Iterative solution of the Helmholtz equation by a fourth-order method

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Contents

The Helmholtz equation

$$-\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \kappa^2 u = 0,$$

where

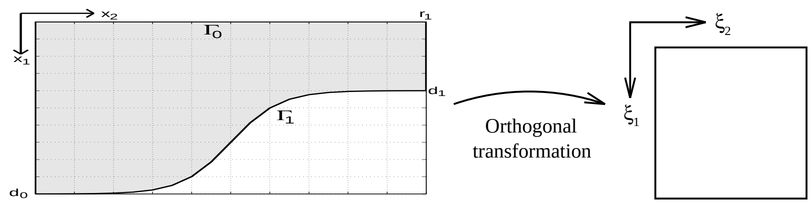
$$\kappa(x_1, x_2) = \frac{2\pi f}{c(x_1, x_2)},$$

f is the frequency,

$c(x_1, x_2)$ is the local speed of sound, and

$\mathrm{Re}\big(u(x_1, x_2)e^{-i2\pi f t}\big)$ is the acoustic pressure.

Sound waves in a 2D duct



Transformed equation

$$\begin{cases} -\dfrac{\partial}{\partial \xi_1}\left(a\dfrac{\partial u}{\partial \xi_1}\right) - \dfrac{\partial}{\partial \xi_2}\left(a^{-1}\dfrac{\partial u}{\partial \xi_2}\right) - eu = 0 \\ u(0,\xi_2) = 0 \\ \dfrac{\partial u}{\partial \xi_1}(1,\xi_2) = 0 \end{cases}$$

with metric coefficients

$$a = \frac{\eta_2}{\eta_1} \equiv \sqrt{\frac{\left(\frac{\partial x_1}{\partial \xi_2}\right)^2 + \left(\frac{\partial x_2}{\partial \xi_2}\right)^2}{\left(\frac{\partial x_1}{\partial \xi_1}\right)^2 + \left(\frac{\partial x_2}{\partial \xi_1}\right)^2}}$$

$$e = \kappa^2 \eta_1 \eta_2 \equiv \kappa^2 \sqrt{\left(\left(\frac{\partial x_1}{\partial \xi_1}\right)^2 + \left(\frac{\partial x_2}{\partial \xi_1}\right)^2\right)\left(\left(\frac{\partial x_1}{\partial \xi_2}\right)^2 + \left(\frac{\partial x_2}{\partial \xi_2}\right)^2\right)}$$

Separability ansatz

By assumption, there are *local* transformations at $x_2 = 0$ and $x_2 = d_2$ with metric coefficients independent of ξ_2.

$$\begin{cases} -\dfrac{\partial}{\partial \xi_1}\left(a\dfrac{\partial u}{\partial \xi_1}\right) - eu = a^{-1}\dfrac{\partial^2 u}{\partial \xi_2^2} \\ u(0, \xi_2) = 0 \\ \dfrac{\partial u}{\partial \xi_1}(1, \xi_2) = 0 \end{cases}$$

Radiation boundary conditions, eigenfunctions

Close to the vertical boundaries the problem is separable, i.e., $u(\xi_1, \xi_2) = \psi(\xi_1)\varphi(\xi_2)$. The solution is built up from *orthonormal eigenfunctions* $\psi_m(\xi_1)$ and traveling waves $\varphi_m(\xi_2)$.
The formal solution is

$$u(\xi_1, \xi_2) = \sum_{m=1}^{\infty} \psi_m(\xi_1)\big(A_m e^{i\sqrt{-\lambda_m}\xi_2} + B_m e^{-i\sqrt{-\lambda_m}\xi_2}\big),$$

where A_m and B_m are unknown coefficients.

Left and right boundaries

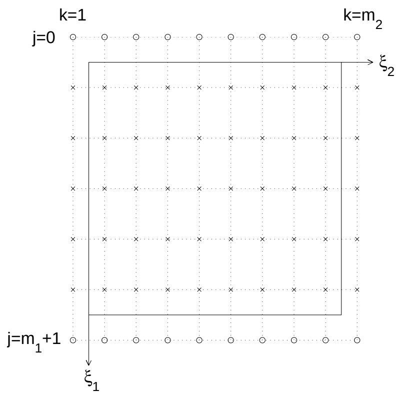
At the *near-zone* boundary, we let A_m be the forcing terms. Exploiting the orthonormality of the eigenfunctions, the remaining coefficients can be determined from the solution.

