Exam in Automatic Control II
Reglerteknik II 5hp (1RT495)

Date: January 2, 2017

Venue: Polacksbacken, exam hall

Responsible teacher: Hans Rosth.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Good luck!

Problem 1 A double tank system (similar to the one in the MPC demo lab) has the linearized model

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} -1 & 1 \end{bmatrix} x(t), \end{cases} \quad \Leftrightarrow \quad y(t) = \frac{0.5}{(p+1)(p+0.5)} u(t).$$

Here $y(t)$ is the water level in the lower tank and $u(t)$ is the flow into the upper tank (from a pump).

(a) Assume that the double tank is controlled by (continuous-time) proportional feedback, $u(t) = K(r(t) - y(t))$, where $r(t)$ is the reference. For which $K \in \mathbb{R}$ is the closed loop system stable? **(2p)**

The controller is implemented as a sampling controller, with a zero-order hold (ZOH) circuit. That is,

$$u(t) = u(kh) \quad \text{for} \quad kh \leq t < kh+h, \quad \text{and} \quad u(kh) = K(r(kh)-y(kh)), \quad (1)$$

where h is the sampling interval. Analysis of the closed loop system then requires a discrete-time model of the system:

$$\begin{cases} x(kh+h) = Fx(kh) + Gu(kh), \\ y(kh) = Hx(kh) \end{cases} \quad \Leftrightarrow \quad y(kh) = H(qI - F)^{-1} G u(kh).$$

(b) Determine the ZOH sampled model of the system, i.e. give F, G and H in the state space model above. The answer should be expressed in the sampling period h. **(3p)**

(c) For a certain choice of sampling interval the ZOH sampled model is

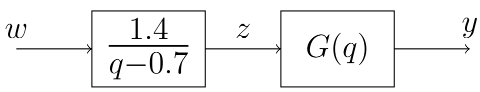
$$y(kh) = \frac{0.16(q+0.6)}{(q-0.6)(q-0.36)} u(kh).$$

For which $K \in \mathbb{R}$ is the closed loop system stable when (1) is used? **(3p)**

(d) What is the static gain of the *closed loop* system in (c)? **(1p)**

Problem 2 The block diagram below represents a stationary discrete-time stochastic process. The transfer operator $G(q)$ is minimum phase (with $G(1) \geq 0$), and w is zero mean white noise.

$\Phi_w(\omega) = 0.5$



$\Phi_y(\omega) = \frac{98}{1.64 - 1.6\cos\omega}$

(a) Determine the spectrum for z. **(2p)**

(b) Determine the transfer operator $G(q)$. **(4p)**

Problem 3 A continuous-time system has the state space representation

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_1(t), \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + v_2(t), \end{cases}$$

where v_1 and v_2 are uncorrelated, zero mean white noise processes with intensities $\Phi_{v_1}(\omega) = R_1 = 48$ and $\Phi_{v_2}(\omega) = R_2 = 1$.

(a) In order to estimate the state vector, the observer

$$\dot{\hat{x}}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \left(y(t) - \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}(t)\right)$$

is used. What are the observer poles? **(1p)**

(b) Determine the covariance matrix, $\Pi_{\tilde{x}} = E\tilde{x}(t)\tilde{x}(t)^T$, of the estimation error, $\tilde{x}(t) = x(t) - \hat{x}(t)$, for the observer in (a). **(3p)**

(c) Determine the spectrum, $\Phi_\nu(\omega)$, for the output innovations,

$$\nu(t) = y(t) - \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \tilde{x}(t) + v_2(t),$$

of the observer in (a).
Hint: First derive the transfer operators/functions from v_1 and v_2 to ν. **(3p)**

(d) The optimal observer is the Kalman filter. What is the covariance matrix of the estimation error $\tilde{x}(t)$ for the Kalman filter of the system?
Hint: In the solution of the associated Riccati equation the lower right diagonal element is $p_2 = 16$. **(4p)**

(e) What is the spectrum, $\Phi_\nu(\omega)$, for the output innovations of the Kalman filter? **(2p)**

Problem 4 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true"/"false" are considered!

(a) Controllability is always preserved under zero-order hold sampling.

(b) For a Kalman filter, based on a correct model and full knowledge of the noise, the *output innovations* are white noise.

(c) White noise is characterized by that its covariance function $r(\tau)$ is constant and non-zero for all time lags τ.

(d) White noise processes always have a constant spectrum.

