

# Exam in Automatic Control II

## Reglerteknik II 5hp (1RT495)

**Date:** May 31, 2017

**Venue:** Fyrishov

**Responsible teacher:** Hans Rosth.

**Aiding material:** Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

**Preliminary grades:** 23p for grade 3, 33p for grade 4, 43p for grade 5.

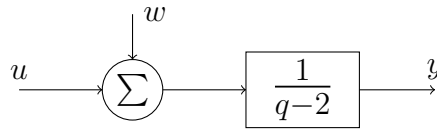
**Use separate sheets** for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

**Important:** Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

**Problem 6** is an alternative to the homework assignments. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Good luck!

**Problem 1** The block diagram below shows an unstable discrete-time system, affected by a disturbance  $w$ .



The disturbance is a stochastic process with  $Ew(k) = 0$ , and with spectral density

$$\Phi_w(\omega) = \frac{1}{1.5 + \sqrt{2} \cos \omega}. \quad (1)$$

In order to keep the output  $y$  close to zero the system must be stabilized by a feedback controller

$$u(k) = -F_y(q)y(k). \quad (2)$$

First, in (a)–(d), we consider proportional control, so that  $F_y(q) = K$ .

(a) For which gains  $K \in \mathbb{R}$  is the closed loop system stable? **(1p)**

(b) Assume that a stabilizing  $K$  is used, so that  $y$  is a stationary stochastic process. Determine the spectral density of the output  $y$ . **(2p)**

(c) Determine the value of  $\alpha$  so that the difference equation

$$w(k+1) + \alpha w(k) = v(k), \quad Ev(k) = 0, \quad \Phi_v(\omega) = 1, \quad (3)$$

is a model of the stochastic process  $w$ , with spectral density as in (1). The disturbance  $w$  is stationary, and so should the model (3) be. **(2p)**

(d) Determine the variance of the control input,  $Eu(k)^2$ , when the proportional control  $u = -2y$  is used. (Assume that the closed loop system is stable.)

*Hint:* Set up a state space model of the closed loop system, by use of the block diagram and (3). Then compute the covariance matrix of the combined state vector. **(3p)**

(e) Assume that the stabilization of the system should be obtained with as small control effort as possible. That is, one would like to find a stabilizing controller  $F_y(q)$  such that, when (2) is used, the criterion/cost function

$$V = Eu(k)^2$$

is minimized.

Describe how the optimal controller is obtained. In order to obtain full score your description should include

- a correct model of the system (based on the block diagram and (3)),
- and the necessary, correctly stated equations,

all with correct numerical values. You need/should not solve for  $F_y(q)$ . **(4p)**

**Problem 2**

(a) Give a state space model for the system

$$y(k) - 1.5y(k-1) + 0.56y(k-2) = 2u_1(k) - 0.6u_1(k-1) - 0.8u_1(k-2) + u_2(k-2). \quad (3p)$$

(b) Now consider the system with state space representation

$$\begin{cases} x(k+1) = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 2.4 & -10 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}, \\ y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}. \end{cases} \quad (4)$$

Is the state space representation (4) a minimal realization? (2p)

(c) Is (4) controllable from  $u_1$ ? (1p)

(d) For the system (4), assume that  $x(0) = [10 \ 10]^T$  and that  $u_1(k) = u_2(k) = 0$  for  $k \geq 0$ . Compute the value of  $x(3)$ . (2p)

**Problem 3** The unstable system

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + v_2(t), \end{cases} \quad (5)$$

is to be stabilized by output feedback. Here  $v_2$  is zero mean white measurement noise with intensity  $\Phi_2(\omega) = R_2 = 1$ . (Notice that there is no process noise affecting the system, and thus  $R_1 = R_{12} = 0$ .)

The intention is to use state feedback control, and therefore an observer is required.

(a) Determine the Kalman filter for the system (5).