At the *far-zone* boundary, there are no reflections; therefore $B_m = 0$.

$$-\frac{\partial u}{\partial \xi_2}(\xi_1, 0) - i\sum_{m=1}^{\mu_\ell} \sqrt{-\lambda_m}\langle \psi_m(\cdot), u(\cdot, 0)\rangle \psi_m(\xi_1) = g(\xi_1)$$

$$\frac{\partial u}{\partial \xi_2}(\xi_1, 1) - i\sum_{m=1}^{\mu_r} \sqrt{-\lambda_m}\langle \psi_m(\cdot), u(\cdot, 1)\rangle \psi_m(\xi_1) = 0$$

Discretization: Grid and resolution



$$h_2 = \frac{(480\tau)^{\frac{1}{4}}}{(\kappa d_2)^{\frac{5}{4}}}, \quad m_2 = \lceil \tfrac{1}{h_2} + 2 \rceil$$

$$h_1 = \frac{(480\tau d_\ell / d_2)^{\frac{1}{4}}}{(\kappa d_\ell)^{\frac{5}{4}}}, \quad m_1 = \lceil \tfrac{1}{h_1} \rceil$$

Resolution strategy: Maximum phase error τ (in the ξ_2-direction).

Discretization: The Numerov scheme

Let $u^{j,k}$ denote the approximate solution at the point (ξ_1^j, ξ_2^k). From a Taylor series expansion we obtain

$$h_1^{-2}\big((u^{j+1,k} - u^{j,k})a^{j+\frac{1}{2},k} - (u^{j,k} - u^{j-1,k})a^{j-\frac{1}{2},k}\big) =$$
$$\tfrac{\partial}{\partial \xi_1}\Big(a\tfrac{\partial u}{\partial \xi_1}\Big)^{j,k} + h_1^2\Big(\tfrac{1}{12}a\tfrac{\partial^4 u}{\partial \xi_1^4} + \tfrac{1}{6}\tfrac{\partial a}{\partial \xi_1}\tfrac{\partial^3 u}{\partial \xi_1^3} + \tfrac{1}{8}\tfrac{\partial^2 a}{\partial \xi_1^2}\tfrac{\partial^2 u}{\partial \xi_1^2} + \tfrac{1}{24}\tfrac{\partial^3 a}{\partial \xi_1^3}\tfrac{\partial u}{\partial \xi_1}\Big)^{j,k} + \mathcal{O}(h_1^4).$$

Using the formulas

$$a\frac{\partial^4 u}{\partial \xi_1^4} = \frac{\partial^3}{\partial \xi_1^3}\Big(a\frac{\partial u}{\partial \xi_1}\Big) - 3\frac{\partial a}{\partial \xi_1}\frac{\partial^3 u}{\partial \xi_1^3} - 3\frac{\partial^2 a}{\partial \xi_1^2}\frac{\partial^2 u}{\partial \xi_1^2} - \frac{\partial^3 a}{\partial \xi_1^3}\frac{\partial u}{\partial \xi_1},$$

$$\frac{\partial^3 a}{\partial \xi_1^3}\frac{\partial u}{\partial \xi_1} = \frac{\partial}{\partial \xi_1}\Big(\frac{\partial^2 a}{\partial \xi_1^2}\frac{\partial u}{\partial \xi_1}\Big) - \frac{\partial^2 a}{\partial \xi_1^2}\frac{\partial^2 u}{\partial \xi_1^2},$$

$$\frac{\partial a}{\partial \xi_1}\frac{\partial^3 u}{\partial \xi_1^3} + \frac{\partial^2 a}{\partial \xi_1^2}\frac{\partial^2 u}{\partial \xi_1^2} = \frac{\partial}{\partial \xi_1}\Big(a^{-1}\frac{\partial a}{\partial \xi_1}\frac{\partial}{\partial \xi_1}\Big(a\frac{\partial u}{\partial \xi_1}\Big) - a^{-1}\Big(\frac{\partial a}{\partial \xi_1}\Big)^2\frac{\partial u}{\partial \xi_1}\Big),$$

