Exam in Automatic Control II
Reglerteknik II 5hp (1RT495)

Date: May 29, 2019

Venue: Fyrislundsgatan 80 sal 1

Responsible teacher: Hans Rosth.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments from the spring semester 2019. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Good luck!

Problem 1 A continuous-time harmonic oscillator can be modeled as

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \end{cases} \quad \Leftrightarrow \quad y(t) = \frac{p}{p^2 + \pi^2} u(t). \tag{1}$$

(a) Show that it is possible to stabilize this harmonic oscillator by use of proportional feedback, $u(t) = K(r(t) - y(t))$. **(1p)**

Now assume that a sampling controller is used, operating with zero-order hold (ZOH) sampling[1].

(b) Give the ZOH sampled, discrete-time state space model for (1), expressed in the sampling period h. **(2p)**

(c) When the sampling period is $h = 0.5$ the transfer function for the discrete-time model in (b) becomes

$$G_{ZOH}(z) = \frac{1}{\pi} \cdot \frac{z-1}{z^2+1}. \tag{2}$$

Compute the transfer function $G_{ZOH}(z)$ when $h = 1$. This transfer function is fundamentally different from (2). Explain what happens in the state space model in (b) for $h = 1$, causing this fundamental difference. **(3p)**

(d) Assume the sampling period is chosen to $h = 0.5$. Is it possible to stabilize (1) by use of a ZOH sampling proportional controller, $u(kh) = K(r(kh) - y(kh))$, with this h? If you answer yes, suggest a stabilizing value for K, if you answer no, show that it is not possible. **(3p)**

Problem 2 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true"/"false" are considered!

(a) A Kalman filter is always stable.

(b) For Kalman filters, based on correct models and noise intensities, the *output innovations* are white noise.

(c) White noise processes always have constant spectral densities.

(d) For white noise the covariance function $r(\tau)$ is constant for $\tau \neq 0$.

(e) MPC can always be represented as an LTI controller.

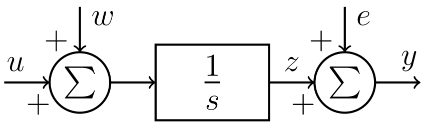
(f) In MPC the *input/control horizon* is typically shorter than the *output/prediction horizon.*

(g) The Nyquist frequency is always 50% of the sampling frequency.

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(7p)**

[1] The control input, $u(t)$, is held constant between the sampling instants.

Problem 3 A simple model describing variations of the level in a water reservoir is given in the block diagram and the state space model below:



$$\Leftrightarrow \quad \begin{cases} \dot{x}(t) = u(t) + w(t), \\ z(t) = x(t), \\ y(t) = x(t) + e(t). \end{cases}$$

Here the input u is the controlled net flow into the reservoir, z is the level and y is the measured level. There are two independent, zero mean noise sources: the measurement noise, e, and uncontrolled variations in the net flow, w.

It is desirable to keep the variations of the level as small as possible, at a minimal cost in terms of control power. Therefore the LQG control law $u = -L\hat{x}$, which minimizes the cost function

$$V = E\left[z^2 + \rho^2 u^2\right], \qquad \rho > 0,$$

is used. (The estimate $\hat{x}$ is obtained from a Kalman filter, which is not considered in this problem.)

(a) In a first approximation it is assumed that both w and e are white noise. Find the optimal feedback gain L, expressed in ρ. **(2p)**

(b) It turns out that

$$w(t) = \frac{1}{p+1} v(t),$$

where v is white noise, is a much better description of w. The measurement noise, e, is still assumed to be white. Give a state space representation of the total model, where the dynamics of w are incorporated. Use the "standard" form, i.e. determine the matrices and vectors A, B, N, M and C in

$$\begin{cases} \dot{\bar{x}}(t) = A\bar{x}(t) + Bu(t) + Nv_1(t), \\ z(t) = M\bar{x}(t), \\ y(t) = C\bar{x}(t) + v_2(t). \end{cases}$$