(e) MPC can always be represented as an LTI controller.

(f) In MPC the *input/control horizon* is typically shorter than the *output/prediction horizon*.

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(6p)**

Problem 5 A discrete-time system is modeled by the difference equation

$$y(k)+y(k-1) = u(k-1)+e(k)-e(k-1) \;\Leftrightarrow\; (q+1)y(k) = u(k)+(q-1)e(k), \tag{2}$$

where $e(k)$ is zero mean white noise with intensity/variance $Ee(k)^2 = R_e = 1$. The system can also be represented in state space form as

$$\begin{cases} x(k+1) = -x(k) - u(k) + 2e(k), \\ y(k) = -x(k) + e(k), \end{cases} \tag{3}$$

or, equivalent, as

$$\begin{cases} x(k+1) = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} v(k), \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k), \end{cases} \tag{4}$$

where $v(k) = e(k+1) = qe(k)$ (i.e. $v(k)$ is white noise with the same properties as $e(k)$).

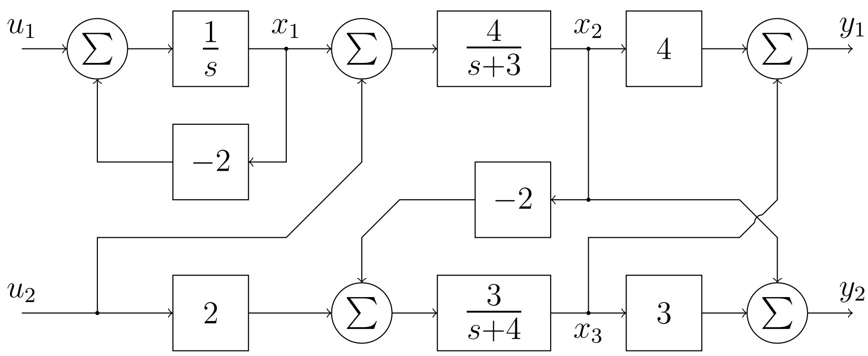
The system should be controlled by output feedback, $u(k) = -F_y(q)y(k)$, so that $V = Ey(k)^2$ is minimized ($\Rightarrow$ LQG with $Q_1 = 1$ and $Q_2 = 0$). The controller, $F_y(q)$, should be strictly proper so that $u(k)$ depends on $y(k-1)$ and previous outputs, but not on $y(k)$ (i.e. do not use $\hat{x}(k|k)$).

(a) Verify that (3) and (4) both are equivalent to (2). **(3p)**

(b) Determine the optimizing controller $F_y(q)$. Use either (3) or (4)[1]. (It suffices to give the corresponding K and L.) **(5p)**

(c) What are the poles of the closed loop system? **(1p)**

Problem 6 *The HW bonus points are exchangeable for this problem.*



(a) Give the state space model of the system in the block diagram above, with $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$, $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ and $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ as input, output and state vector, and with x_1, x_2 and x_3 as in the block diagram. **(4p)**

(b) Is the state space model in (a) stable and/or a minimal realization? **(3p)**

[1] Either way you will end up with the same $F_y(q)$.

Solutions to the exam in Automatic Control II, 2017-01-02:

1. (a)

$$y(t) = \frac{0.5}{(p+1)(p+0.5)}K(r(t) - y(t)) \Leftrightarrow y(t) = \frac{0.5K}{(p+1)(p+0.5)+0.5K}r(t).$$

For stability all poles must be in the left half plane. The poles are given by

$$0 = (p+1)(p+0.5) + 0.5K = p^2 + 1.5p + 0.5 + 0.5K.$$

For second order systems it is sufficient to have all coefficient positive, so the stability condition here is $0.5 + 0.5K > 0$, i.e. $K > -1$.

(b)

$$F = e^{Ah} = \begin{bmatrix} e^{-h} & 0 \\ 0 & e^{-0.5h} \end{bmatrix}, G = \int_0^h e^{At}Bdt = \int_0^h \begin{bmatrix} e^{-t} \\ e^{-0.5h} \end{bmatrix} dt = \begin{bmatrix} 1 - e^{-h} \\ 2(1 - e^{-0.5h}) \end{bmatrix}.$$

(c)

$$y(k) = \frac{0.16(q+0.6)}{(q-0.6)(q-0.36)}K(r(k) - y(k))$$

$$\Leftrightarrow \quad y(k) = \frac{0.16K(q+0.6)}{(q-0.6)(q-0.36) + 0.16K(q+0.6)}r(k).$$