we can rewrite the difference approximation above as

$$\begin{aligned}
&-h_1^{-2}\big((u^{j+1,k} - u^{j,k})a^{j+\frac{1}{2},k} - (u^{j,k} - u^{j-1,k})a^{j-\frac{1}{2},k}\big) \\
&+ \tfrac{h_1^2}{12}\tfrac{\partial^3}{\partial \xi_1^3}\Big(a\tfrac{\partial u}{\partial \xi_1}\Big)^{j,k} - \tfrac{h_1^2}{12}\tfrac{\partial}{\partial \xi_1}\Big(a^{-1}\tfrac{\partial a}{\partial \xi_1}\tfrac{\partial}{\partial \xi_1}\Big(a\tfrac{\partial u}{\partial \xi_1}\Big)\Big)^{j,k} \\
&+ \tfrac{h_1^2}{12}\tfrac{\partial}{\partial \xi_1}\Big(a^{-1}\Big(\tfrac{\partial a}{\partial \xi_1}\Big)^2\tfrac{\partial u}{\partial \xi_1}\Big)^{j,k} - \tfrac{h_1^2}{24}\tfrac{\partial}{\partial \xi_1}\Big(\tfrac{\partial^2 a}{\partial \xi_1^2}\tfrac{\partial u}{\partial \xi_1}\Big)^{j,k} = \\
&-\frac{\partial}{\partial \xi_1}\Big(a\frac{\partial u}{\partial \xi_1}\Big)^{j,k} + \mathcal{O}(h_1^4).
\end{aligned}$$

A similar formula can be derived for $\frac{\partial}{\partial \xi_2}\big(a^{-1}\frac{\partial u}{\partial \xi_2}\big)^{j,k}$. Combining those two formulas and performing the substitutions

$$\begin{aligned}\frac{\partial}{\partial \xi_1}\Big(a\frac{\partial u}{\partial \xi_1}\Big) &= -\frac{\partial}{\partial \xi_2}\Big(a^{-1}\frac{\partial u}{\partial \xi_2}\Big) - eu,\\ \frac{\partial}{\partial \xi_2}\Big(a^{-1}\frac{\partial u}{\partial \xi_2}\Big) &= -\frac{\partial}{\partial \xi_1}\Big(a\frac{\partial u}{\partial \xi_1}\Big) - eu,\end{aligned}$$

in the correction terms, we obtain a Numerov method for the Helmholtz equation:

$$\begin{aligned}&-h_1^{-2}\big((u^{j+1,k}-u^{j,k})a^{j+\frac{1}{2},k}-(u^{j,k}-u^{j-1,k})a^{j-\frac{1}{2},k}\big)\\ &-h_2^{-2}\big((u^{j,k+1}-u^{j,k})/a^{j,k+\frac{1}{2}}-(u^{j,k}-u^{j,k-1})/a^{j,k-\frac{1}{2}}\big)-e^{j,k}u^{j,k}\\ &-\tfrac{h_1^2}{12}\tfrac{\partial^2}{\partial \xi_1^2}\Big(\tfrac{\partial}{\partial \xi_2}\Big(a^{-1}\tfrac{\partial u}{\partial \xi_2}\Big)\Big)^{j,k}-\tfrac{h_2^2}{12}\tfrac{\partial^2}{\partial \xi_2^2}\Big(\tfrac{\partial}{\partial \xi_1}\Big(a\tfrac{\partial u}{\partial \xi_1}\Big)\Big)^{j,k}\\ &-\tfrac{h_1^2}{12}\Big(\tfrac{\partial^2(eu)}{\partial \xi_1^2}\Big)^{j,k}-\tfrac{h_2^2}{12}\Big(\tfrac{\partial^2(eu)}{\partial \xi_2^2}\Big)^{j,k}\\ &+\tfrac{h_1^2}{12}\tfrac{\partial}{\partial \xi_1}\Big(a^{-1}\tfrac{\partial a}{\partial \xi_1}\tfrac{\partial}{\partial \xi_2}\Big(a^{-1}\tfrac{\partial u}{\partial \xi_2}\Big)\Big)^{j,k}+\tfrac{h_2^2}{12}\tfrac{\partial}{\partial \xi_2}\Big(a\tfrac{\partial a^{-1}}{\partial \xi_2}\tfrac{\partial}{\partial \xi_1}\Big(a\tfrac{\partial u}{\partial \xi_1}\Big)\Big)^{j,k}\\ &+\tfrac{h_1^2}{12}\tfrac{\partial}{\partial \xi_1}\Big(a^{-1}\tfrac{\partial a}{\partial \xi_1}eu\Big)^{j,k}+\tfrac{h_2^2}{12}\tfrac{\partial}{\partial \xi_2}\Big(a\tfrac{\partial a^{-1}}{\partial \xi_2}eu\Big)^{j,k}\\ &+\tfrac{h_1^2}{12}\tfrac{\partial}{\partial \xi_1}\Big(a^{-1}\Big(\tfrac{\partial a}{\partial \xi_1}\Big)^2\tfrac{\partial u}{\partial \xi_1}\Big)^{j,k}+\tfrac{h_2^2}{12}\tfrac{\partial}{\partial \xi_2}\Big(a\Big(\tfrac{\partial a^{-1}}{\partial \xi_2}\Big)^2\tfrac{\partial u}{\partial \xi_2}\Big)^{j,k}\\ &-\tfrac{h_1^2}{24}\tfrac{\partial}{\partial \xi_1}\Big(\tfrac{\partial^2 a}{\partial \xi_1^2}\tfrac{\partial u}{\partial \xi_1}\Big)^{j,k}-\tfrac{h_2^2}{24}\tfrac{\partial}{\partial \xi_2}\Big(\tfrac{\partial^2 a^{-1}}{\partial \xi_2^2}\tfrac{\partial u}{\partial \xi_2}\Big)^{j,k}=0.\end{aligned}$$

