Uppsala University
Department of Information Technology
Division of Systems and Control

Homework Assignment no. 3

Rational Parametric Methods for Line Spectra

This third homework is based on **Exercise C3.17** (old book) / **C3.18** (new book). For your convenience, code for the different spectral estimators can be downloaded at `http://www.prenhall.com/stoica`. Make sure that you use these functions correctly (use "`help`". If still in doubt how to use the functions, try by inspecting the code directly). Here follow some explanations and clarifications for the corresponding parts of **Exercise C3.17/C3.18**:

AR and ARMA Estimators for Line Spectral Estimation

(a)

The true spectrum is given. Verify that the expression for $\phi(\omega)$ is correct. Use the true spectrum as a reference for the remaining parts.

(b)

The Yule-Walker and modified Yule-Walker methods use the ACS sequence to compute the parameters. Usually the ACS sequence is estimated from the data. Here you have to use the *true* ACS. Use (4.1.6) to compute the true ACS for the given process. Now modify the m-files so as to use this true ACS sequence instead of an estimated sequence. For example, you can modify the function so that the true ACS sequence is passed rather than the data. In that case the computation of $r(k)$ inside the m-file is unnecessary. Using the true ACS helps to eliminate the effects of estimation errors and makes it easy to study the resolution properties of various methods. Plot the locations of the roots of $A(z)$ in a separate figure for each example. **Users of the old book: Note that the last sentence in Exercise C3.17b belongs to Exercise C3.17c (see the errata).**

(c)

Note that in this noise-free case ($\sigma^2 = 0$) the covariance matrix R will have rank = 4 (the number of complex sinusoids in the data). This means that for model orders > 4 there will be a rank deficiency which shows up as warnings in Matlab. Does this fact seem to cause any problems?

(d)

Experimenting with the model orders, SNR, K and M is highly recommended in order to understand their impact on the results. Show the plots you think is most relevant to motivate your answers and conclusions.

(e)

This should be straight-forward.

In your handed in solutions, show only the relevant plots and motivate why you show them!

Exercise C3.18: AR and ARMA Estimators for Line Spectral Estimation

The ARMA methods can also be used to estimate line spectra (estimation of line spectra by other methods is the topic of Chapter 4). In this application, AR(MA) techniques are often said to provide *super–resolution* capabilities because they are able to resolve sinusoids too closely spaced in frequency to be resolved by periodogram–based methods.

We again consider the four AR and ARMA estimators described above.

(a) Generate realizations of the signal

$$y(t) = 10\sin(0.24\pi t + \varphi_1) + 5\sin(0.26\pi t + \varphi_2) + e(t), \qquad t = 1, \ldots, N$$

where $e(t)$ is (real) white Gaussian noise with variance σ^2, and where φ_1, φ_2 are independent random variables each uniformly distributed on $[0, 2\pi]$. From the results in Chapter 4, we find the spectrum of $y(t)$ to be

$$\begin{aligned}\phi(\omega) = {} & 50\pi\left[\delta(\omega - 0.24\pi) + \delta(\omega + 0.24\pi)\right] \\ & + 12.5\pi\left[\delta(\omega - 0.26\pi) + \delta(\omega + 0.26\pi)\right] + \sigma^2\end{aligned}$$

(b) Compute the "true" AR polynomial (using the true ACS sequence; see equation (4.1.6)) using the Yule–Walker equations for both AR(4), AR(12), ARMA(4,4) and ARMA(12,12) models when $\sigma^2 = 1$. This experiment corresponds to estimates obtained as $N \to \infty$. Plot $1/|A(\omega)|^2$ for each case, and find the roots of $A(z)$. Which method(s) are able to resolve the two sinusoids?

(c) Consider now $N = 64$, and set $\sigma^2 = 0$; this corresponds to the finite data length but infinite SNR case. Compute estimated AR polynomials using the four spectral estimators and the AR and ARMA model orders described above; for the MYW technique consider both $M = n$ and $M = 2n$, and for the LS ARMA technique use both $K = n$ and $K = 2n$. Plot $1/|A(\omega)|^2$, overlaid, for 50 different Monte–Carlo simulations (using different values of φ_1 and φ_2 for each). Also plot the zeroes of $A(z)$, overlaid, for these 50 simulations. Which method(s) are reliably able to resolve the sinusoids? Explain why. Note that as $\sigma^2 \to 0$, $y(t)$ corresponds to a (limiting) AR(4) process. How does the choice of M or K in the ARMA methods affect resolution or accuracy of the frequency estimates?

(d) Obtain spectral estimates ($\hat{\sigma}^2|\hat{B}(\omega)|^2/|\hat{A}(\omega)|^2$ for the ARMA estimators and $\hat{\sigma}^2/|\hat{A}(\omega)|^2$ for the AR estimators) for the four methods when $N = 64$ and $\sigma^2 = 1$. Plot ten overlaid spectral estimates and overlaid polynomial zeroes of the $\hat{A}(z)$ estimates. Experiment with different AR and ARMA model orders to see if the true frequencies are estimated more accurately; note also the appearance and severity of "spurious" sinusoids in the estimates for higher model orders. Which method(s) give reliable "super–resolution" estimation of the sinusoids? How does the model order influence the resolution properties? Which method appears to have the best resolution?

You may want to experiment further by changing the SNR and the relative amplitudes of the sinusoids to gain a better understanding of the relative differences between the methods. Also, experiment with different model orders and parameters K and M to understand their impact on estimation accuracy.

(e) Compare the estimation results with periodogram–based estimates obtained from the same signals. Discuss differences in resolution, bias, and variance of the techniques.