Use the state vector $\bar{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T = \begin{bmatrix} z & w \end{bmatrix}^T$. Also, define v_1 and v_2 in terms of v and e. **(4p)**

(c) Find the optimal state feedback gain L for the total model in (b). **(3p)**

Problem 4 Consider the discrete-time stochastic process

$$\begin{cases} x(k+1) = \begin{bmatrix} 0.4 & 1 \\ -0.6 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0.6 \\ 0 \end{bmatrix} v(k), \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + v(k), \end{cases} \qquad \begin{aligned} Ev(k) &= 0, \\ \Phi_v(\omega) &= 1. \end{aligned} \tag{3}$$

(a) What is the covariance matrix, $\Pi_x = Ex(k)x(k)^T$, of the state vector in the state space representation (3)? **(3p)**

(b) What is the variance of the output, $Ey(k)^2$? **(1p)**

(c) Give the Kalman filter for (3).
Hint: If necessary, use that the solution of the associated Riccati equation is a diagonal matrix. **(5p)**

(d) The state space representation (3) has a particular property (apart from being on standard form and observer canonical form), and representations with this property are said to be on a certain form. What is the name for this certain form? **(1p)**

Problem 5

(a) What is the output spectrum, $\Phi_y(\omega)$, for the system given by (3) in Problem 4? **(3p)**

(b) Consider a stochastic process $w(k) = G(q)v(k)$, where $v(k)$ is zero mean white noise with unit intensity. Is there a rational transfer operator $G(q)$ such that $w(k)$ is a stationary stochastic process with spectral density

$$\Phi_w(\omega) = \frac{8.5 - 4\cos\omega}{1.49 - 1.4\cos\omega}?$$

If your answer is yes, give the corresponding stable and minimum phase transfer function $G(q)$. Otherwise, explain why not. **(2p)**

(c) Consider (again) a stochastic process $w(k) = G(q)v(k)$, where $v(k)$ is zero mean white noise with unit intensity. Is there a rational transfer operator $G(q)$ such that $w(k)$ is a stationary stochastic process with spectral density

$$\Phi_w(\omega) = \frac{2 + 4\cos\omega}{1.52 - 1.28\cos\omega + 1.2\cos 2\omega}?$$

If your answer is yes, give the corresponding stable and minimum phase transfer function $G(q)$. Otherwise, explain why not. **(3p)**

Problem 6 *The HW bonus points (from the spring 2019) are exchangeable for this problem.*
A DC motor is used for a position servo, implemented by use of a sampling state feedback controller. The DC motor is described by the following (ZOH sampled) discrete-time state space model:

$$\begin{cases} x(k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 0.9 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k), \\ y(k) = \begin{bmatrix} 2.107 & -2 \end{bmatrix} x(k). \end{cases}$$

The control law is $u(k) = -Lx(k) + \tilde{r}(k)$.

(a) Determine the state feedback gain vector L so that the closed loop pole polynomial becomes $p(z) = z^2 - 1.6z + 0.65$. **(2p)**

(b) Let $\tilde{r}(k) = l_0 r(k)$, where $r(k)$ is the reference. Determine the scalar l_0 so that $y = r$ in steady state (with $r(k) =$ constant) when the controller in (a) is used. **(1p)**

(c) Assume that one would like to make the position servo twice as fast as in (a). How should the closed loop pole polynomial $p(z)$ be chosen in order to achieve this? **(2p)**

(d) Now assume that the feedback gain vector L was obtained by LQ optimization, that is, by minimizing the criterion

$$V = \sum_{k=0}^{\infty} \left(Q_1 y(k)^2 + Q_2 u(k)^2 \right),$$

for a certain choice of $Q_1, Q_2 > 0$. How should Q_1 and Q_2 be changed in order to make the response of the closed loop system faster? A thorough motivation is required! **(2p)**

Solutions to the exam in Automatic Control II, 2019-05-29:

1. (a) The closed loop pole polynomial, which can be obtained eg. by $0 = 1 + KG(s)$ or by $\det(sI - A + CK)$, becomes $s^2 + Ks + \pi^2$. The closed loop system is stable (all poles in LHP) for $K > 0$.
(b) The ZOH sampled system is (by Theorem 4.1)

$$\begin{cases} qx = Fx + Gu, \\ y = Cx, \end{cases} \qquad \text{where} \quad F = e^{Ah},\ G = \int_0^h e^{At} B dt.$$

Use that $e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$:

$$e^{At} = \mathcal{L}^{-1}\left[\begin{bmatrix} s & -\pi \\ \pi & s \end{bmatrix}^{-1}\right] = \mathcal{L}^{-1}\left[\begin{bmatrix} \frac{s}{s^2+\pi^2} & \frac{\pi}{s^2+\pi^2} \\ \frac{-\pi}{s^2+\pi^2} & \frac{s}{s^2+\pi^2} \end{bmatrix}\right] = \begin{bmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{bmatrix}$$

$$\Rightarrow \quad F = \begin{bmatrix} \cos \pi h & \sin \pi h \\ -\sin \pi h & \cos \pi h \end{bmatrix}, \quad G = \int_0^h \begin{bmatrix} \sin \pi t \\ \cos \pi t \end{bmatrix} dt = \frac{1}{\pi} \begin{bmatrix} 1 - \cos \pi h \\ \sin \pi h \end{bmatrix}.$$

(c) The transfer function is $G_{ZOH}(z) = C(zI - F)^{-1}G$, and with $h = 1$ we get

$$G_{ZOH}(z) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z+1 & 0 \\ 0 & z+1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{2}{\pi} \\ 0 \end{bmatrix} = 0.$$

Checking for controllability and observability for the case $h = 1$:

$$\mathcal{S} = \begin{bmatrix} G & FG \end{bmatrix} = \begin{bmatrix} \frac{2}{\pi} & -\frac{2}{\pi} \\ 0 & 0 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} C \\ CF \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}.$$

Both $\mathcal{S}$ and $\mathcal{O}$ are rank-deficient, meaning that the system is neither controllable, nor observable. This is due to the particular choice of sampling period h. Because of this the system order decreases (here to order zero). (What happens here is that the sampling period coincides exactly with the system's natural frequency, so that the sampling instants occur exactly in the zero crossings of the system's impulse response.)
(d) The closed loop pole polynomial becomes $z^2 + 1 + \frac{K}{\pi}(z-1) = z^2 + \frac{K}{\pi}z + 1 - \frac{K}{\pi}$. A polynomial $z^2 + az + b$ has both its zeroes inside the unit circle if and only if $|a| - 1 < b < 1$. Here:

$$\begin{aligned} b < 1: &\quad 1 - \frac{K}{\pi} < 1 &\Leftrightarrow&\quad K > 0, \\ a - 1 < b: &\quad \frac{K}{\pi} - 1 < 1 - \frac{K}{\pi} &\Leftrightarrow&\quad K < \pi, \\ -a - 1 < b: &\quad -\frac{K}{\pi} - 1 < 1 - \frac{K}{\pi} &\Leftrightarrow&\quad -1 < 1. \end{aligned}$$

Thus, the closed loop system is stable for $0 < K < \pi$.

2. (a) True; **(b)** True; **(c)** True (By definition); **(d)** True ($r(\tau) = 0$ for $\tau \neq 0$); **(e)** False (Always time-varying, typically nonlinear); **(f)** True; **(g)**

True (By definition);

3. **(a)** Theorem 9.1 $\Rightarrow L = Q_2^{-1}B^T S$, where $S = S^T > 0$ solves the CARE $0 = A^T S + SA + M^T Q_1 M - SBQ_2^{-1}B^T S$. Here $Q_1 = 1$, $Q_2 = \rho^2$, $A = 0$, $B = 1$ and $M = 1$, leading to

$$0 = 1 - S^2/\rho^2 \qquad \Rightarrow \qquad S = \rho,$$

and then $L = \rho^{-2}\rho = 1/\rho$.