By approximating the correction derivatives to second-order accuracy

with centered differences, we obtain a fourth-order accurate scheme.

Discretization: Inner grid points

$$\begin{aligned}&B^{j,k}_{\underline{j}-1,\underline{k}-1}u^{j-1,k-1}+B^{j,k}_{\underline{j}-1,\underline{k}}u^{j-1,k}+B^{j,k}_{\underline{j}-1,\underline{k}+1}u^{j-1,k+1}+\\ &B^{j,k}_{\underline{j},\underline{k}-1}u^{j,k-1}+B^{j,k}_{\underline{j},\underline{k}}u^{j,k}+B^{j,k}_{\underline{j},\underline{k}+1}u^{j,k+1}+\\ &B^{j,k}_{\underline{j}+1,\underline{k}-1}u^{j+1,k-1}+B^{j,k}_{\underline{j}+1,\underline{k}}u^{j+1,k}+B^{j,k}_{\underline{j}+1,\underline{k}+1}u^{j+1,k+1}=0\end{aligned}$$

$$\begin{aligned}&\big(a^{\frac{1}{2},k}-\tfrac{1}{4}(a^{1,k}-a^{0,k})\big)u^{0,k}=\\ &-\big(a^{\frac{1}{2},k}+\tfrac{1}{4}(a^{1,k}-a^{0,k})\big)u^{1,k},\\ &\big(a^{m_1+\frac{1}{2},k}+\tfrac{h_1^2}{48}(e^{m_1+1,k}-e^{m_1,k})\big)u^{m_1+1,k}=\\ &\big(a^{m_1+\frac{1}{2},k}-\tfrac{h_1^2}{48}(e^{m_1+1,k}-e^{m_1,k})\big)u^{m_1,k},\\ &k=2,\dots,m_2-1\end{aligned}$$

Discretization: Near- and far-zone boundaries

By a fourth-order accurate FEM, we compute the vertical eigenfunctions $\psi_m(\xi_1)$ for all propagating modes ($\lambda_m < 0$). Form the column vectors

$$\psi_m \equiv \left\{\psi_m(\xi_1^j)\right\}_{j=1}^{m_1}, \quad m = 1, \ldots, \mu.$$

