Exam in Automatic Control II
Reglerteknik II 5hp (1RT495)

Date: May 29, 2018

Venue: Polacksbacken, skrivsalen

Responsible teacher: Hans Rosth.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

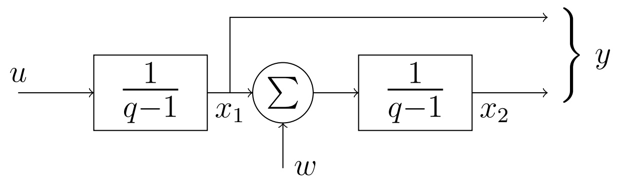
Problem 6 is an alternative to the homework assignments from the spring semester 2018. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Please use English in your solutions when possible, that would be appreciated!

Have a nice summer!

Please use English in your solutions!

Problem 1 The block diagram below shows a version of a discrete-time double integrator.



A state space model of the system, with state vector $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, is

$$\begin{cases} x(k+1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(k), \\ y(k) = x(k). \end{cases} \tag{1}$$

Here w is a process disturbance. The double integrator should be stabilized, and since the full state vector is measured (with no measurement noise) pure state feedback can be used, ie. the control law $u(k) = -Lx(k)$ can be applied.

(a) Initially it was assumed that the process disturbance is negligible, and it was set to $w \equiv 0$ so that the model (1) became purely deterministic. The feeback gain L was then computed by solving the LQ problem where the criterion function

$$V_a = \sum_{k=0}^{\infty} \left\{ y^T(k) Q_1 y(k) + 0.1 u^2(k) \right\}, \quad Q_1 \in \mathbb{R}^{2\times 2},$$

is minimized. It turned out that the solution of the associated Riccati equation is

$$S = \begin{bmatrix} 0.9 & 0.4 \\ 0.4 & 0.52 \end{bmatrix}.$$

How was the weighting matrix Q_1 chosen? Determine the value of Q_1 (a 2×2 matrix) such that S above solves the associated Riccati equation. **(3p)**

(b) Determine the poles of the closed loop system when the LQ controller in (a) is used. **(2p)**

(c) Later it turned out that the process disturbance is *not* negligible, but that it can be modeled as

$$w(k) = \frac{2}{q+0.9} v(k), \quad Ev(k) = 0, \quad \Phi_v(\omega) = 1. \tag{2}$$

Combine (1) and (2) into a state space model on "standard form", with u as input, y as output and $\bar{x} = \begin{bmatrix} x_1 & x_2 & w \end{bmatrix}^T$ as state vector. **(3p)**

(d) It also turned out that w can be measured (without measurement noise), so that the pure state feedback $u(k) = \bar{L}\bar{x}(k)$ is applicable. State the *equations* needed to find the feedback gain $\bar{L} \in \mathbb{R}^{1\times 3}$ that minimizes

$$V_d = E\left\{ y^T(k) Q_1 y(k) + 0.1 u^2(k) \right\}, \quad \text{with the same } Q_1 \text{ as in (a).}$$

(You need/should *not* solve this LQ problem.) **(2p)**

Please use English in your solutions!

Problem 2 A continuous-time system has the state space representation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \quad \text{where} \quad e^{At} = \begin{bmatrix} e^{-4t} & 0 \\ e^{-2t} - e^{-4t} & e^{-2t} \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{3}$$

The system is to be controlled by a sampling controller.

(a) Give the discrete-time model of (3) obtained by zero-order-hold (ZOH) sampling, expressed in the sampling period h. **(2p)**

(b) For a certain h the ZOH sampled version of (3) becomes

$$y(kh) = \frac{0.25(-q+1)}{(q-0.5)(q-0.25)} u(kh).$$

Suppose that the proportional feedback $u(kh) = -Ky(kh)$ is used. For which $K \in \mathbb{R}$ is the closed loop system stable? **(2p)**

(c) Assume that pole placement is used for design of a state feedback for the system (3). It is found that a double pole in -5 would be a good choice *if* a continuous-time controller could be used. How should the closed loop pole polynomial, $p(z) = z^2 + p_1 z + p_2$, be chosen in order to have an equivalent bahaviour to that, when using a sampling controller? Express you answer in the sampling period h. **(2p)**

(d) Determine the matrix A in (3). **(1p)**

Problem 3 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true"/"false" are considered!

(a) White noise is always periodic.

(b) The Nyquist frequency should always be greater than the bandwidth of the white noise in the measurements.

(c) A drawback with MPC is that it is only applicable for systems with sufficiently small time constants.