(b) We have that $pw = -w + v$, and therefore

$$\begin{cases} \dot{z} = w + u, \\ \dot{w} = -w + v, \\ z = z, \\ y = z + e \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v, \\ z = \begin{bmatrix} 1 & 0 \end{bmatrix} x, \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + e, \end{cases}$$

with $x = \begin{bmatrix} z & w \end{bmatrix}^T$. We also note that $v_1 = v$ and $v_2 = e$.

(c) The CARE becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix} + \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$- \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rho^{-2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix},$$

which, taken element by element, leads to the equation system

$$\begin{cases} 0 = 1 - s_1^2/\rho^2, \\ 0 = s_1 - s_{12} - s_1 s_{12}/\rho^2, \\ 0 = 2s_{12} - 2s_2 - s_{12}^2/\rho^2, \end{cases} \qquad \Rightarrow \qquad \begin{cases} s_1 = \rho, \\ s_{12} = \dfrac{s_1}{1 + s_1/\rho^2} = \dfrac{\rho^2}{\rho + 1}, \\ s_2 = \left(2 - s_{12}/\rho^2\right) s_{12}/2 = \dfrac{\rho^2(2\rho + 1)}{2(\rho + 1)^2}. \end{cases}$$

The state feedback gain is then $L = \rho^{-2} \begin{bmatrix} s_1 & s_{12} \end{bmatrix} = \begin{bmatrix} 1/\rho & 1/(\rho + 1) \end{bmatrix}$.

4. **(a)** The covariance matrix solves the discrete-time Lyapunov equation,$\Pi_x = F\Pi_x F^T + NRN^T$ (can be used directly since v is white noise), where $R = 1$ in this case. Here we have

$$F = \begin{bmatrix} 0.4 & 1 \\ -0.6 & 0 \end{bmatrix}, \ N = \begin{bmatrix} 0.6 \\ 0 \end{bmatrix}. \quad \text{Setting} \quad \Pi_x = \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix}$$

and spelling out the Lyapunov equation, the following equation system is obtained:

$$\begin{cases} \pi_1 = 0.16\pi_1 + 0.8\pi_{12} + \pi_2 + 0.36, \\ \pi_{12} = -0.24\pi_1 - 0.6\pi_{12}, \\ \pi_2 = 0.36\pi_1, \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} \pi_1 = \dfrac{0.36}{0.6} = 0.6, \\ \pi_{12} = -\dfrac{0.24}{1.6} \cdot 0.6 = -0.09, \\ \pi_2 = 0.36 \cdot 0.6 = 0.216. \end{cases}$$

(b) Since $y = Hx+v$ we get $Ey^2 = E[(Hx+v)(Hx+v)^T] = H\Pi_x H^T + R_v = \pi_1 + R_v$ (since $x(k)$ and $v(k)$ are uncorrelated). Hence

$$\Pi_x = \begin{bmatrix} 0.6 & -0.09 \\ -0.09 & 0.216 \end{bmatrix}, \quad Ey^2 = \pi_1 + R_v = 0.6 + 1 = 1.6.$$

(c) Notice that (3) is on *innovations form* if $F - NH$ is stable (has all eigenvalues inside the unit circle). Here $\det(qI - F + NH) = q^2 + 0.2q + 0.6$ $\Rightarrow$ eigenvalues in $-0.1 \pm i\sqrt{0.59}$, which is inside the unit circle. Hence the system is on innovations form, and then the Kalman gain is $K = N$ — see Eq. (5.83) in Glad/Ljung (Eq. (5.84) in the English version) (this holds for discrete-time systems as well). Thus, without solving any DARE we can directly state that

$$K = N = \begin{bmatrix} 0.6 \\ 0 \end{bmatrix}.$$

(Of course it is possible and correct to obtain K by solving the DARE. The solution then is $P = 0$! Note though that $R_1 = R_2 = R_{12} = 1$!)
(d) As mentioned above, (3) is on *innovations form.*