The discrete boundary conditions in block form become

$$(B_j^{:,1} - C_j^{:,1})u^{j,1} + (-B_j^{:,1} - C_j^{:,1})u^{j,2} = g^{:,1},$$
$$(-B_j^{:,m_2} - C_j^{:,m_2})u^{j,m_2-1} + (B_j^{:,m_2} - C_j^{:,m_2})u^{j,m_2} = 0,$$

where

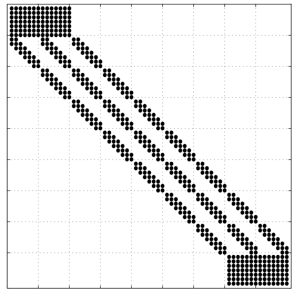
$$C_:^{:,1} = ih_2 \sum_{m=1}^{\mu_\ell} \sqrt{-\lambda_m}\psi_m\psi_m^* W_:^{:,1}.$$

System of equations, structure of B

$M^{-1}Bu = M^{-1}g,$

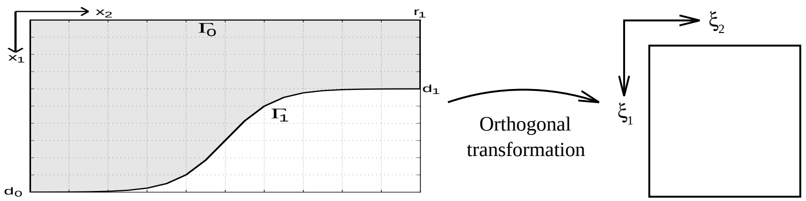
M preconditioner,

$$u \equiv \begin{pmatrix} u^{:,1} \\ u^{:,2} \\ \vdots \\ u^{:,m_2} \end{pmatrix}, \quad g = \begin{pmatrix} g^{:,1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$



The system of equations is solved using preconditioned GMRES.

Rectangular domain



Assume that $d_r = d_\ell \equiv d_1$.

$$\textit{Global} \text{ transformation } \begin{cases} x_1 = \xi_1 d_1, & 0 \le \xi_1 \le 1 \\ x_2 = \xi_2 d_2, & 0 \le \xi_2 \le 1 \end{cases}$$

with metric coefficients

$$a = d_2/d_1, \quad e = \kappa^2 d_1 d_2$$

Physical boundary conditions

$$\text{Dirichlet–Neumann} \begin{cases} u(0,\xi_2) = 0 \\ \dfrac{\partial u}{\partial \xi_1}(1,\xi_2) = 0 \end{cases}$$

$$\text{Neumann–Neumann} \begin{cases} \dfrac{\partial u}{\partial \xi_1}(0,\xi_2) = 0 \\ \dfrac{\partial u}{\partial \xi_1}(1,\xi_2) = 0 \end{cases}$$

$$\text{Dirichlet–Dirichlet} \begin{cases} u(0,\xi_2) = 0 \\ u(1,\xi_2) = 0 \end{cases}$$

Radiation boundary conditions

Dirichlet–to–Neumann maps yield *nonlocal* boundary conditions

$$-\frac{\partial u}{\partial \xi_2}(\xi_1, 0) - i \sum_{m=1}^{\mu} \sqrt{-\lambda_m} \langle \psi_m(\cdot), u(\cdot, 0)\rangle \psi_m(\xi_1) = g(\xi_1),$$

$$\frac{\partial u}{\partial \xi_2}(\xi_1, 1) - i \sum_{m=1}^{\mu} \sqrt{-\lambda_m} \langle \psi_m(\cdot), u(\cdot, 1)\rangle \psi_m(\xi_1) = 0,$$

where the eigenmodes and cutoff limits are

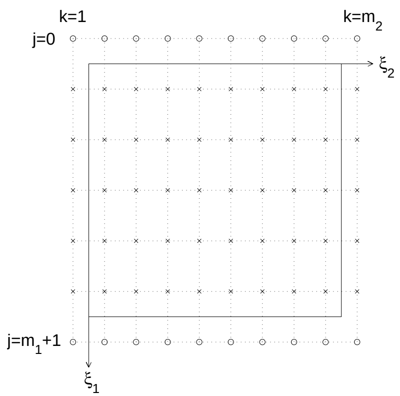
Eigenmodes

D–N $\begin{cases} \psi_m(\xi_1) = \sqrt{2}\sin\big((m-\frac{1}{2})\pi\xi_1\big) \\ \lambda_m = \big((m-\frac{1}{2})\pi d_2/d_1\big)^2 - (\kappa d_2)^2 \end{cases}, \quad \mu = \lfloor \kappa d_1/\pi + \frac{1}{2} \rfloor$