(d) MPC yields a linear time-invariant controller.

(e) LQG yields a linear time-invariant controller.

(f) LQG handles control constraints better than MPC does.

(g) The weighting matrices $\{Q_1, Q_2\}$ and $\{kQ_1, kQ_2\}$ (with $k > 0$) give identical LQG controllers.

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(7p)**

Please use English in your solutions!

Problem 4 In a scientific project the properties of massive bodies in very distant galaxies are studied. A certain aspect of the motion of a black hole is described by the (scalar) model

$$\begin{cases} \dot{x}(t) = w(t), \\ y(t) = x(t) + e(t). \end{cases} \tag{4}$$

Here x is the entity of interest, and y is the measurement of it, provided by radio telescopes on earth. The process disturbance w and the measurement noise e are modeled as uncorrelated, zero mean stochastic processes.

(a) The disturbances w and e are assumed to be white noise with intensities $\Phi_w(\omega) = 16$ and $\Phi_e(\omega) = 9$. Determine the Kalman filter that gives the optimal estimate $\hat{x}$ under this assumption. **(3p)**

(b) What is the covariance of the estimation error $\tilde{x} = x - \hat{x}$ for the Kalman filter under the assumption in (a)? That is, determine $E\tilde{x}^2$. **(1p)**

(c) The project gets a proposal to incorporate their radio telescopes into an international world wide grid of radio telescopes. This would improve the sensitivity of the measurements in the sense that the measurement noise intensity would decrease to $\Phi_e(\omega) = 1$. Assume that the Kalman filter in (a) still would be used (as an observer), what would $E\tilde{x}^2$ then be (with $\Phi_w(\omega)$ as in (a))? **(3p)**

(d) What is the smallest possible value of $E\tilde{x}^2$ when $\Phi_e(\omega) = 1$ (and $\Phi_w(\omega)$ as in (a))? **(2p)**

(e) Another research group finds out that w is *not* really white noise, but that its spectrum is

$$\Phi_w(\omega) = \frac{K(\omega^2 + \beta)}{\omega^4 + \alpha_1\omega^2 + \alpha_2} \qquad \text{with} \qquad \begin{cases} K = 1.44 \cdot 10^6, \\ \beta = 1.6 \cdot 10^3, \\ \alpha_1 = 1.609 \cdot 10^5, \\ \alpha_2 = 1.44 \cdot 10^8. \end{cases} \tag{5}$$

Find a stable, minimum phase transfer operator $G(p)$, with $G(0) \geq 0$, such that

$$w(t) = G(p)v(t), \quad Ev(t) = 0 \quad \text{and} \quad \Phi_v(\omega) = 1,$$

as a model of the stochastic process $w(t)$, has $\Phi_w(\omega)$ in (5) as spectrum. **(3p)**

Please use English in your solutions!

Problem 5 A discrete-time system with one input and two outputs is described by the difference equations

$$\begin{cases} y_1(k+1) - 0.7y_1(k) + y_2(k) = 2u(k), \\ y_2(k+1) + 0.2y_2(k) - y_1(k) = 2u(k+1) - u(k). \end{cases}$$

(a) Give the transfer operator $G(q)$ in the following model of the system:

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = G(q)u(k).$$

(3p)

(b) Give a state space representation for the system. **(3p)**

(c) Is your state space model in (b) observable from y_2? **(1p)**

Problem 6 *The HW bonus points (from the spring 2018) are exchangeable for this problem.*
Consider the scalar discrete-time system

$$\begin{cases} x(k+1) = 0.5x(k) + 2.5v(k), \\ \quad y(k) = x(k) + v(k), \end{cases}$$

where $v(k)$ is zero mean, Gaussian white noise, and $E[v(k)^2] = 1$.

(a) Determine the *one-step predictor* version of the Kalman filter (the "standard observer" form), for producing the estimate $\hat{x}(k|k-1)$. **(4p)**

(b) Assume that $\hat{x}(k|k-1) = -0.5$, and that $y(k) = 2$. Compute the optimal prediction $\hat{x}(k+1|k)$. **(1p)**

(c) Assume again that $\hat{x}(k|k-1) = -0.5$, and that $y(k) = 2$. Compute the optimal estimate $\hat{x}(k|k)$. **(2p)**

Solutions to the exam in Automatic Control II, 2018-05-29:

1. (a) The associated Riccati equation for the LQ problem is the DARE $S = F^TSF + M^TQ_1M - F^TSG(G^TSG + Q_2)^{-1}G^TSF$. Here we notice that $z = y = x \Leftrightarrow M = I$, and that $G = \begin{bmatrix}1 & 0\end{bmatrix}^T$ and $Q_2 = 0.1 \Rightarrow G^TSG + Q_2 = s_1 + 0.1 = 1$. Thus we get

$$Q_1 = S - F^TSF + F^TSGG^TSF = \begin{bmatrix}0.9 & 0.4\\ 0.4 & 0.52\end{bmatrix} - \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}0.9 & 0.4\\ 0.4 & 0.52\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} +$$
$$+ \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}0.9 & 0.4\\ 0.4 & 0.52\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0.9 & 0.4\\ 0.4 & 0.52\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} = \begin{bmatrix}0.37 & 0\\ 0 & 0.16\end{bmatrix}.$$

(b) The feedback gain is $L = (G^TSG + Q_2)^{-1}G^TSF = \begin{bmatrix}1.3 & 0.4\end{bmatrix}$. The poles are then given by

$$0 = \det(zI - F + GL) = \det\left(\begin{bmatrix}z & 0\\ 0 & z\end{bmatrix} - \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} + \begin{bmatrix}1.3 & 0.4\\ 0 & 0\end{bmatrix}\right)$$
$$= \det\begin{bmatrix}z + 0.3 & 0.4\\ -1 & -z - 1\end{bmatrix} = (z + 0.3)(z - 1) + 0.4 = z^2 - 0.7z + 0.1.$$

The poles are $z = 0.35 \pm \sqrt{0.35^2 - 0.1} = 0.35 \pm \sqrt{0.0225} = 0.35 \pm 0.15$. Thus, there is one pole in $+0.2$ and one in $+0.5$.

(c) From (2) we get $qw = -0.9w + 2v$. Combining this with (1) gives

$$\begin{cases} qx_1 = x_1 + u,\\ qx_2 = x_1 + x_2 + w,\\ qw = -0.9w + 2v,\\ y_1 = x_1,\\ y_2 = x_2, \end{cases} \Leftrightarrow \begin{cases} q\bar{x} = \overbrace{\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 1\\ 0 & 0 & -0.9\end{bmatrix}}^{=\bar{F}}\bar{x} + \overbrace{\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}}^{=\bar{G}}u + \overbrace{\begin{bmatrix}0\\ 0\\ 2\end{bmatrix}}^{=\bar{N}}v,\\ y = \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}}_{=\bar{H}}\bar{x}. \end{cases}$$

(d) Now $z = y = \bar{H}\bar{x} \Leftrightarrow M = \bar{H}$. Then $\bar{L} = (\bar{G}^T\bar{S}\bar{G} + 0.1)^{-1}\bar{G}^T\bar{S}\bar{F}$, where $\bar{S} = \bar{S}^T \geq 0$ solves the DARE

$$\bar{S} = \bar{F}^T\bar{S}\bar{F} + \bar{H}^TQ_1\bar{H} - \bar{F}^T\bar{S}\bar{G}(\bar{G}^T\bar{S}\bar{G} + 0.1)^{-1}\bar{G}^T\bar{S}\bar{F},$$

with matrices given in (c). (See Theorem 9.4.)

2. (a) Theorem 4.1 gives the ZOH sampled system as

$$\begin{cases} qx = Fx + Gu,\\ y = Cx, \end{cases} \quad \text{with} \quad F = e^{Ah}, \quad G = \int_0^h e^{At}Bdt.$$

Thus, here we have

$$F = \begin{bmatrix}e^{-4h} & 0\\ e^{-2h} - e^{-4h} & e^{-2h}\end{bmatrix}, \quad G = \int_0^h \begin{bmatrix}4e^{-4t}\\ 2e^{-2t} - 4e^{-4t}\end{bmatrix}dt = \begin{bmatrix}1 - e^{-4h}\\ e^{-4h} - e^{-2h}\end{bmatrix}.$$