5. (a) The output spectrum is $\Phi_y(\omega) = |G(e^{i\omega})|^2 \Phi_v(\omega)$. The system (3) is on observer canonical form so we get $G(q) = \frac{0.6q}{q^2-0.4q+0.6} + 1 = \frac{q^2+0.2q+0.6}{q^2-0.4q+0.6}$ (can also use $G(q) = H(qI - F)^{-1}N + 1$), and since $\Phi_v(\omega) = 1$ we have

$$\Phi_y(\omega) = \frac{e^{-i2\omega} + 0.2e^{-i\omega} + 0.6}{e^{-i2\omega} - 0.4e^{-i\omega} + 0.6} \cdot \frac{e^{i2\omega} + 0.2e^{i\omega} + 0.6}{e^{i2\omega} - 0.4e^{i\omega} + 0.6}$$
$$= \frac{1 + 0.2^2 + 0.6^2 + (0.2 + 0.2 \cdot 0.6)(e^{i\omega} + e^{-i\omega}) + 0.6(e^{i2\omega} + e^{i2\omega})}{1 + (-0.4)^2 + 0.6^2 + (-0.4 - 0.4 \cdot 0.6)(e^{i\omega} + e^{-i\omega}) + 0.6(e^{i2\omega} + e^{i2\omega})}$$
$$= \frac{1.40 + 0.32(e^{i\omega} + e^{-i\omega}) + 0.6(e^{i2\omega} + e^{-i2\omega})}{1.52 - 0.64(e^{i\omega} + e^{-i\omega}) + 0.6(e^{i2\omega} + e^{-i2\omega})} = \frac{1.40 + 0.64\cos\omega + 1.2\cos 2\omega}{1.52 - 1.28\cos\omega + 1.2\cos 2\omega}.$$

(b) Theorem 5.1, in its discrete-time version (see p. 152 in the textbook), states that for $0 \le \Phi(\omega) < \infty$, rational in $\cos\omega$, there exists a rational, stable and minimum phase transfer function such that $\Phi(\omega) = G(e^{i\omega})G(e^{-i\omega})$. Here these conditions hold for $\Phi_w(\omega)$, and it is meaningful to find a transfer function $G(q)$. Both numerator and denominator of $\Phi_w(\omega)$ are first order polynomials in $\cos\omega$, so we try with $G(q) = K\frac{q+\beta}{q+\alpha}$, with $|\alpha| < 1$ and $|\beta| < 1$ (for stability and minimum phase). Then

$$G(e^{i\omega})G(e^{-i\omega}) = K^2 \frac{(e^{i\omega} + \beta)(e^{-i\omega} + \beta)}{(e^{i\omega} + \alpha)(e^{-i\omega} + \alpha)}$$
$$= K^2 \frac{1 + \beta^2 + 2\beta\cos\omega}{1 + \alpha^2 + 2\alpha\cos\omega} = \frac{K^2\beta}{\alpha} \cdot \frac{\frac{1+\beta^2}{2\beta} + \cos\omega}{\frac{1+\alpha^2}{2\alpha} + \cos\omega}.$$

Here we have normalized so that the coefficient for $\cos\omega$ is $+1$, and by normalizing the given spectrum the same way we get

$$\Phi_w(\omega) = \frac{8.5 - 4\cos\omega}{1.49 - 1.4\cos\omega} = \frac{4}{1.4} \cdot \frac{-8.5/4 + \cos\omega}{-1.49/1.4 + \cos\omega}.$$

