

Exam in Automatic Control II

Reglerteknik II 5hp

Date: May 27, 2016

Venue: Polacksbacken

Responsible teacher: Hans Norlander.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Good luck!

Problem 1 In a certain manufacturing process it is important to keep track of a particular level, $z(t)$, described as

$$\begin{cases} \dot{z}(t) = -z(t) + w(t), \\ y(t) = z(t) + e(t). \end{cases} \quad (1)$$

Here $w(t)$ is a zero mean stochastic process, and the measurement noise $e(t)$ is zero mean white noise with intensity/spectrum $\Phi_e(\omega) = 1$.

(a) Initially it was assumed that $w(t)$ is zero mean white noise with intensity/spectrum $\Phi_w(\omega) = 3$. Based on this assumption the corresponding Kalman filter was computed and used for estimation of $z(t)$. Determine this Kalman filter. **(3p)**

Later it was discovered that $w(t)$ is *not* white noise, but instead the stochastic process

$$w(t) = \frac{1}{p+4}v(t) \quad \Leftrightarrow \quad \dot{w}(t) = -4w(t) + v(t), \quad (2)$$

where $v(t)$ is zero mean white noise with intensity/spectrum $\Phi_v(\omega) = 48$.

(b) Determine the spectrum *and* the variance of $w(t)$ in (2). **(2p)**

(c) Give a state space model in the “standard form” of the system (1) combined with the disturbance model (2). Both the process noise and the measurement noise should be zero mean white noise. Choose the state vector $x(t)$ so that its first element is $x_1(t) = z(t)$. **(3p)**

(d) When computing the Kalman filter for the model in (c) the solution of the corresponding algebraic Riccati equation is

$$P = \begin{bmatrix} 0.7446 & 1.0217 \\ 1.0217 & 5.8695 \end{bmatrix}.$$

What is the variance of the estimation error $\tilde{z}(t) = z(t) - \hat{z}(t)$ when this Kalman filter is used for estimation? **(2p)**

(e) While the Kalman filter in (a) is not the optimal observer for estimation of $z(t)$ (since it is computed for an erroneous model of $w(t)$), it is still useful for that. We can then regard it as an ad hoc observer. What is the (true) variance of the estimation error $\tilde{z}(t) = z(t) - \hat{z}(t)$ in this case¹? **(2p)**

¹The variance obtained in (a) is not valid since $w(t)$ is not white noise.

Problem 2 A discrete-time system is modeled by the difference equation

$$y(k) - y(k-1) = u(k-1) + e(k) - 2e(k-1) \Leftrightarrow (q-1)y(k) = u(k) + (q-2)e(k), \quad (3)$$

where $e(k)$ is zero mean white noise with intensity/variance $Ee(k)^2 = R_e = 1$. The system can also be represented in state space form as

$$\begin{cases} x(k+1) = x(k) + u(k) - e(k), \\ y(k) = x(k) + e(k), \end{cases} \quad (4)$$

or, equivalent, as

$$\begin{cases} x(k+1) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} v(k), \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k), \end{cases} \quad (5)$$

where $v(k) = e(k+1) = qe(k)$ (i.e. $v(k)$ is white noise with the same properties as $e(k)$).

The system should be controlled by output feedback, $u(k) = -F_y(q)y(k)$, so that $V = Ey(k)^2$ is minimized (\Rightarrow LQG with $Q_1 = 1$ and $Q_2 = 0$). The controller, $F_y(q)$, should be strictly proper so that $u(k)$ depends on $y(k-1)$ and previous outputs, but not on $y(k)$ (i.e. do not use $\hat{x}(k|k)$).

- (a) Verify that (4) and (5) both are equivalent to (3). **(3p)**
- (b) Determine the optimizing controller $F_y(q)$. Use either (4) or (5)². (It suffices to give the corresponding K and L .) **(6p)**
- (c) What are the poles of the closed loop system? **(1p)**

Problem 3 Specify for each of the following statements whether it is true or false. No motivations required — only answers “true”/“false” are considered!

- (a) Controllability is always preserved under zero-order hold sampling.
- (b) For a Kalman filter, based on a correct model and full knowledge of the noise, the *output innovations* are white noise.
- (c) The best performance is obtained when the Nyquist frequency is chosen equal to, or higher than the sampling frequency.
- (d) A drawback of MPC is that it cannot handle bounds on the control input.
- (e) In general one should choose the input/control horizon longer than the output/prediction horizon when using MPC.

Each correct answer scores +1, each incorrect answer scores -1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(5p)**

²Either way you will end up with the same $F_y(q)$.