*Hint:* The solution of the associated Riccati equation is strictly positive definite. (4p)

(b) Determine the observer poles for the Kalman filter in (a). Also give the spectral density for the output innovations,

$$\nu(t) = y(t) - [1 \ 0] \hat{x}(t) = [1 \ 0] \tilde{x}(t) + v_2(t), \quad (6)$$

of the Kalman filter. (2p)

(c) Assume that, instead of the Kalman filter, an observer based on pole placement is used for estimation of  $x$  in (5). Determine the spectral density for the output innovations (given by (6)) for the observer with observer polynomial  $s^2 + 4s + 8$ . (4p)

**Problem 4** A double tank system (similar to the one in the MPC demo lab) has the linearized model

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 1 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t). \end{cases}$$

Here  $y(t)$  is the water level in the lower tank and  $u(t)$  is the flow into the upper tank (from a pump).

The controller is implemented as a sampling state feedback controller, with a zero-order hold (ZOH) circuit. That is,

$$u(t) = u(kh) \text{ for } kh \leq t < kh + h, \text{ and } u(kh) = -Lx(kh) + mr(kh), \quad (7)$$

where  $h$  is the sampling interval. Discrete-time control design then requires a discrete-time model of the system:

$$\begin{cases} x(kh + h) = Fx(kh) + Gu(kh), \\ y(kh) = Hx(kh) \end{cases} \Leftrightarrow y(kh) = H(qI - F)^{-1}Gu(kh).$$

(a) Determine the ZOH sampled model of the system, i.e. give  $F$ ,  $G$  and  $H$  in the state space model above. The answer should be expressed in the sampling period  $h$ . **(4p)**

(b) The state feedback gain  $L$  in (7) should be determined by use of the pole placement technique. It is found that the performance of the closed loop system would be satisfactory for the *continuous-time* pole polynomial  $s^2 + 2s + 1$  (if a continuous-time controller is used).

What *discrete-time* closed loop pole polynomial,  $q^2 + aq + b$ , should one aim for in order to have an equivalent performance when the sampling controller (7) is used? The answer should be expressed in the sampling period  $h$ . (You need not compute  $L$ .) **(2p)**

(c) For a certain choice of sampling period the ZOH sampled model is

$$y(kh) = \frac{0.5(q + 0.5)}{q^2 - 0.75q + 0.125}u(kh).$$

The desired closed loop pole polynomial is then  $q^2 - 0.5q + 0.0625$ . Determine the value of  $m$  in (7) such that there is unit static gain from  $r$  to  $y$ . **(1p)**

**Problem 5** Specify for each of the following statements whether it is true or false. No motivations required — only answers “true”/“false” are considered!

- (a)  $\Phi(\omega) = \frac{4}{1.44+2\cos\omega}$  is the spectrum of a discrete-time stochastic process.
- (b) A standard model of continuous-time white noise has the spectrum  $\Phi(\omega) = \frac{1}{\omega^2+\epsilon^2}$ , with  $\epsilon \rightarrow 0$ .
- (c) White noise is characterized by that its covariance function  $r(\tau)$  is constant and non-zero for all time lags  $\tau$ .
- (d) The sampling frequency is 50% of the Nyquist frequency.
- (e) In MPC the computational load typically increases with the *control/input horizon*.
- (f) In MPC the *control/input horizon* is typically shorter than the *prediction/output horizon*.

Each correct answer scores +1, each incorrect answer scores -1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(6p)**

**Problem 6** *The HW bonus points are exchangeable for this problem.*

Consider the continuous-time stochastic process

$$\dot{x}(t) = Ax(t) + Nv(t), \quad A = \begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix}, \quad N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Ev(t) = 0, \quad \Phi_v(\omega) = 32.$$

- (a) What is the covariance matrix,  $\Pi_x^c = Ex(t)x(t)^T$ , of the state vector for this stochastic process? **(2p)**
- (b) A discrete-time model representation of the stochastic process above is

$$x(kh+h) = Fx(kh) + w(kh), \quad F = e^{Ah}, \quad R_w = Ew(kh)w(kh)^T = \begin{bmatrix} r_1 & r_{12} \\ r_{12} & r_2 \end{bmatrix}.$$