Equating the coefficients in $G(e^{i\omega})G(e^{-i\omega})$ and $\Phi_w(\omega)$ gives the equation system

$$\begin{cases} \dfrac{K^2\beta}{\alpha} = \dfrac{4}{1.4}, \\ \dfrac{1+\beta^2}{2\beta} = -\dfrac{8.5}{4}, \\ \dfrac{1+\alpha^2}{2\alpha} = -\dfrac{1.49}{1.4}, \end{cases} \Leftrightarrow \begin{cases} K = \sqrt{\dfrac{4\alpha}{1.4\beta}}, \\ 0 = \beta^2 + 4.25\beta + 1, \\ 0 = \alpha^2 + \dfrac{1.49}{0.7}\alpha + 1, \end{cases} \Leftrightarrow \begin{cases} K = 2\sqrt{2}, \\ \beta = -0.25, \\ \alpha = -0.7. \end{cases}$$

(Here roots $|\alpha| > 1$ and $|\beta| > 1$ were omitted.) Thus, $G(q) = 2\sqrt{2}\frac{q-0.25}{q-0.7}$.

(c) Here we see that $\Phi_w(\omega) \geq 0$ only for $|\omega| \leq \frac{2\pi}{3}$, and $\Phi_w(\omega) < 0$ outside that interval (on $-\pi \leq \omega < \pi$). Therefore $\Phi_w(\omega)$ is not a spectrum, and there is no $G(q)$ ...

6. (a) The transfer function for the DC motor is $G(z) = \frac{0.107z+0.1037}{(z-1)(z-0.9)} = \frac{b(z)}{a(z)}$. When using state feedback the closed loop system becomes $Y(z) = G_c(z)\tilde{R}(z)$, where $G_c(z) = H(zI - F + GL)^{-1}G = \frac{b(z)}{p(z)}$ and $p(z) = \det(zI - F + GL)$. Here

$$\det(zI-F+GL) = \det\begin{bmatrix} z-1+l_1 & l_2 \\ l_1 & z-0.9+l_2 \end{bmatrix} = z^2+(-1.9+l_1+l_2)z+0.9-0.9l_1-l_2,$$

and by equating the coefficients with the ones in the desired pole polynomial we get the equation system

$$\begin{cases} -1.9 + l_1 + l_2 = -1.6, \\ 0.9 - 0.9l_1 - l_2 = 0.65 \end{cases} \Leftrightarrow \begin{cases} l_1 = 0.5, \\ l_2 = -0.2 \end{cases} \Leftrightarrow L = \begin{bmatrix} 0.5 & -0.2 \end{bmatrix}.$$

(b) $\tilde{R}(z) = l_0R(z) \Rightarrow Y(z) = G_c(z)l_0R(z)$. In order to have $y = r$ in steady state we need to have

$$1 = G_c(1)l_0 = \frac{(0.107 + 0.1037)l_0}{1 - 1.6 + 0.65} = \frac{0.2107l_0}{0.05} \Leftrightarrow l_0 = \frac{0.05}{0.2107} = 0.2373.$$

(c) In continuous time we achieve a twice as fast system by doubling the poles' distance from the origin. That is, a continuous-time pole p_c is made twice as fast by multiplication with two, $2p_c$. By ZOH sampling a continuous-time pole p_c is mapped on a discrete-time pole $p_d = e^{p_ch}$, where h is the sampling period. Thus, a twice as fast continuous-time pole $2p_c$ is mapped onto $e^{2p_ch} = \left(e^{p_ch}\right)^2 = p_d^2$. The conclusion is that a twice as fast discrete-time pole is obtained by taking the square.

Here the (discrete-time) poles are given by $0 = z^2 - 1.6z + 0.65 \Leftrightarrow p_d = 0.8 \pm i0.1$. Then $p_d^2 = (0.8 \pm i0.1)^2 = 0.63 \pm i0.16$, and the corresponding pole polynomial is

$$p(z) = (z - 0.63)^2 + 0.16^2 = z^2 - 1.26z + 0.4225.$$

(d) It is only the ratio Q_1/Q_2 that will affect the closed loop system. Increasing this ration emphasizes the size of y above u in the optimization, and vice versa. Increasing Q_1/Q_2 allows bigger inputs, and therefore the response from the system will be faster. Thus, to make the system faster one should either increase Q_1 or decrease Q_2.