Breaking All the Symmetries in Matrix Models
Results, Conjectures, and Directions

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1. Matrix Models

Example: Sport schedule in $Periods \times Weeks \rightarrow Teams \times Teams$
for:

- $|Teams| = n$
- $|Weeks| = n - 1$
- $|Periods| = n / 2$

such that:

- every team plays every other team once;
- every team plays exactly once per week;
- every team plays at most twice per period.

A solution for $n = 8$:

	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7
Period 1	0 vs 1	0 vs 2	1 vs 5	2 vs 4	3 vs 6	3 vs 7	4 vs 7
Period 2	2 vs 3	1 vs 7	0 vs 6	5 vs 6	5 vs 7	1 vs 4	0 vs 3
Period 3	4 vs 5	3 vs 5	2 vs 7	0 vs 7	0 vs 4	2 vs 6	1 vs 6
Period 4	6 vs 7	4 vs 6	3 vs 4	1 vs 3	1 vs 2	0 vs 5	2 vs 5

2. Symmetries (in Matrix Models)

	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7
Period 1	0 vs 1	0 vs 2	1 vs 5	2 vs 4	3 vs 6	3 vs 7	4 vs 7
Period 2	2 vs 3	1 vs 7	0 vs 6	5 vs 6	5 vs 7	1 vs 4	0 vs 3
Period 3	4 vs 5	3 vs 5	2 vs 7	0 vs 7	0 vs 4	2 vs 6	1 vs 6
Period 4	6 vs 7	4 vs 6	3 vs 4	1 vs 3	1 vs 2	0 vs 5	2 vs 5

The periods, weeks, and teams are *indistinguishable*, because:

(1) the periods (rows) can be permuted (*variable symmetry*);

(2) the weeks (columns) can be permuted (*variable symmetry*);

(3) the teams of any game can be permuted (*variable symmetry*);

(4) the teams can be permuted (*value symmetry*);

without affecting the solution status of any assignment.

Definition: A *symmetry class* (or *orbit*, in group theory) is an equivalence class of assignments under *all* the symmetries (including their compositions).

3. Symmetry-Breaking Before Search

Add (*lexicographic*) *ordering constraints* so that (ideally) each orbit has exactly one element:

(1) every row is lexicographically smaller than or equal to (denoted $\leq_{lex}$) the next, if any;

(2) every column is lexicographically smaller than or equal to the next, if any;

(3) the first team of every game has a smaller number than the second team of the game.

When lexicographically ordering along every dimension with indistinguishable indices:

- *No* orbit is of size 0.
- However, in general, *not* all orbits are of size 1, except if all the matrix values are distinct, etc.

Counterexample: symmetric matrices with lexicographically ordered rows *and* columns:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \xleftarrow[\textit{swap columns 1 \& 2}]{\textit{swap rows 2 \& 3}} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xleftarrow[\textit{swap columns 2 \& 3}]{\textit{swap rows 1 \& 2}} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