N–N $\begin{cases} \psi_m(\xi_1) = \sqrt{2}\cos\big((m-1)\pi\xi_1\big) \\ \lambda_m = \big((m-1)\pi d_2/d_1\big)^2 - (\kappa d_2)^2 \end{cases}, \quad \mu = \lfloor \kappa d_1/\pi + 1 \rfloor$

D–D $\begin{cases} \psi_m(\xi_1) = \sqrt{2}\sin(m\pi\xi_1) \\ \lambda_m = (m\pi d_2/d_1)^2 - (\kappa d_2)^2 \end{cases}, \quad \mu = \lfloor \kappa d_1/\pi \rfloor$

Discretization: Grid



$$\xi_2^k = \big(k - \tfrac{3}{2}\big)h_2, \quad h_2 = \frac{1}{m_2 - 2}$$

$$\xi_1^j = \big(j - \tfrac{1}{2}\big)h_1, \quad h_1 = \frac{1}{m_1} \quad \text{(N)}$$

$$\xi_1^j = j h_1, \quad h_1 = \frac{1}{m_1 + 1} \quad \text{(D–D)}$$

Discretization: Physical boundaries

D–N $\begin{cases} u^{0,k} = -u^{1,k} \\ u^{m_1+1,k} = u^{m_1,k} \end{cases}$

N–N $\begin{cases} u^{0,k} = u^{1,k} \\ u^{m_1+1,k} = u^{m_1,k} \end{cases}$

D–D $\begin{cases} u^{0,k} = 0 \\ u^{m_1+1,k} = 0 \end{cases}, \quad k = 1, \ldots, m_2$

Phase error analysis

For a rectangular domain, the true solution is

$$u(\xi_1,\xi_2) = \sum_{m=1}^{\mu} A_m \psi_m(\xi_1)\exp(i\sqrt{-\lambda_m}\xi_2),$$
$$\lambda_m = a(a(m-N)^2\pi^2 - e), \quad a = d_2/d_1, \quad e = \kappa^2 d_1 d_2,$$

where $N = \frac{1}{2}, 1$, or 0 depending on the boundary conditions.

The Numerov operator becomes

$$P = -\big(ah_1^{-2} + \frac{e}{12}\big)\Delta_1 - \big(a^{-1}h_2^{-2} + \frac{e}{12}\big)\Delta_2 - \frac{ah_1^{-2}}{12}\Delta_2\Delta_1 - \frac{a^{-1}h_2^{-2}}{12}\Delta_1\Delta_2 - e,$$

where Δ_d is a second difference in direction d.

Notice that

$$h_1^{-2}\Delta_1\psi_m(\xi_1^j) \equiv h_1^{-2}\big(\psi_m(\xi_1^j + h_1) - 2\psi_m(\xi_1^j) + \psi_m(\xi_1^j - h_1)\big)$$
$$= 2h_1^{-2}\big(\cos\big((m-N)\pi h_1\big) - 1\big)\psi_m(\xi_1^j),$$

i.e., $\{\psi_m(\xi_1^j)\}_{j=1}^{m_1}$ is an eigenvector to $h_1^{-2}\Delta_1$ with eigenvalue

$$\alpha_m = 2h_1^{-2}\big(\cos\big((m-N)\pi h_1\big) - 1\big)$$
$$= -(m-N)^2\pi^2 + \tfrac{1}{12}(m-N)^4\pi^4h_1^2 - \tfrac{1}{360}(m-N)^6\pi^6h_1^4 + \mathcal{O}(m^8h_1^6)$$

Thus, we are led to the ansatz

$$u^{j,k} = \sum_{m=1}^{\mu} \psi_m(\xi_1^j)v_m^k$$

for the discrete solution to $Pu^{j,k} = 0$.