The poles are given by

$$0 = (q-0.6)(q-0.36) + 0.16K(q+0.6) = q^2 + \underbrace{(0.16K - 0.96)}_{=\alpha} q + \underbrace{0.216 + 0.096K}_{=\beta}.$$

For stability all poles must lie within the unit circle, which for a second order system is equivalent to $|\alpha| - 1 < \beta < 1$:

$$\begin{aligned} \beta < 1: &\quad 0.216 + 0.096K < 1 \Leftrightarrow K < \frac{0.784}{0.096} = \frac{49}{6} \approx 8.167, \\ \alpha - 1 < \beta: &\quad 0.16K - 1.96 < 0.216 + 0.096K \Leftrightarrow K < \frac{2.176}{0.064} = 34, \\ -\alpha - 1 < \beta: &\quad -0.04 - 0.16K < 0.216 + 0.096K \Leftrightarrow K > \frac{-0.256}{0.256} = -1. \end{aligned}$$

Stable for $-1 < K < \frac{49}{6}$.

(d) The static gain is $G_c(1)$, and from (c) we that $G_c(1) = \frac{0.16K \cdot 1.6}{0.4 \cdot 0.64 + 0.16K \cdot 1.6} = \frac{0.256K}{0.256(1+K)} = \frac{K}{1+K}$.

2. (a) We have

$$\Phi_z(\omega) = \left|\frac{1.4}{e^{i\omega} - 0.7}\right|^2 \Phi_w(\omega) = \frac{1.4}{e^{i\omega} - 0.7} \cdot \frac{1.4}{e^{-i\omega} - 0.7} \cdot 0.5$$

$$= \frac{1.4^2 \cdot 0.5}{1 + 0.7^2 - 0.7(e^{i\omega} + e^{-i\omega})} = \frac{0.98}{1.49 - 1.4\cos\omega}.$$

(b) Again we can use that $\Phi_y(\omega) = |G(e^{i\omega})|^2\Phi_z(\omega)$, and from (a) we have $\Phi_z(\omega)$. Thus,

$$|G(e^{i\omega})|^2 = \frac{\Phi_y(\omega)}{\Phi_z(\omega)} = \frac{\frac{98}{1.64-1.6\cos\omega}}{\frac{0.98}{1.49-1.4\cos\omega}} = \frac{100(1.49-1.4\cos\omega)}{1.64-1.6\cos\omega}$$
$$= 100\cdot\frac{1.4}{1.6}\cdot\frac{-\frac{1.49}{1.4}+\cos\omega}{-\frac{1.64}{1.6}+\cos\omega}.$$

We try with

$$G(q) = K\cdot\frac{q+b}{q+a} \quad\Rightarrow\quad |G(e^{i\omega})|^2 = K^2\cdot\frac{(e^{i\omega}+b)(e^{-i\omega}+b)}{(e^{i\omega}+a)(e^{-i\omega}+a)}$$
$$= K^2\cdot\frac{1+b^2+b(e^{i\omega}+e^{-i\omega})}{1+a^2+a(e^{i\omega}+e^{-i\omega})} = K^2\cdot\frac{1+b^2+2b\cos\omega}{1+a^2+2a\cos\omega} = K^2\cdot\frac{b}{a}\cdot\frac{\frac{1+b^2}{2b}+\cos\omega}{\frac{1+a^2}{2a}+\cos\omega}.$$

Comparison with the expression above gives

$$\begin{cases} K^2\cdot\dfrac{b}{a} = 100\cdot\dfrac{1.4}{1.6}, \\ \dfrac{1+b^2}{2b} = -\dfrac{1.49}{1.4}, \\ \dfrac{1+a^2}{2a} = -\dfrac{1.64}{1.6}, \end{cases} \quad\Rightarrow\quad \begin{cases} b = \dfrac{-1.49\pm 0.51}{1.4} = -0.7, \\ a = \dfrac{-1.64\pm 0.36}{1.6} = -0.8, \\ K = (\pm)10. \end{cases}$$

Stationarity $\Leftrightarrow$ stability $\Leftrightarrow |a| < 1$, minimum phase $\Leftrightarrow |b| < 1$, and $G(1) \geq 0$ $\Leftrightarrow K \geq 0$. Thus $G(q) = 10\cdot\frac{q-0.7}{q-0.8}$