Compute the matrix  $F$  when the sampling period is  $h = 0.5$ . **(1p)**

- (c) In the discrete-time model the process noise is white and vector valued, and its covariance  $R_w$  is a  $2 \times 2$  matrix. The discrete-time model is such that the covariance matrix for its state vector is identical to the one for the continuous-time stochastic process, i.e.,  $\Pi_x^d = Ex(kh)x(kh)^T = Ex(t)x(t)^T = \Pi_x^c$ . Compute  $R_w$  so that this property holds. **(4p)**

**Solutions to the exam in Automatic Control II, 2017-05-31:**

1. (a) From the block diagram, and from  $u = -Ky$ , we get

$$y = \frac{1}{q-2}(u+w) = \frac{1}{q-2}(-Ky+w) \Leftrightarrow \left(1 + \frac{K}{q-2}\right)y = \frac{1}{q-2}w$$

$$\Leftrightarrow y = \frac{\frac{1}{q-2}}{1 + \frac{K}{q-2}}w = \frac{1}{q-2+K}w.$$

The pole of the closed loop system is given by  $0 = q-2+K$ , and for stability the pole must be inside the unit circle. Thus,

$$|2-K| < 1 \Leftrightarrow 1 < K < 3.$$

(b) For  $y = G(q)w$  the output spectrum is  $\Phi_y(\omega) = |G(e^{i\omega})|^2\Phi_w(\omega)$ , and from (a) we have  $G(q) = \frac{1}{q-2+K}$ . Hence,

$$\begin{aligned} \Phi_y(\omega) &= \frac{1}{|e^{i\omega}-2+K|^2}\Phi_w(\omega) = \frac{1}{(e^{i\omega}-2+K)(e^{-i\omega}-2+K)}\Phi_w(\omega) \\ &= \frac{1}{1+(K-2)^2+(K-2)(e^{i\omega}+e^{-i\omega})}\Phi_w(\omega) \\ &= \frac{1}{(1+(K-2)^2+2(K-2)\cos\omega)(1.5+\sqrt{2}\cos\omega)}. \end{aligned}$$

(c) Again, use that  $\Phi_w(\omega) = |G_w(e^{i\omega})|^2\Phi_v(\omega)$ . Here

$$G_w(q) = \frac{1}{q+\alpha} \Rightarrow \Phi_w(\omega) = \frac{1}{1+\alpha^2+2\alpha\cos\omega},$$

and by comparing with (1) we get  $1+\alpha^2=1.5$  and  $2\alpha=\sqrt{2} \Rightarrow \alpha=\sqrt{0.5}$ .

(d) From the block diagram, and from (3), we have

$$\begin{cases} qy = 2y + w + (-2y) = w, \\ qw = -\alpha w + v, \end{cases} \Leftrightarrow \begin{cases} qx = Fx + Nv, \\ y = Hx, \end{cases}$$

with

$$x = \begin{bmatrix} y \\ w \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H = [1 \quad 0].$$

The covariance matrix  $\Pi_x = Exx^T$  solves the discrete-time Lyapunov equation  $\Pi_x = F\Pi_x F^T + NR_v N^T$ , and since  $u = -2y = -2Hx$ , the sought variance is  $Eu^2 = E(-2Hx)(-2Hx)^T = 2^2 H\Pi_x H^T = 4\pi_1$ . Spelling out the Lyapunov equation one obtains the equation system

$$\begin{cases} \pi_1 = \pi_2, \\ \pi_{12} = -\alpha\pi_2, \\ \pi_2 = \alpha^2\pi_2 + 1, \end{cases} \Leftrightarrow \begin{cases} \pi_1 = \frac{1}{1-\alpha^2} = 2, \\ \pi_{12} = -\frac{\alpha}{1-\alpha^2} = -2\sqrt{0.5}, \\ \pi_2 = \frac{1}{1-\alpha^2} = 2, \end{cases}$$

Hence,  $Eu^2 = 4 \cdot 2 = 8$ .