We study every mode separately:

$$0 = P\psi_m(\xi_1^j)v^k$$
$$= \big(-(a\alpha_m + \frac{e\alpha_m h_1^2}{12} + e)v^k - (a^{-1}h_2^{-2} + \frac{e}{12} + \frac{a\alpha_m}{12} + \frac{a^{-1}h_2^{-2}\alpha_m h_1^2}{12})\Delta_2 v^k\big)\psi_m(\xi_1^j),$$

leading to a difference equation

$$v^{k+1} - 2\big(1 + \frac{\beta_m h_2^2}{2}\big)v^k + v^{k-1} = 0,$$
$$\beta_m = \frac{\nu_m - \frac{ae\alpha_m h_1^2}{12}}{1 - \frac{\nu_m h_2^2}{12} + \frac{\alpha_m h_1^2}{12}}, \quad \nu_m \equiv a(-a\alpha_m - e), \quad m = 1,\ldots,\mu.$$

The characteristic equation reads

$$r^2 - 2\big(1 + \frac{\beta_m h_2^2}{2}\big)r + 1 = 0, \quad m = 1,\ldots,\mu,$$

where the roots corresponding to right-going modes are

$$r = 1 + \frac{\beta_m h_2^2}{2} + \big(1 + \beta_m h_2^2 + \frac{\beta_m^2 h_2^4}{4} - 1\big)^{\frac{1}{2}}$$
$$= \exp\big(ih_2(\sqrt{-\beta_m} + \frac{(\sqrt{-\beta_m})^3h_2^2}{24} + \frac{3(\sqrt{-\beta_m})^5h_2^4}{640} + \mathcal{O}(h_2^5))\big),$$
$$m = 1,\ldots,\mu.$$

Some further Taylor expansions yield

$$\sqrt{-\beta_m} = \sqrt{-\lambda_m} - \frac{(\sqrt{-\lambda_m})^3 h_2^2}{24} + \frac{(\sqrt{-\lambda_m})^5 h_2^4}{384} + \frac{a^2(m-N)^6\pi^6 h_1^4}{480\sqrt{-\lambda_m}} - \frac{\sqrt{-\lambda_m}(m-N)^2\pi^2 a e h_1^2 h_2^2}{288} + \mathcal{O}(h^6), \quad m = 1,\ldots,\mu.$$

Thus,

$$\begin{aligned} v^k &= C_m r^{k-\frac{3}{2}} \\ &= C_m \exp\Big(i\xi_2^k\big(\sqrt{-\lambda_m} + \frac{(\sqrt{-\lambda_m})^5 h_2^4}{480} + \frac{a^2(m-N)^6\pi^6 h_1^4}{480\sqrt{-\lambda_m}} \\ &\qquad - \frac{\sqrt{-\lambda_m}((\kappa d_1)^2 + a^{-2}\lambda_m)(\kappa d_2)^2}{288} h_1^2 h_2^2 + \mathcal{O}(h^5)\big)\Big), \\ &\qquad m = 1,\ldots,\mu, \end{aligned}$$

where the amplitude C_m is A_m by assumption.

For every mode labeled $m = 1,\ldots,\mu$, there is a phase error given by

$$\Big(\frac{(\sqrt{-\lambda_m})^5 h_2^4}{480} + \frac{a^2(m-N)^6\pi^6 h_1^4}{480\sqrt{-\lambda_m}} - \frac{\sqrt{-\lambda_m}((\kappa d_1)^2 + a^{-2}\lambda_m)(\kappa d_2)^2}{288} h_1^2 h_2^2\Big)\xi_2^k.$$

Observations

- The phase error grows linearly with ξ_2.
- The h_2- and h_1-terms complement each other.
- The mixed error term is considerably smaller than the other two.
- When $\lambda_m \to 0$, i.e., for standing waves, the analysis breaks down.

Resolution criteria

Based on these observations, appropriate resolution criteria would be

$$\frac{(\sqrt{-\lambda_1})^5 h_2^4}{480} \leq \tau, \qquad \frac{a^2(\kappa d_1)^6 h_1^4}{480\max(\sqrt{-\lambda_\mu}, \frac{1}{\sqrt{5}})} \leq \tau,$$

where τ is a relative error tolerance.