Problem 4 A double tank system (similar to the one in the MPC demo lab) has the linearized model

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 2 & -2 \end{bmatrix} x(t), \end{cases} \Leftrightarrow y(t) = \frac{2}{(p+1)(p+2)} u(t).$$

Here $y(t)$ is the water level in the lower tank and $u(t)$ is the flow into the upper tank (from a pump).

(a) Assume that the double tank is controlled by (continuous-time) proportional feedback, $u(t) = K(r(t) - y(t))$, where $r(t)$ is the reference. For which $K \in \mathbb{R}$ is the closed loop system stable? **(2p)**

The controller is implemented as a sampling controller, with a zero-order hold (ZOH) circuit. That is

$$u(t) = u(kh) \quad \text{for } kh \leq t < kh+h, \quad \text{and } u(kh) = K(r(kh) - y(kh)), \quad (6)$$

where h is the sampling interval. Analysis of the closed loop system then requires a discrete-time model of the system:

$$\begin{cases} x(kh+h) = Fx(kh) + Gu(kh), \\ y(kh) = Hx(kh) \end{cases} \Leftrightarrow y(kh) = H(qI - F)^{-1}Gu(kh).$$

(b) Determine the ZOH sampled model of the system, i.e. give F , G and H in the state space model above. The answer should be expressed in the sampling period h . **(4p)**

(c) For a certain choice of sampling interval the ZOH sampled model is

$$y(kh) = \frac{0.04(q+0.8)}{(q-0.8)(q-0.64)} u(kh).$$

For which $K \in \mathbb{R}$ is the closed loop system stable when (6) is used? **(3p)**

(d) What is the static gain of the closed loop system in (c)? **(1p)**

Problem 5 Consider the discrete-time system

$$y(k) - 1.2y(k-1) + 0.36y(k-2) = 2u_1(k-1) + u_1(k-2) + u_2(k-1).$$

(a) Determine the transfer operator $G(q)$ so that the system can be represented as

$$y(k) = G(q)u(k) \quad \text{with} \quad u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}. \quad (2\text{p})$$

(b) Give a state space representation for the system. (4p)

Problem 6 *The HW bonus points are exchangeable for this problem.*

(a) A discrete-time stochastic process, $w(k)$, has the spectrum

$$\Phi_w(\omega) = \frac{6 + 4\sqrt{2} \cos \omega}{1.25 - \cos 2\omega}.$$

Find a stable, minimum phase transfer operator $G(q)$ such that the model

$$w(k) = G(q)v(k),$$

with $v(k)$ being zero mean white noise with unit variance, has $\Phi_w(\omega)$ as spectrum. (5p)

(b) Another discrete-time process is

$$x(k+1) = \begin{bmatrix} -0.8 & 1 \\ 0 & -0.6 \end{bmatrix} x(k) + v(k),$$

where $v(k) = [v_1(k) \ v_2(k)]^T$ is zero mean white noise. The covariance matrix of the *state vector* is

$$Ex(k)x^T(k) = \begin{bmatrix} 600 & -75 \\ -75 & 65 \end{bmatrix}.$$

Determine the covariance matrix $R_v = Ev(k)v^T(k)$. (2p)

Solutions to the exam in Automatic Control II, 2016-05-27:

1. (a) The Kalman filter is $\dot{\hat{z}} = A\hat{z} + K(y - C\hat{z})$. The Kalman gain is $K = PC^T R_2^{-1}$, where P solves the CARE

$$0 = AP + PA^T + NR_1 N^T - PC^T R_2^{-1} CP.$$

Here $A = -1$, $N = C = R_2 = 1$ and $R_1 = 3 \Rightarrow$ the CARE becomes

$$0 = -2P + 3 - P^2 \quad \Leftrightarrow \quad P = -1 \pm 2,$$

where the negative root is rejected. Thus, $P = 1$ and $K = P = 1$.