(e) From the block diagram, and from (3), we have

$$\begin{cases} qy = 2y + w + u, \\ qw = -\alpha w + v, \end{cases} \Leftrightarrow \begin{cases} qx = Fx + Gu + Nv, \\ y = Hx, \end{cases}$$

with

$$x = \begin{bmatrix} y \\ w \end{bmatrix}, \quad F = \begin{bmatrix} 2 & 1 \\ 0 & -\alpha \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H = [1 \ 0].$$

No measurement noise  $\Rightarrow R_1 = \Phi_v(\omega) = 1$ ,  $R_{12} = R_2 = 0$ . The criterion  $V = Eu^2 \Leftrightarrow Q_1 = 0$  and  $Q_2 = 1$ . The optimal controller is the state feedback  $u = -L\hat{x}(k|k-1)$ , see Theorem 9.4 (or, actually  $u = -L\hat{x}(k|k)$  — see Corollary 9.2) where  $\hat{x}$  is obtained from a Kalman filter. The optimal state feedback gain is  $L = (G^T S G + Q_2)^{-1} G^T S F = \frac{1}{s_1+1} [2s_1 \ s_1 - \alpha s_{12}]$ , where  $S$  is the solution of the DARE

$$S = F^T S F - F^T S G (G^T S G + Q_2)^{-1} G^T S F.$$

The Kalman gain is  $K = F P H^T (H P H^T)^{-1} = \frac{1}{p_1} [2p_1 + p_{12} \ -\alpha p_{12}]^T$ , where  $P$  is the solution of the DARE

$$P = F P F^T + N R_1 N^T - F P H^T (H P H^T)^{-1} H P F^T.$$

(... and the estimate  $\hat{x}(k|k) = \hat{x}(k|k-1) + \tilde{K}(y(k) - H\hat{x}(k|k-1))$ , with  $\tilde{K} = P H^T (H P H^T)^{-1}$ .)

**2. (a)** Only one output  $\Rightarrow$  the observer canonical forms works. Rewrite the difference equation:

$$\begin{aligned} (q^2 - 1.5q + 0.56)y(k) &= (2q^2 - 0.6q - 0.8)u_1(k) + u_2(k) \Leftrightarrow \\ y(k) &= \frac{2q^2 - 0.6q - 0.8}{q^2 - 1.5q + 0.56}u_1(k) + \frac{1}{q^2 - 1.5q + 0.56}u_2(k) \\ &= \left(2 + \frac{2.4q - 1.92}{q^2 - 1.5q + 0.56}\right)u_1(k) + \frac{1}{q^2 - 1.5q + 0.56}u_2(k). \end{aligned}$$

The observer canonical form then is

$$\begin{cases} x(k+1) = \begin{bmatrix} 1.5 & 1 \\ -0.56 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 2.4 & 0 \\ -1.92 & 1 \end{bmatrix} u(k), \\ y(k) = [1 \ 0] x(k) + [2 \ 0] u(k). \end{cases}$$

**(b)** Minimal realization  $\Leftrightarrow$  both controllable and observable. The controllability matrix is is

$$\mathcal{S} = [G \ FG] = \begin{bmatrix} 2.4 & -10 & 1.68 & -7 \\ 0 & 10 & 0 & 8 \end{bmatrix},$$

and it has full rank  $\Leftrightarrow$  the system is controllable. The observability matrix is

$$\mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0.7 & 0.8 \end{bmatrix},$$

and has full rank  $\Leftrightarrow$  the system is observable. Hence the state space representation is minimal.

(c) The controllability matrix for  $u_1$  alone is

$$\mathfrak{S}_1 = \begin{bmatrix} 2.4 & 1.68 \\ 0 & 0 \end{bmatrix},$$

which has not full rank. It is not controllable from  $u_1$  alone.