(b) The closed loop poles are given by

$$0 = 1 + KG(q) = 1 + \frac{0.25K(-q+1)}{(q-0.5)(q-0.25)} \quad \Leftrightarrow$$

$$0 = (q-0.5)(q-0.25) + 0.25K(-q+1) = q^2 + (-0.25K - 0.75)q + 0.125 + 0.25K$$

The roots of $0 = z^2 + \alpha z + \beta$ lies inside the unit circle if and only if $|\alpha| - 1 < \beta < 1$. Here $\alpha = -0.25K - 0.75$ and $\beta = 0.125 + 0.25K$, so stability depends on the conditions

$$\begin{aligned} \beta < 1: &\quad 0.125 + 0.25K < 1 &\Leftrightarrow&\quad K < 3.5, \\ \alpha - 1 < \beta: &\, -0.25K - 0.75 - 1 < 0.125 + 0.25K &\Leftrightarrow&\quad K > -3.75, \\ -\alpha - 1 < \beta: &\quad 0.25K + 0.75 - 1 < 0.125 + 0.25K &\Leftrightarrow&\quad -0.25 < 0.125. \end{aligned}$$

The closed loop system is stable for $-3.75 < K < 3.5$.
(c) Under ZOH sampling a continuous-time pole p is mapped onto e^{ph}. The double pole in -5 is then mapped onto e^{-5h}, and the desired closed loop polynomial then is $(z - e^{-5h})^2 = z^2 - 2e^{-5h}z + e^{-10h}$.
(d) Let $\phi(t) = e^{At}$. Then $\phi(0) = I$ and $\frac{d}{dt}\phi(t) = A\phi(t)$. Thus, $\frac{d}{dt}\phi(0) = A$. Here

$$\frac{d}{dt}\begin{bmatrix} e^{-4t} & 0 \\ e^{-2t} - e^{-4t} & e^{-2t} \end{bmatrix} = \begin{bmatrix} -4e^{-4t} & 0 \\ -2e^{-2t} + 4e^{-4t} & -2e^{-2t} \end{bmatrix} \quad \Rightarrow \quad A = \begin{bmatrix} -4 & 0 \\ 2 & -2 \end{bmatrix}.$$

3. (a) False (For white noise $Ew(t)w(s) = 0$ for all $s \neq t$); **(b)** False (Impossible, white noise has constant spectrum $\Rightarrow \omega_B = \infty$); **(c)** False (Rather the other way around); **(d)** False (MPC = nonlinear and time-varying); **(e)** True; **(f)** False (The other way around); **(g)** True (Only the relative sizes of Q_1 and Q_2 matters)

4. (a) The Kalman filter is $\dot{\hat{x}} = A\hat{x} + K(y - C\hat{x})$, with $K = PC^TR_2^{-1}$ where $P = P^T \geq 0$ solves the CARE $0 = AP + PA^T + NR_1N^T - PC^TR_2^{-1}CP$. Here $A = 0$, $N = C = 1$, $R_1 = 16$ and $R_2 = 9$, so the CARE becomes

$$0 = 16 - \frac{P^2}{9} \quad \Rightarrow \quad P = 12 \quad \Rightarrow \quad K = \frac{12}{9} = \frac{4}{3}.$$

(b) We have $E\tilde{x}^2 = P = 12$.
(c) The estimation error is governed by the state equation $\dot{\tilde{x}} = (A - KC)\tilde{x} + Nw - Ke$, and its covariance solves the continuous-time Lyapunov equation $0 = (A - KC)\Pi_{\tilde{x}} + \Pi_{\tilde{x}}(A - KC)^T + NR_wN^T + KR_eK^T$. Here this gives

$$0 = -2K\Pi_{\tilde{x}} + 16 + K^2 \quad \Leftrightarrow \quad \Pi_{\tilde{x}} = \frac{16 + K^2}{2K} = \frac{16 + (4/3)^2}{2 \cdot 4/3} = \frac{20}{3} = 6\frac{2}{3}.$$

(d) The smallest possible covariance is the P that solves the corresponding CARE (compare with (a)):

$$0 = 16 - P^2 \quad \Rightarrow \quad P = 4.$$

(e) The polynomial degrees of ω^2 in the numerator and denominator suggest that $G(p) = \frac{b_1 p + b_2}{p^2 + a_1 p + a_2}$. Then we must have

$$\Phi_w(\omega) = |G(i\omega)|^2 = \frac{(b_1\omega)^2 + b_2^2}{(a_2 - \omega^2)^2 + (a_1\omega)^2} = \frac{b_1^2[\omega^2 + (b_2/b_1)^2]}{\omega^4 + (a_1^2 - 2a_2)\omega^2 + a_2^2}.$$