(b) The spectrum is

$$\Phi_w(\omega) = |G(i\omega)|^2 \Phi_v(\omega) = \frac{1}{(i\omega + 4)(-i\omega + 4)} \cdot 48 = \frac{48}{\omega^2 + 16}.$$

The variance, $Ew^2 = \Pi_w$, is the solution to the continuous-time Lyapunov equation:

$$0 = A\Pi_w + \Pi_w A^T + NR_v N^T = 2 \cdot (-4)\Pi_w + 48 \quad \Leftrightarrow \quad \Pi_w = \frac{48}{8} = 6.$$

(c) Introduce $x = [z \quad w]^T$.

$$\begin{cases} \dot{z} = -z + w, \\ \dot{w} = -4w + v, \\ y = z + e \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -4 \end{bmatrix}^{=A} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{=N} v, \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix}_{=C} x + e. \end{cases}$$

(d) We have $z = Cx \Rightarrow \tilde{z} = C\tilde{x}$, and thus

$$E\tilde{z}\tilde{z}^T = EC\tilde{x}\tilde{x}^T C^T = C(E\tilde{x}\tilde{x}^T)C^T = CPC^T = p_1 = 0.7446.$$

(e) From (a) and (1) we get

$$\dot{\hat{z}} = -2\hat{z} + y \quad \Rightarrow \quad \dot{\tilde{z}} = \dot{z} - \hat{z} = -z + w - (-2\hat{z} + z + e) = -2\tilde{z} + w - e.$$

Combined with (2) this gives

$$\begin{cases} \dot{\tilde{z}} = -2\tilde{z} + w - e, \\ \dot{w} = -4w + v \end{cases} \quad \stackrel{\xi = \begin{bmatrix} \tilde{z} \\ w \end{bmatrix}}{\Rightarrow} \begin{cases} \dot{\xi} = \begin{bmatrix} -2 & 1 \\ 0 & -4 \end{bmatrix}^{=A} \xi + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{=N} \begin{bmatrix} e \\ v \end{bmatrix}^{=\nu}, \\ \tilde{z} = \begin{bmatrix} 1 & 0 \end{bmatrix}_{=C} \xi. \end{cases}$$

The variance of \tilde{z} is then $C\Pi_\xi C^T$, and Π_ξ is the solution of the Lyapunov equation $0 = A\Pi_\xi + \Pi_\xi A^T + NR_\nu N^T$. Set up and solve this Lyapunov equation:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} + \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & -4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 48 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Element by element we get

$$\begin{cases} 0 = -4\pi_1 + 2\pi_{12} + 1, \\ 0 = -6\pi_{12} + \pi_2, \\ 0 = -8\pi_2 + 48, \end{cases} \Leftrightarrow \begin{cases} \pi_1 = 0.75, \\ \pi_{12} = 1, \\ \pi_2 = 6. \end{cases}$$

Then $E\tilde{z}^2 = C\Pi_\xi C^T = \pi_1 = 0.75$. Thus the optimal estimate yields a less than 1% improvement!

2. (a) Use that

$$\begin{cases} qx = Fx + Gu + Nv_1, \\ y = Hx + v_2, \end{cases} \Rightarrow y = H(qI - F)^{-1}(Gu + Nv_1) + v_2,$$

and then retrieve the difference equation from that. With (4) we have

$$F = G = H = 1, \quad N = -1 \text{ and } v_1 = v_2 = e. \quad (4')$$

Thus,

$$y = \frac{1}{q-1}(u - e) + e \Leftrightarrow (q-1)y = u + (q-2)e,$$

which is equivalent to (3).

With (5) we have

$$F = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad N = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad H = [1 \quad 0], \quad v_1 = v \text{ and } v_2 = 0. \quad (5')$$

Then we get

$$\begin{aligned} y &= [1 \quad 0] \begin{bmatrix} q-1 & 1 \\ 0 & q \end{bmatrix}^{-1} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} v \right) = \frac{q}{q(q-1)}u + \frac{q-2}{q(q-1)}v \\ &\Leftrightarrow q(q-1)y = qu + (q-2)v \Leftrightarrow (q-1)y = u + (q-2)e, \end{aligned}$$

since $v = qe$. Again we have equivalence with (3).

(b) Let \hat{x} denote $\hat{x}(k|k-1)$. In LQG the control law is $u = -L\hat{x}$, where a Kalman filter (KF), $q\hat{x} = F\hat{x} + Gu + K(y - H\hat{x})$ (see Theorem 9.4) provides \hat{x} . The Kalman gain is $K = (FPH^T + NR_{12})(HPH^T + R_2)^{-1}$, where P solves the DARE

$$P = FPF^T + NR_1N^T - (FPH^T + NR_{12})(HPH^T + R_2)^{-1}(FPH^T + NR_{12})^T.$$

The feedback gain is $L = (G^T SG + Q_2)^{-1}G^T SF$, where S solves the DARE

$$S = F^T SF + M^T Q_1 M - F^T SG(G^T SG + Q_2)^{-1}G^T SF.$$

Here we note that $Q_1 = 1$ and $Q_2 = 0$.

