\Phi_u(\omega) = 1$. The spectrum of $y(t)$ is

$$\Phi_y(\omega) = \frac{16\omega^2 + 4}{\omega^4 - 15\omega^2 + 64}.$$

Determine the transfer operator $G(p)$. **(4p)**

[1] This means that $u(k)$ depends on $y(k-1)$ and older outputs, but not on $y(k)$. It also means that the numerator of $F_y(q)$ must be of lower degree than the denominator.

Solutions to the exam in Automatic Control II, 2018-10-24:

1. (a) Use Theorem 4.1: $F = e^{Ah}$, $G = \int_0^h e^{At} B dt$ and $H = C$. Use the Laplace transform to compute e^{At}:

$$\mathcal{L}[e^{At}] = (sI - A)^{-1} = \begin{bmatrix} s-1 & 1 \\ -2 & s+1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s+1}{s^2+1} & \frac{-1}{s^2+1} \\ \frac{2}{s^2+1} & \frac{s-1}{s^1+1} \end{bmatrix}.$$

Thus,

$$e^{At} = \begin{bmatrix} \cos t + \sin t & -\sin t \\ 2\sin t & \cos t - \sin t \end{bmatrix} \quad \Rightarrow \quad F = \begin{bmatrix} \cos h + \sin h & -\sin h \\ 2\sin h & \cos h - \sin h \end{bmatrix},$$

$$\text{and} \quad G = \int_0^h \begin{bmatrix} \cos t + \sin t \\ 2\sin t \end{bmatrix} dt = \begin{bmatrix} 1 + \sin h - \cos h \\ 2 - 2\cos h \end{bmatrix}.$$

(b) The closed loop poles are given by

$$0 = 1 + KG(s) = 1 + \frac{2K}{s^2+1} \quad \Leftrightarrow \quad 0 = s^2 + 1 + 2K \quad \Leftrightarrow \quad s = \pm\sqrt{-1-2K}.$$

Thus, the poles are either on the imaginary axis, or there is one pole in the right half plane.
(c) Now the closed loop poles are given by

$$0 = 1 + KG(q) = 1 + \frac{K(q+1)}{q^2 - q + 1} \quad \Leftrightarrow \quad 0 = q^2 - q + 1 + K(q+1) = q^2 + (-1+K)q + 1 + K.$$

The roots of $0 = z^2 + \alpha z + \beta$ lies inside the unit circle if and only if $|\alpha| - 1 < \beta < 1$. Here $\alpha = -1 + K$ and $\beta = 1 + K$, so stability depends on the conditions

$$\begin{aligned} \beta < 1: \quad & 1 + K < 1 & \Leftrightarrow \quad & K < 0, \\ \alpha - 1 < \beta: \quad & -1 + K - 1 < 1 + K & \Leftrightarrow \quad & -2 < 1, \\ -\alpha - 1 < \beta: \quad & 1 - K - 1 < 1 + K & \Leftrightarrow \quad & -0.5 < K. \end{aligned}$$

The closed loop system is stable for $-0.5 < K < 0$.
(d) We have

$$\begin{aligned} x(kh+h) = Fx(kh) + Gu(kh) \quad \Rightarrow \quad x(k(2h)+2h) &= Fx(k(2h)+h) + Gu(k(2h)+h) \\ &= F^2 x(k(2h)) + FGu(k(2h)) + Gu(k(2h)+h). \end{aligned}$$

Doubling the sampling period from h to $2h \Rightarrow u(k(2h)+h) = u(k(2h))$, and thus

$$x(k(2h)+2h) = F^2 x(k(2h)) + (F+I)Gu(k(2h)).$$

Hence, $F_{2h} = F_h^2$ and $G_{2h} = (F_h + I)G_h$.