For the second-order scheme, the corresponding formulas are

$$\frac{(\sqrt{-\lambda_1})^3 h_2^2}{24} \leq \tau, \qquad \frac{a^2(\kappa d_1)^4 h_1^2}{24\max(\sqrt{-\lambda_\mu}, \frac{1}{\sqrt{5}})} \leq \tau.$$

Normal block preconditioner

$$Q \equiv [q_1, \dots, q_{m_1}], \quad q_m = \left\{ \sqrt{2h_1} \sin\big((m - \tfrac{1}{2})\pi(j - \tfrac{1}{2})h_1\big) \right\}_{j=1}^{m_1}$$

The preconditioner M is block tridiagonal and $M_{:,s}^{:,k} \equiv Q\Lambda_{:,s}^{:,k}Q^*$.

Choose

$$\Lambda_{:,s}^{:,k} = \mathrm{diag}(Q^* B_{:,s}^{:,k} Q),$$

which is optimal in the Frobenius norm.

Preconditioner solve

$$(I \otimes Q^*)M(I \otimes Q) \equiv \Lambda$$

$$M^{-1} = (I \otimes Q)\Lambda^{-1}(I \otimes Q^*)$$

1. $v = (I \otimes Q^*)y$
2. solve $\Lambda z = v$
3. $x = (I \otimes Q)z$

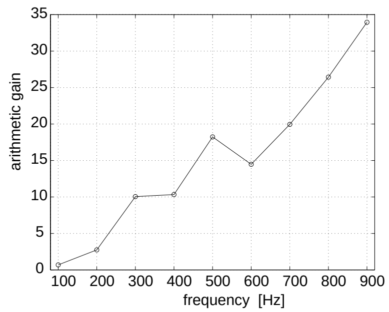
Arithmetic complexity/unknown

band GE	$8m_1^2 + 25m_1$
GMRES(ℓ)	$150 \log_2 m_1 + 685 + n_{it}(40 \log_2 m_1 + 8\ell + 267)$

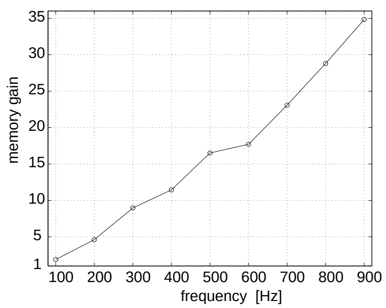
Memory requirement/unknown

band GE	$4m_1 + 4$
GMRES(ℓ)	$2\ell + 27$

Comparison with band GE

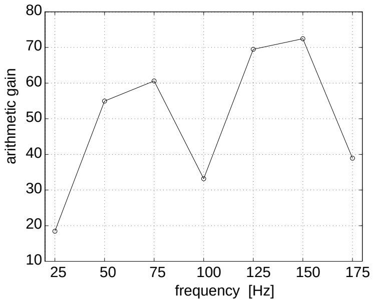


(i) Arithmetic operations

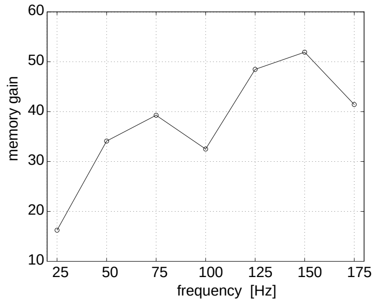


(ii) Memory

Comparison with a second-order scheme

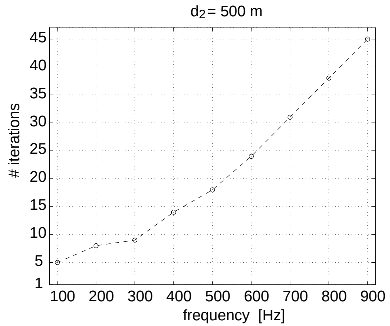


(iii) Arithmetic operations

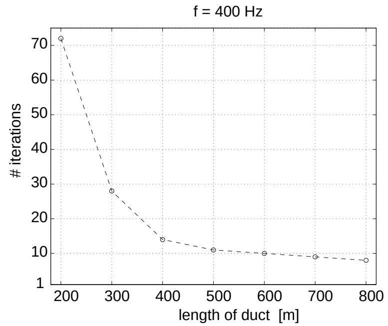


(iv) Memory

Number of iterations as a function of



(v) frequency



(vi) geometry

Conclusions

- We have reduced the storage requirement and the arithmetic work drastically compared with band GE.
- We have constructed a “fast Helmholtz” preconditioner that has proven to be very effective.
- The fourth-order Numerov scheme is more efficient than a standard second-order scheme.