With (4): From (4') we note that $R_1 = R_2 = R_{12} = 1$, so for the KF we get $K = \frac{P-1}{P+1}$ and the DARE

$$P = P + 1 - \frac{(P-1)^2}{P+1} \Leftrightarrow P = 3 \text{ (and } = 0).$$

Thus $K = 0.5$. The feedback gain is $L = \frac{S}{s} = 1$ (we need not solve the DARE).

With (5): From (5') we get that $R_1 = 1$ and $R_2 = R_{12} = 0 \Rightarrow HPH^T + r_2 = \frac{p_1}{p_1}$. To set up the DARE we can start by computing FP , and then we readily get that

$$FPF^T = \begin{bmatrix} p_1 - 2p_{12} + p_2 & 0 \\ 0 & 0 \end{bmatrix}, FPH^T = \begin{bmatrix} p_1 - p_{12} \\ 0 \end{bmatrix}, NR_1N^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

The DARE then spells out as

$$\begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} = \begin{bmatrix} p_1 - 2p_{12} + p_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \frac{(p_1 - p_{12})^2}{p_1} & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow P = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}.$$

The Kalman gain then is $K = \begin{bmatrix} \frac{p_1 - p_{12}}{p_1} \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$. Since $Q_2 = 0$ we get $G^TSG + Q_2 = s_1$, and $L = \frac{1}{s_1}G^T SF = \frac{1}{s_1} [s_1 \quad -s_1] = [1 \quad -1]$ (so neither here we need to solve the DARE).

(The feedback filter is $F_y(q) = L(qI - F + KH + GL)^{-1}K$, and in both cases we get $F_y(q) = \frac{0.5}{q+0.5}$.)

(c) The closed loop poles are those from the state feedback, solving $0 = \det(zI - F + GL)$, and the observer poles, solving $0 = \det(zI - F + KH)$. With the (4) solution we get $zI - F + GL = z - 1 + 1 = z$, i.e. a pole in the origin, and $zI - F + KH = z - 1 + 0.5 = z - 0.5$, i.e. a pole in 0.5. (For the (5) solution we get two additional poles in the origin, but these are cancelled out in the closed loop system.)

3. (a) False (Controllability may be lost for certain sampling intervals); **(b)** True (See Theorem 5.5); **(c)** False (Nonsense - by definition $\omega_n = 0.5\omega_s$); **(d)** False (MPC handles bounds and constraint); **(e)** False ($N \leq M$ and in general $N \ll M$);

4. (a)

$$y(t) = \frac{2}{(p+1)(p+2)}K(r(t) - y(t)) \quad y(t) = \frac{2k}{(p+1)(p+2) + 2K}r(t).$$

For stability all poles must be in the left half plane. The poles are given by

$$0 = (p+1)(p+2) + 2K = p^2 + 3p + 2 + 2K.$$

For second order systems it is sufficient to have all coefficient positive, so the stability condition here is $2 + 2K > 0$, i.e. $K > -1$.

(b)

$$F = e^{Ah} = \begin{bmatrix} e^{-h} & 0 \\ 0 & e^{-2h} \end{bmatrix}, G = \int_0^h e^{At}Bdt = \int_0^h \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix} dt = \begin{bmatrix} 1 - e^{-h} \\ 0.5(1 - e^{-2h}) \end{bmatrix}.$$

(c)

$$y(k) = \frac{0.04(q+0.8)}{(q-0.8)(q-0.64)}K(r(k) - y(k))$$

$$\Leftrightarrow y(k) = \frac{0.04K(q+0.8)}{(q-0.8)(q-0.64) + 0.04K(q+0.8)}r(k).$$

The poles are given by

$$0 = (q-0.8)(q-0.64) + 0.04K(q+0.8) = q^2 + \underbrace{(0.04K - 1.44)}_{=\alpha}q + \underbrace{0.512 + 0.032K}_{=\beta}.$$

For stability all poles must lie within the unit circle, which for a second order system is equivalent to $|\alpha| - 1 < \beta < 1$:

$$\beta < 1: \quad 0.512 + 0.032K < 1 \Leftrightarrow K < \frac{0.488}{0.032} = 15.25,$$

$$\alpha - 1 < \beta: \quad -2.44 + 0.04K < 0.512 + 0.032K \Leftrightarrow K < \frac{2.952}{0.008} = 369,$$

$$-\alpha - 1 < \beta: \quad 0.44 - 0.04K < 0.512 + 0.032K \Leftrightarrow K > \frac{-0.072}{0.072} = -1.$$

Stable for $-1 < K < 15.25$.