2. **(a)** The covariance $Ex^2 = \Pi_x$ solves the continuous-time Lyapunov equation

$$0 = A\Pi_x + \Pi_x A^T + NR_1N^T = 2(-2)\Pi_x + 5 \quad \Leftrightarrow \quad \Pi_x = 1.25.$$

(b) We have

$$\Phi_y(\omega) = \begin{bmatrix} G(i\omega) & 1 \end{bmatrix} \Phi_\eta(\omega) \begin{bmatrix} G(-i\omega) \\ 1 \end{bmatrix}, \quad \text{and} \quad \Phi_\eta(\omega) = \begin{bmatrix} R_1 & R_{12} \\ R_{12} & R_2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \Phi_y(\omega) = 5G(i\omega)G(-i\omega) - G(i\omega) - G(-i\omega) + 1.$$

Furthermore, $G(p) = C(pI - A)^{-1}N = \frac{2}{p+2}$, so

$$\Phi_y(\omega) = 5\frac{2}{i\omega + 2} \cdot \frac{2}{-i\omega + 2} - \frac{2}{i\omega + 2} - \frac{2}{-i\omega + 2} + 1 = \frac{\omega^2 + 16}{\omega^2 + 4}.$$

(c) The Kalman filter is $\dot{\hat{x}} = A\hat{x} + K(y - C\hat{x})$, with $K = (PC^T + NR_{12})R_2^{-1}$ where $P = P^T \geq 0$ solves the CARE $0 = AP + PA^T + NR_1N^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T$. Here we get

$$0 = -2P - 2P + 5 - (2P - 1)^2 = -4P^2 + 4 \quad \Rightarrow \quad P = 1, \quad \text{and} \quad K = (2P - 1) = 1.$$

(d) The minimal covariance $E\tilde{x}^2 = P$ solves the CARE, now with $C = 0$,

$$0 = AP + PA^T + NR_1N^T - NR_{12}R_2^{-1}R_{12}^TN^T = 2(-2)P + 5 - 1 = -4P + 4.$$

Thus, the smallest possible covariance is $P = 1$.

3. **(a)** True (Lemma 5.1); **(b)** True (Theorem 5.5); **(c)** True ($F - NH = 1 - 1 \cdot 1 = 0$); **(d)** False ($F - NH = -1 - 1 \cdot 1 = -2$); **(e)** False; **(f)** True (see e.g. equation (4.22))

4. **(a)** From the block diagram and the models of w and n we have:

$$\begin{aligned} x_1 &= \frac{1}{q-1}u & &\Leftrightarrow \quad qx_1 = x_1 + u, \\ x_2 &= \frac{3}{q-0.7}(x_1 + x_3) & &\Leftrightarrow \quad qx_2 = 3x_1 + 0.7x_2 + 3x_3, \\ x_3 &= \frac{4}{q+0.6}v & &\Leftrightarrow \quad qx_3 = -0.6x_3 + 4v, \\ & & &\qquad\;\; qx_4 = 0.2x_4 + 0.3e. \end{aligned}$$

Put this together and we get

$$\begin{cases} qx_1 = x_1 + u, \\ qx_2 = 3x_1 + 0.7x_2 + 3x_3, \\ qx_3 = -0.6x_3 + 4v, \\ qx_4 = 0.2x_4 + 0.3v, \\ \quad z = x_2, \\ \quad y = x_2 + x_4 + e, \end{cases} \Leftrightarrow \begin{cases} qx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0.7 & 3 & 0 \\ 0 & 0 & -0.6 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 4 & 0 \\ 0 & 0.3 \end{bmatrix} \begin{bmatrix} v \\ e \end{bmatrix}, \\ z = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x, \\ y = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} x + e. \end{cases}$$

(b) Here we have $v_1 = \begin{bmatrix} v \\ e \end{bmatrix}$ and $v_2 = e \Rightarrow$

$$R_1 = E\begin{bmatrix} v \\ e \end{bmatrix}\begin{bmatrix} v & e \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_2 = Ee^2 = 1, \quad R_{12} = E\begin{bmatrix} v \\ e \end{bmatrix} e = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