3. (a) The observer poles are given by

$$0 = \det(sI - A + KC) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s\end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -1\end{bmatrix} + \begin{bmatrix} 2 \\ 1\end{bmatrix}\begin{bmatrix} 1 & 0\end{bmatrix}\right)$$
$$= \det\begin{bmatrix} s+3 & -1 \\ 1 & s+1\end{bmatrix} = s^2+4s+4 = (s+2)^2.$$

Hence, the observer poles coincide in a double pole in -2.
(b) The estimation error is governed by the state equation

$$\dot{\tilde{x}} = (A-KC)\tilde{x} + Nv_1 - Kv_2, \tag{5}$$

and $\Pi_{\tilde{x}}$ solves the continuous-time Lyapunov equation

$$0 = (A-KC)\Pi_{\tilde{x}} + \Pi_{\tilde{x}}(A-KC)^T + NR_1N^T + KR_2K^T.$$

By noting that

$$A - KC = \begin{bmatrix} -1 & 1 \\ 0 & -1\end{bmatrix} - \begin{bmatrix} 2 \\ 1\end{bmatrix}\begin{bmatrix} 1 & 0\end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -1 & -1\end{bmatrix}$$
$$\Rightarrow\quad (A-KC)\Pi_{\tilde{x}} = \begin{bmatrix} -3 & 1 \\ -1 & -1\end{bmatrix}\begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2\end{bmatrix} = \begin{bmatrix} -3\pi_1+\pi_{12} & -3\pi_{12}+\pi_2 \\ -\pi_1-\pi_{12} & -\pi_{12}-\pi_2\end{bmatrix}$$

the Lyapunov equation can be spelled out as

$$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}-3\pi_1+\pi_{12} & -3\pi_{12}+\pi_2\\ -\pi_1-\pi_{12} & -\pi_{12}-\pi_2\end{bmatrix} + \begin{bmatrix}-3\pi_1+\pi_{12} & -\pi_1-\pi_{12}\\ -3\pi_{12}+\pi_2 & -\pi_{12}-\pi_2\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 48\end{bmatrix} + \begin{bmatrix}4 & 2\\ 2 & 1\end{bmatrix}.$$

Element by element this gives the linear equation system

$$\begin{cases} 0 = -6\pi_1 + 2\pi_{12} + 4, \\ 0 = -\pi_1 - 4\pi_{12} + \pi_2 + 2, \\ 0 = -2\pi_{12} - 2\pi_2 + 49, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \pi_1 = \dfrac{73}{32} = 2.28125, \\ \pi_{12} = \dfrac{155}{32} = 4.84375, \\ \pi_2 = \dfrac{629}{32} = 19.65625. \end{cases}$$

(c) We note that $\nu = y - C\hat{x} = C\tilde{x} + v_2$, which together with (5) gives

$$\begin{cases} \dot{\tilde{x}} = (A - KC)\tilde{x} + Nv_1 - Kv_2, \\ \nu = C\tilde{x} + v_2, \end{cases}$$

as state space representation for ν. Thus,

$$\begin{aligned} \nu(t) &= \underbrace{C(pI - A + KC)^{-1}N}_{=G_1(p)} v_1(t) + \underbrace{[1 - C(pI - A + KC)^{-1}K]}_{=G_2(p)} v_2(t) \\ &= \begin{bmatrix} G_1(p) & G_2(p) \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}. \end{aligned}$$

The spectrum of ν then is (in accordance with Eq. (5.15))

$$\Phi_\nu(\omega) = \begin{bmatrix} G_1(i\omega) & G_2(i\omega) \end{bmatrix} \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} G_1(-i\omega) \\ G_2(-i\omega) \end{bmatrix} = |G_1(i\omega)|^2 R_1 + |G_2(i\omega)|^2 R_2.$$

Since $C(pI - A + KC)^{-1}$ is part of both $G_1(p)$ and $G_2(p)$ we compute that first:

$$C(pI - A + KC)^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p+3 & -1 \\ 1 & p+1 \end{bmatrix}^{-1} = \frac{1}{(p+2)^2} \begin{bmatrix} p+1 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} G_1(p) &= \frac{1}{(p+2)^2} \begin{bmatrix} p+1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(p+2)^2} \\ \Rightarrow \quad |G_1(i\omega)|^2 &= \frac{1}{|(i\omega^2+2)^2|^2} = \frac{1}{(\omega^2+4)^2}, \end{aligned}$$

and

$$\begin{aligned} G_2(p) &= 1 - \frac{1}{(p+2)^2} \begin{bmatrix} p+1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1 - \frac{2p+3}{(p+2)^2} \\ &= \frac{p^2+2p+1}{(p+2)^2} = \frac{(p+1)^2}{(p+2)^2} \qquad \Rightarrow \qquad |G_2(i\omega)|^2 = \frac{(\omega^2+1)^2}{(\omega^2+4)^2}. \end{aligned}$$