In order to have $G(p)$ stable, minimum phase and $G(0) \geq 0$ we must have $a_1, a_2, b_1, b_2 \geq 0$. Comparing the spectrum of the model with the given spectrum gives the equation system

$$\begin{cases} b_1^2 = K, \\ (b_2/b_1)^2 = \beta, \\ a_1^2 - 2a_2 = \alpha_1, \\ a_2^2 = \alpha_2 \end{cases} \Leftrightarrow \begin{cases} b_1 = \sqrt{K} = 1.2 \cdot 10^3, \\ b_2 = \sqrt{\beta} \cdot b_1 = 4.8 \cdot 10^4 \\ a_1 = \sqrt{2a_2 + \alpha_1} = 4.3 \cdot 10^2, \\ a_2 = \sqrt{\alpha_2} = 1.2 \cdot 10^4. \end{cases}$$

Thus,

$$G(p) = \frac{1.2 \cdot 10^3 p + 4.8 \cdot 10^4}{p^2 + 4.3 \cdot 10^2 p + 1.2 \cdot 10^4} = \frac{1200(p+40)}{p^2 + 430p + 12000} = \frac{1200(p+40)}{(p+30)(p+400)}.$$

5. (a) Rewrite the difference equations with the shift operator q:

$$\begin{cases} (q - 0.7)y_1 + y_2 = 2u, \\ (q + 0.2)y_2 - y_1 = (2q - 1)u, \end{cases} \Leftrightarrow \begin{bmatrix} q - 0.7 & 1 \\ -1 & q + 0.2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2q - 1 \end{bmatrix} u$$

$$\Leftrightarrow$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} q - 0.7 & 1 \\ -1 & q + 0.2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2q - 1 \end{bmatrix} u = \begin{bmatrix} \frac{1.4}{q^2 - 0.5q + 0.86} \\ \frac{2q^2 - 2.4q + 2.7}{q^2 - 0.5q + 0.86} \end{bmatrix} u = \begin{bmatrix} \frac{1.4}{q^2 - 0.5q + 0.86} \\ 2 + \frac{-1.4q + 0.98}{q^2 - 0.5q + 0.86} \end{bmatrix} u.$$

(b) One input $\Rightarrow$ controller canonical form works. Notice that y_2 requires a direct term:

$$\begin{cases} qx = \begin{bmatrix} 0.5 & -0.86 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1.4 \\ -1.4 & 0.98 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u. \end{cases}$$

(c) Observability from $y_2 \Leftrightarrow$ check whether or not the observability matrix $\mathcal{O}_2 = \begin{bmatrix} H_2 \\ H_2 F \end{bmatrix}$ has full rank:

$$\mathcal{O}_2 = \begin{bmatrix} -1.4 & 0.98 \\ 0.28 & 1.204 \end{bmatrix} \quad \text{full rank} \quad \Leftrightarrow \quad \text{observable from } y_2.$$

6. (a) For a discrete-time system, with no input, the Kalman predictor is

$$\hat{x}(k+1|k) = F\hat{x}(k|k-1) + K(y(k) - H\hat{x}(k|k-1))$$

where $K = (FPH^T + NR_{12})(HPH^T + R_2)^{-1}$, and $P = P^T \geq 0$ is a solution to the DARE

$$P = FPF^T + NR_1N^T - (FPH^T + NR_{12})(HPH + R_2)^{-1}(FPH^T + NR_{12})^T.$$

Here $F = 0.5$, $N = 2.5$, $H = 1$ and $R_1 = R_{12} = R_2 = 1$, so the DARE becomes

$$P = 0.5^2P + 2.5^2 - \frac{(0.5P + 2.5)^2}{P + 1} \quad \Leftrightarrow$$
$$P(P{+}1) = (0.25P{+}6.25)(P{+}1){-}(0.25P^2{+}2.5P{+}6.25) \quad \Leftrightarrow \quad P(P{-}3) = 0.$$

Thus $P = 3$ (the largest solution), and $K = \frac{0.5P+2.5}{P+1} = 1$.
(b) From (a) we have that

$$\hat{x}(k{+}1|k) = 0.5\hat{x}(k|k{-}1){+}1{\cdot}(y(k){-}\hat{x}(k|k{-}1)) = 0.5(-0.5){+}2{-}(-0.5) = 2.25.$$

(c) The estimate is $\hat{x}(k|k) = \hat{x}(k|k-1) + \tilde{K}(y(k) - H\hat{x}(k|k-1))$, where $\tilde{K} = PH^T(HPH^T + R_2)^{-1}$ (see Eq. (5.100) in Glad/Ljung). Here $\tilde{K} = \frac{P}{P+1} = 0.75$, so

$$\hat{x}(k|k) = -0.5 + 0.75(2 - (-0.5)) = 1.375.$$