5. (a) The optimizing controller is the LQG controller in Theorem 9.4:

$$\begin{gathered} u = -L\hat{x}, \quad \text{with} \quad L = (G^T SG + Q_2)^{-1} G^T SF, \\ S = F^T SF + MQ_1 M - F^T SG(G^T SG + Q_2)^{-1} G^T SF, \quad \text{and} \\ q\hat{x} = F\hat{x} + Gu + K(y - K\hat{x}), \quad \text{with} \quad K = FPH^T(HPH^T + R_2)^{-1}, \\ P = FPF^T + NR_1 N^T - FPH^T(HPH^T + R_2)^{-1} HPF^T. \end{gathered}$$

(Here we used that $R_{12} = 0$.) We have

$$F = G = N = M = H = 1, \quad Q_2 = 1, \quad R_1 = 12, \quad R_2 = 45 \quad \text{and} \quad R_{12} = 0.$$

The LQ-problem: $L = (G^T SG + Q_2)^{-1} G^T SF = \frac{S}{S+1}$ and the DARE is

$$S = S + Q_1 - \frac{S^2}{S+1} \quad \Leftrightarrow \quad S^2 - Q_1 S - Q_1 = 0 \quad \Leftrightarrow \quad S = 0.5(Q_1 + \sqrt{Q_1^2 + 4Q_1}),$$

$$\Rightarrow \quad L = \frac{S}{S+1} = \frac{0.5(Q_1 + \sqrt{Q_1^2 + 4Q_1})}{0.5(Q_1 + \sqrt{Q_1^2 + 4Q_1}) + 1} = 0.5\left(\sqrt{Q_1^2 + 4Q_1} - Q_1\right).$$

The Kalman filter problem: $K = FPH^T(HPH^T + R_2)^{-1} = \frac{P}{P+45}$ and the DARE is

$$P = P + 12 - \frac{P^2}{P+45} \quad \Leftrightarrow \quad P^2 - 12P - 540 = 0 \quad \Leftrightarrow \quad P = 30,$$

$$K = \frac{P}{P+45} = \frac{30}{30+45} = \frac{30}{75} = 0.4.$$

(b) The closed loop poles are those from the state feedback, $0 = \det(qI - F + GL) = q - 1 + L$, and the observer poles, $0 = \det(qI - F + KH) = q - 1 + 0.4 = q - 0.6$. Hence, the poles are in $1 - L = 1 - 0.5\left(\sqrt{Q_1^2 + 4Q_1} - Q_1\right)$ and in $+0.6$.

(c) We need to compute $Ez^2 = \pi_1$ and $Eu^2 = \pi_2$ in order to evaluate $V = Q_1\pi_1 + \pi_2$. The notation here is based on the idea that if we represent the closed loop system in state space form, with state vector $\bar{x} = \begin{bmatrix} x & u \end{bmatrix}^T$, we can compute the covariance matrix for $\bar{x}$ by solving a Lyapunov equation. Then π_1 and π_2 will be the diagonal elements of that covariance matrix $\Pi_{\bar{x}}$. The state equation for x is given by (4). To find a state equation for u we can use

$$u = -F_y(q)y = -\frac{0.2}{q - 0.1}(x + e) \quad \Leftrightarrow \quad qu = -0.2x + 0.1u - 0.2e.$$

Thus,

$$\begin{cases} qx = x + u + v, \\ qu = -0.2x + 0.1u - 0.2e, \end{cases} \quad \Leftrightarrow \quad q\bar{x} = \bar{F}\bar{x} + \bar{N}\eta, \quad \eta = \begin{bmatrix} v \\ e \end{bmatrix}$$

where

$$\bar{F} = \begin{bmatrix} 1 & 1 \\ -0.2 & 0.1 \end{bmatrix}, \quad \bar{N} = \begin{bmatrix} 1 & 0 \\ 0 & -0.2 \end{bmatrix}, \quad R_\eta = E\eta\eta^T = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & 45 \end{bmatrix}.$$