The spectrum is then

$$\Phi_\nu(\omega) = 48|G_1(i\omega)|^2 + |G_2(i\omega)|^2 = \frac{48 + (\omega^2+1)^2}{(\omega^2+4)^2} = 1 + \frac{-6\omega^2+33}{(\omega^2+4)^2}.$$

(d) The covariance of the estimation error for the Kalman filter, $E\tilde{x}\tilde{x}^T = P$, is the solution of the CARE

$$0 = AP + PA^T + NR_1N^T - PC^TR_2^{-1}CP.$$

Set $P = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$ $\Rightarrow$ the CARE spells out as

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} + \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} + 48\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$- \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}.$$

Element by element this gives the equation system

$$\begin{cases} 0 = -2p_1 + 2p_{12} - p_1^2, \\ 0 = -2p_{12} + p_2 - p_1p_{12}, \\ 0 = -2p_2 + 48 - p_{12}^2, \end{cases}$$

Since we know that $p_2 = 16$ we get

$$\begin{cases} 0 = -2p_1 + 2p_{12} - p_1^2, \\ 0 = -2p_{12} + 16 - p_1p_{12} = 16 - p_{12}(2+p_1), \\ 0 = -32 + 48 - p_{12}^2 = 16 - p_{12}^2. \end{cases}$$

The last equation gives that $p_{12} = \pm 4$, and since $p_1 \geq 0$ must hold ($\Leftarrow P \geq 0$), the second equation gives that $p_{12} > 0$. Thus,

$$\begin{cases} p_1 = 2, \\ p_{12} = 4, \\ p_2 = 16, \end{cases} \quad \Leftrightarrow \quad P = \begin{bmatrix} 2 & 4 \\ 4 & 16 \end{bmatrix}.$$

(e) Theorem 5.5 $\Rightarrow \Phi_\nu(\omega) = R_2 = 1$.

4. **(a)** False (It may be lost for certain choices of sampling interval); **(b)** True (Theorem 5.5); **(c)** False ($r(\tau) = 0$ for $\tau \neq 0$); **(d)** True (Definition 5.2); **(e)** False (In general MPC is nonlinear and time-varying); **(f)** True (Simply true...)

5. **(a)** Use that

$$\begin{cases} qx = Fx + Gu + Nv_1, \\ y = Hx + v_2, \end{cases} \quad \Rightarrow y = H(qI - F)^{-1}(Gu + Nv_1) + v_2,$$

and then retrieve the difference equation from that. With (3) we have

$$F = G = H = -1,\ N = 2 \quad \text{and} \quad v_1 = v_2 = e. \tag{3'}$$

Thus,

$$y = \frac{-1}{q+1}(-u + 2e) + e \quad \Leftrightarrow \quad (q+1)y = u + (q-1)e,$$

which is equivalent to (2). With (4) we have

$$F = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, N = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \end{bmatrix}, v_1 = v \text{ and } v_2 = 0. \tag{4'}$$

Then we get

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q+1 & -1 \\ 0 & q \end{bmatrix}^{-1} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ -1 \end{bmatrix} v \right) = \frac{q}{q(q+1)} u + \frac{q-1}{q(q+1)} v$$
$$\Leftrightarrow \quad q(q+1)y = qu + (q-1)v \quad \Leftrightarrow \quad (q+1)y = u + (q-1)e,$$

since $v = qe$. Again we have equivalence with (2).