(d) The solution of the state equation is (see eq. (3.41))

$$x(k) = F^k x(0) + \sum_{l=0}^{k-1} F^{k-1-l} G u(l),$$

and since  $u(k) = 0$  we get

$$x(3) = F^3 x(0) = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.8 \end{bmatrix}^3 \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 0.7^3 & 0 \\ 0 & 0.8^3 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \cdot 0.7^3 \\ 10 \cdot 0.8^3 \end{bmatrix} = \begin{bmatrix} 3.43 \\ 5.12 \end{bmatrix}.$$

**3. (a)** Continuous-time system  $\Rightarrow$  use Theorem 5.4:

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}), \quad K = (PC^T + NR_{12})R_2^{-1}$$

where  $P = P^T \geq 0$  solves the CARE

$$0 = AP + PA^T - NR_1 N^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T.$$

Here we have  $R_1 = R_{12} = 0$ , so the CARE spells out like

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} + \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_{12} \end{bmatrix} \begin{bmatrix} p_1 & p_{12} \end{bmatrix}.$$

By viewing element by element we obtain the equation system

$$\begin{cases} 0 = -2p_1 + 2p_{12} - p_1^2, \\ 0 = 2p_1 - p_{12} + p_2 - p_1 p_{12}, \\ 0 = 4p_{12} - p_{12}^2. \end{cases}$$

The last equation gives that  $p_{12} = 0$  or  $p_{12} = 4$ . With  $p_{12} = 0$  the condition  $P = P^T \geq 0 \Rightarrow P = 0 \Rightarrow A - KC = A$  is unstable. Thus  $p_{12} = 4$ , and we get

$$P = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \quad \Rightarrow \quad K = PC^T R_2^{-1} = \begin{bmatrix} p_1 \\ p_{12} \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$



(b) The poles are the zeros of the observer polynomial,

$$0 = \det(sI - A + KC) = \det \begin{bmatrix} s + 1 + 2 & -1 \\ -2 + 4 & s \end{bmatrix} = s(s+3) + 2 = s^2 + 3s + 2 = (s+1)(s+2).$$

Thus, the observer poles are  $-1$  and  $-2$ . Theorem 5.5 gives that  $\Phi_\nu(\omega) = R_2 = 1$ .

(c) For any observer the following holds:

$$\begin{cases} \dot{\tilde{x}} = (A - KC)\tilde{x} + Nv_1 - Kv_2, \\ \nu = C\tilde{x} + v_2. \end{cases}$$

Since  $v_1 = 0$  in this case, we get  $\nu = G_\nu(p)v_2$ , with  $G_\nu(p) = 1 - C(pI - A + KC)^{-1}K$ , and then  $\Phi_\nu(\omega) = |G_\nu(i\omega)|^2 R_2$ . Here

$$\begin{aligned} G_\nu(s) &= 1 - [1 \ 0] \begin{bmatrix} s + 1 + k_1 & -1 \\ -2 + k_2 & s \end{bmatrix}^{-1} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 1 - \frac{k_1 s + k_2}{s^2 + (1 + k_1)s - 2 + k_2} \\ &= \frac{s^2 + (1 + k_1)s - 2 + k_2 - (k_1 s + k_2)}{s^2 + (1 + k_1)s - 2 + k_2} = \frac{s^2 + s - 2}{s^2 + (1 + k_1)s - 2 + k_2}. \end{aligned}$$

The observer polynomial is chosen to  $s^2 + 4s + 8$  (corresponds to  $k_1 = 3$  and  $k_2 = 10$ ), and thus  $G_\nu(s) = \frac{s^2 + s - 2}{s^2 + 4s + 8}$ . The spectrum becomes

$$\Phi_\nu(\omega) = \frac{((i\omega)^2 + i\omega - 2)((-i\omega)^2 - i\omega - 2)}{((i\omega)^2 + i4\omega + 8)((-i\omega)^2 - i4\omega + 8)} = \frac{\omega^4 + 5\omega^2 + 4}{\omega^4 + 64}.$$

4. (a) Use Theorem 4.1. Then we need  $e^{At}$ , which is obtained by the Laplace transform:

$$\begin{aligned} \mathcal{L}[e^{At}] &= (sI - A)^{-1} = \begin{bmatrix} s + 1 & 0 \\ -1 & s + 0.5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{(s+1)(s+0.5)} & \frac{1}{s+0.5} \end{bmatrix}, \\ &\Leftrightarrow e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 2(e^{-0.5t} - e^{-t}) & e^{-0.5t} \end{bmatrix} \end{aligned}$$