The covariance matrix $\Pi_{\bar{x}} = E\bar{x}\bar{x}^T$ solves the discrete-time Lyapunov equation $\Pi_{\bar{x}} = \bar{F}\Pi_{\bar{x}}\bar{F}^T + \bar{N}R_\eta\bar{N}^T$, which spelled out is

$$\begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -0.2 & 0.1 \end{bmatrix} \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} \begin{bmatrix} 1 & -0.2 \\ 1 & 0.1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -0.2 \end{bmatrix} \begin{bmatrix} 12 & 0 \\ 0 & 45 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -0.2 \end{bmatrix},$$

and gives the equation system

$$\begin{cases} \pi_1 = \pi_1 + 2\pi_{12} + \pi_2 + 12 \\ \pi_{12} = -0.2\pi_1 - 0.1\pi_{12} + 0.1\pi_2 \\ \pi_2 = 0.04\pi_1 - 0.04\pi_{12} + 0.01\pi_2 + 1.8 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \pi_1 = 46 \\ \pi_{12} = -8 \\ \pi_2 = 4 \end{cases}$$

Thus, $V = 0.5\pi_1 + \pi_2 = 0.5 \cdot 46 + 4 = 27$.

6. **(a)** Under ZOH sampling a continuous-time pole p_{ct} is mapped onto $p_{ZOH} = e^{p_{ct}h}$, where h is the sampling period. Here the continuous-time poles are given by $0 = s + 1$ and $0 = s^2 + 2s + 2$. Thus, poles in -1 and $-1 \pm i$. The discrete-time poles then should be in $e^{-h} = \alpha$ and $e^{-h \pm ih} = e^{-h}\cos h \pm ie^{-h}\sin h = \sigma \pm i\omega$, and (by noting that $\sigma^2 + \omega^2 = \alpha^2$)

$$\begin{aligned} p_d(q) = (q - \alpha)(q - \sigma - i\omega)(q - \sigma + i\omega) &= q^3 - (\alpha + 2\sigma)q^2 + \alpha(\alpha - 2\sigma)q - \alpha^3 \\ &q^3 - e^{-h}(1 + 2\cos h)q^2 + e^{-2h}(1 - 2\cos h)q - e^{-3h}. \end{aligned}$$

(b) $G(p)$ stationary and minimum phase $\Leftrightarrow$ all poles in the left half plane and no zeros in the right half plane. The numerator is a first order polynomial, and the denominator is a second order polynomial in ω^2 $\Rightarrow$ try with $G(p) = \frac{b_1p + b_2}{p^2 + a_1p + a_2}$, with $b_1, b_2, a_1, a_2 > 0$ (to also have $G(0) > 0$). The spectrum then is

$$\begin{aligned} \Phi_y(\omega) = G(i\omega)G(-i\omega) &= \frac{(b_2 + ib_1\omega)(b_2 - ib_1\omega)}{(-\omega^2 + a_2 + ia_1\omega)(-\omega^2 + a_2 - ia_1\omega)} \\ &= \frac{b_2^2 + b_1^2\omega^2}{(-\omega^2 + a_2)^2 + a_1^2\omega^2} = \frac{b_2^2 + b_1^2\omega^2}{\omega^4 + (a_1^2 - 2a_2)\omega^2 + a_2^2}. \end{aligned}$$

Comparing with the given spectrum, and equating equal powers of ω^2 in both numerator and denominator, gives

$$\begin{cases} b_1^2 = 16, \\ b_2^2 = 4, \\ a_1^2 - 2a_2 = -15, \\ a_2^2 = 64, \end{cases} \quad \Rightarrow \quad \begin{cases} b_1 = 4, \\ b_2 = 2, \\ a_1 = 1, \\ a_2 = 8. \end{cases}$$

Thus $G(p) = \frac{4p+2}{p^2+p+8}$.