(b) Let $\hat{x}$ denote $\hat{x}(k|k-1)$. In LQG the control law is $u = -L\hat{x}$, where a Kalman filter (KF), $q\hat{x} = F\hat{x} + Gu + K(y - H\hat{x})$ (see Theorem 9.4) provides $\hat{x}$. The Kalman gain is $K = (FPH^T + NR_{12})(HPH^T + R_2)^{-1}$, where P solves the DARE

$$P = FPF^T + NR_1N^T - (FPH^T + NR_{12})(HPH^T + R_2)^{-1}(FPH^T + NR_{12})^T.$$

The feedback gain is $L = (G^TSG + Q_2)^{-1}G^TSF$, where S solves the DARE

$$S = F^TSF + M^TQ_1M - F^TSG(G^TSG + Q_2)^{-1}G^TSF.$$

Here we note that $Q_1 = 1$ and $Q_2 = 0$. With (3): From (3') we note that $R_1 = R_2 = R_{12} = 1$, so for the KF we get $K = \frac{P+2}{P+1}$ and the DARE

$$P = P + 4 - \frac{(P+2)^2}{P+1} \quad \Leftrightarrow \quad P^2 = 0 \quad \Leftrightarrow \quad P = 0.$$

Thus $K = 2$. The feedback gain is $L = \frac{S}{S} = 1$ (we need not solve the DARE). With (4): From (4') we get that $R_1 = 1$ and $R_2 = R_{12} = 0 \Rightarrow HPH^T + R_2 = p_1$. To set up the DARE we can start by computing FP, and then we readily get that

$$FPF^T = \begin{bmatrix} p_1 - 2p_{12} + p_2 & 0 \\ 0 & 0 \end{bmatrix}, FPH^T = \begin{bmatrix} -p_1 + p_{12} \\ 0 \end{bmatrix}, NR_1N^T = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The DARE then spells out as

$$\begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} = \begin{bmatrix} p_1 - 2p_{12} + p_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} \frac{(-p_1+p_{12})^2}{p_1} & 0 \\ 0 & 0 \end{bmatrix} \quad \Leftrightarrow \quad P = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The Kalman gain then is $K = \begin{bmatrix} \frac{-p_1+p_{12}}{p_1} \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$. Since $Q_2 = 0$ we get $G^T SG + Q_2 = s_1$, and $L = \frac{1}{s_1} G^T SF = \frac{1}{s_1} \begin{bmatrix} -s_1 & s_1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$ (so neither here we need to solve the DARE). (The feedback filter is $F_y(q) = L(qI - F + KH + GL)^{-1}K$, and in both cases we get $F_y(q) = \frac{2}{q-2}$.)
(c) The closed loop poles are those from the state feedback, solving $0 = \det(zI - F + GL)$, and the observer poles, solving $0 = \det(zI - F + KH)$. With the (3) solution we get $zI - F + GL = z + 1 - 1 = z$, i.e. a pole in the origin, and $zI - F + KH = z + 1 - 2 = z - 1$, i.e. a pole in 1. (For the (4) solution we get two additional poles in the origin, but these are cancelled out in the closed loop system.)
6. (a) From the block diagram we get

$$\begin{aligned}
x_1 &= \frac{1}{p}(-2x_1 + u_1) &&\Leftrightarrow\quad px_1 = -2x_1 + u_1, \\
x_2 &= \frac{4}{p+3}(x_1 + u_2) &&\Leftrightarrow\quad px_2 = 4x_1 - 3x_2 + 4u_2, \\
x_3 &= \frac{3}{p+4}(-2x_2 + 2u_2) &&\Leftrightarrow\quad px_3 = -6x_2 - 4x_3 + 6u_2.
\end{aligned}$$

We also see that $y_1 = 4x_2 + x_3$ and $y_2 = x_2 + 3x_3$. All together we get

$$\begin{cases}
\dot{x}(t) = \overbrace{\begin{bmatrix} -2 & 0 & 0 \\ 4 & -3 & 0 \\ 0 & -6 & -4 \end{bmatrix}}^{=A} x(t) + \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 6 \end{bmatrix}}^{=B} u(t), \\
y(t) = \underbrace{\begin{bmatrix} 0 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}}_{=C} x(t).
\end{cases}$$

(b) The poles are the eigenvalues of A. Since A is triangular all eigenvalues are on the diagonal, and these are all negative. Hence the system is stable. Minimal realization $\Leftrightarrow$ both controllable and observable:

$$\mathcal{S} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & \\ 0 & 4 & 4 & \cdots \\ 0 & 6 & 0 & \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 1 \\ 0 & 1 & 3 \\ 16 & -18 & -12 \\ & \vdots & \end{bmatrix}$$

Both $\mathcal{S}$ and $\mathcal{O}$ have full rank (it suffices to check the three first columns and rows). Thus, the system is both controllable and observable, and thereby a minimal realization.