Hence,

$$\begin{aligned} F &= e^{Ah} = \begin{bmatrix} e^{-h} & 0 \\ 2(e^{-0.5h} - e^{-h}) & e^{-0.5h} \end{bmatrix}, \quad H = C = [0 \ 1], \\ G &= \int_0^h e^{At} B dt = \int_0^h \begin{bmatrix} e^{-t} \\ 2(e^{-0.5t} - e^{-t}) \end{bmatrix} dt = \begin{bmatrix} 1 - e^{-h} \\ 2 - 4e^{-0.5h} + 2e^{-h} \end{bmatrix}. \end{aligned}$$

(b) For ZOH sampling continuous-time poles  $p$  are mapped on discrete-time poles  $e^{ph}$ . The continuous-time pole polynomial here corresponds to a double pole in  $-1$ , so the corresponding discrete-time poles should be a double pole in  $e^{-h}$ . This corresponds to the discrete-time pole polynomial

$$(q - e^{-h})^2 = q^2 - 2e^{-h}q + e^{-2h}.$$

(c) The closed loop system will be

$$y(kh) = \frac{0.5(q + 0.5)}{q^2 - 0.5q + 0.0625}mr(kh) = G_c(q)mr(kh).$$

Static gain corresponds to setting  $q = 1$ . In order to have unit static gain we must have

$$1 = G_c(1)m = \frac{0.5(1 + 0.5)m}{1 - 0.5 + 0.0625} \Leftrightarrow m = \frac{1 - 0.5 + 0.0625}{0.5(1 + 0.5)} = \frac{0.5625}{0.75} = 0.75.$$

5. (a) False (A spectrum is always non-negative); (b) False (White noise has constant spectrum); (c) False (For white noise  $r(\tau) = 0$  for  $\tau \neq 0$ ); (d) False (The opposite holds by definition); (e) True (Control horizon = dimension of the optimization problem); (f) True (Simply true)

6. (a)  $\Pi_x^c$  solves the continuous-time Lyapunov equation,  $0 = A\Pi_x^c + \Pi_x^c A^T + NR_v N^T$ :

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} + \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} 32 \begin{bmatrix} 1 & 0 \end{bmatrix} \Leftrightarrow \\ &\begin{cases} 0 = -4\pi_1 + 32, \\ 0 = \pi_1 - 4\pi_{12}, \\ 0 = 2(\pi_{12} - 2\pi_2), \end{cases} \Leftrightarrow \begin{cases} \pi_1 = 8, \\ \pi_{12} = 2, \\ \pi_2 = 1, \end{cases} \Leftrightarrow \Pi_x^c = \begin{bmatrix} 8 & 2 \\ 2 & 1 \end{bmatrix}. \end{aligned}$$

(b) Use that  $\mathcal{L}[e^{At}] = (sI - A)^{-1}$ :

$$(sI - A)^{-1} = \begin{bmatrix} s+2 & 0 \\ -1 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+2)^2} & \frac{1}{s+2} \end{bmatrix} \Leftrightarrow e^{At} = \begin{bmatrix} e^{-2t} & 0 \\ te^{-2t} & e^{-2t} \end{bmatrix}$$

Thus

$$F = \begin{bmatrix} e^{-1} & 0 \\ 0.5e^{-1} & e^{-1} \end{bmatrix} = e^{-1} \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}.$$

(c) Use the discrete-time Lyapunov equation,  $\Pi_x^d = F\Pi_x^d F^T + R_w$  and that  $\Pi_x^d = \Pi_x^c \Rightarrow$

$$R_w = \Pi_x^c - F\Pi_x^c F^T = \begin{bmatrix} 8 & 2 \\ 2 & 1 \end{bmatrix} - e^{-1} \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} 8 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} e^{-1} = \begin{bmatrix} 8(1 - e^{-2}) & 2 - 6e^{-2} \\ 2 - 6e^{-2} & 1 - 5e^{-2} \end{bmatrix}.$$