(d) The static is $G_c(1)$, and from (c) we that $G_c(1) = \frac{0.04K \cdot 1.8}{0.2 \cdot 0.36 + 0.04K \cdot 1.8} = \frac{0.072K}{0.072(1+K)} = \frac{K}{1+K}$.

5. (a) Rewriting the equation with the shift operator gives

$$(q^2 - 1.2q + 0.36)y = (2q + 1)u_1 + qu_2 \quad \Leftrightarrow$$

$$y = \frac{2q+1}{q^2-1.2q+0.36}u_1 + \frac{q}{q^2-1.2q+0.36}u_2 = \overbrace{\begin{bmatrix} \frac{2q+1}{q^2-1.2q+0.36} & \frac{q}{q^2-1.2q+0.36} \end{bmatrix}}^{=G(q)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

(b) One output \Rightarrow observer canonical form applies:

$$\begin{cases} qx = \begin{bmatrix} 1.2 & 1 \\ -0.36 & 0 \end{bmatrix} x + \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{cases}$$

6. (a) We notice that $\cos 2\omega = 2\cos^2 \omega - 1$. Thus, in the numerator $\cos \omega$ has degree one, and in the denominator it has degree two, which suggests that

we should try with

$$\begin{aligned}
G(q) = \frac{b_1 q + b_2}{q^2 + a_1 q + a_2} &\Rightarrow \Phi_w(\omega) = |G(e^{i\omega})|^2 \\
&= \frac{(b_1 e^{i\omega} + b_2)(b_1 e^{-i\omega} + b_2)}{(e^{i2\omega} + a_1 e^{i\omega} + a_2)(e^{-i2\omega} + a_1 e^{-i\omega} + a_2)} \\
&= \frac{b_1^2 + b_2^2 + b_1 b_2 (e^{i\omega} + e^{-i\omega})}{1 + a_1^2 + a_2^2 + a_1(1 + a_2)(e^{i\omega} + e^{-i\omega}) + a_2(e^{i2\omega} + e^{-i2\omega})} \\
&= \frac{b_1^2 + b_2^2 + 2b_1 b_2 \cos \omega}{1 + a_1^2 + a_2^2 + 2a_1(1 + a_2) \cos \omega + 2a_2 \cos 2\omega} \\
&= \frac{\frac{b_1^2 + b_2^2}{2a_2} + \frac{b_1 b_2}{a_2} \cos \omega}{\frac{1 + a_1^2 + a_2^2}{2a_2} + \frac{a_1(1 + a_2)}{a_2} \cos \omega + \cos 2\omega}.
\end{aligned}$$

This should be compared to the given spectrum:

$$\Phi_w(\omega) = \frac{6 + 4\sqrt{2} \cos \omega}{1.25 - \cos \omega} = \frac{-6 - 4\sqrt{2} \cos 2\omega}{-1.25 + \cos 2\omega}$$

Equating numerator and denominator, term by term, yields the equation system

$$\left\{ \begin{array}{l} \frac{b_1^2 + b_2^2}{2a_2} = -6, \\ \frac{b_1 b_2}{a_2} = -4\sqrt{2}, \\ \frac{1 + a_1^2 + a_2^2}{2a_2} = -1.25, \\ \frac{a_1(1 + a_2)}{a_2} = 0, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} b_1 = 2, \\ b_2 = \sqrt{2}, \\ a_1 = 0, \\ a_2 = -0.5, \end{array} \right. \Rightarrow G(q) = \frac{2q + \sqrt{2}}{q^2 - 0.5}.$$

Stability $\Rightarrow |a_2| < 1$, and minimum phase $\Rightarrow |b_2/b_1| < 1$ — this provides the unique solution. (Actually b_1 and b_2 could be negative, but the additional choice to have $G(1) > 0$ rejects that solution.)

(b) Use that $Exx^T = \Pi_x$ solves the Lyapunov equation $\Pi_x = F\Pi_x F^T + R_v$

$$\begin{aligned}
&\Rightarrow R_v = \Pi_x - F\Pi_x F^T = \\
&\begin{bmatrix} 600 & -75 \\ -75 & 65 \end{bmatrix} - \begin{bmatrix} -0.8 & 1 \\ 0 & -0.6 \end{bmatrix} \cdot \begin{bmatrix} 600 & -75 \\ -75 & 65 \end{bmatrix} \cdot \begin{bmatrix} -0.8 & 0 \\ 1 & -0.6 \end{bmatrix} = \begin{bmatrix} 31 & 0 \\ 0 & 41.6 \end{bmatrix}.
\end{aligned}$$