4. The Crawford *et al.* Method for Breaking *All* the Symmetries

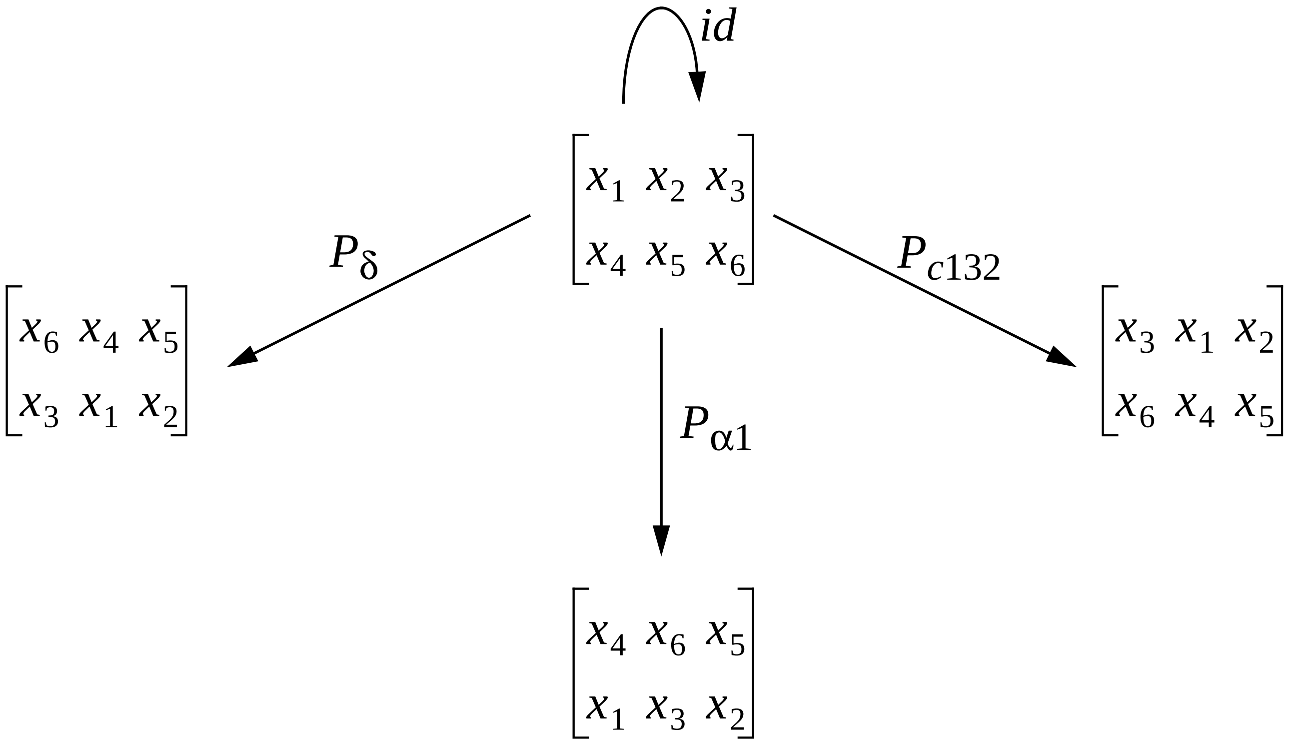
Consider a matrix with total row and column symmetry: $\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}$

Group *Sym* of 12 symmetries (permutations):

Permutation	Name	Order
(1,2)(4,5)	P_{c12}	2
(2,3)(5,6)	P_{c23}	2
(1,4)(2,5)(3,6)	P_{r12}	2
( )	*id*	1
(1,6,2,4,3,5)	P_{δ}	6
(1,5,3,4,2,6)	P_{σ}	6
(1,4)(2,6)(3,5)	$P_{\alpha 1}$	2
(1,5)(2,4)(3,6)	$P_{\alpha 2}$	2
(1,6)(2,5)(3,4)	$P_{\alpha 3}$	2
(1,3)(4,6)	P_{c13}	2
(1,2,3)(4,5,6)	P_{c123}	3
(1,3,2)(4,6,5)	P_{c132}	3

Cycle notation: (1,2,3)(4,5) denotes the function $\{x_1 \rightarrow x_2, x_2 \rightarrow x_3, x_3 \rightarrow x_1, x_4 \rightarrow x_5, x_5 \rightarrow x_4, x_6 \rightarrow x_6\}$.

Illustration



Induced Symmetry-Breaking Constraints (SBCs)

(1) Pick a variable ordering m of the matrix.

(2) Add the constraint $m \leq_{lex} \sigma(m)$ for each $\sigma \in Sym \setminus \{id\}$.

Example: Take $m = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}$

(1,2)(4,5)	x_1	x_2	x_3	x_4	x_5	x_6	$\leq_{lex}$	x_2	x_1	x_3	x_5	x_4	x_6	(c_{12})
(2,3)(5,6)	x_1	x_2	x_3	x_4	x_5	x_6	$\leq_{lex}$	x_1	x_3	x_2	x_4	x_6	x_5	(c_{23})
(1,4)(2,5)(3,6)	x_1	x_2	x_3	x_4	x_5	x_6	$\leq_{lex}$	x_4	x_5	x_6	x_1	x_2	x_3	(r_{12})
(1,6,2,4,3,5)	x_1	x_2	x_3	x_4	x_5	x_6	$\leq_{lex}$	x_6	x_4	x_5	x_3	x_1	x_2	(δ)
(1,5,3,4,2,6)	x_1	x_2	x_3	x_4	x_5	x_6	$\leq_{lex}$	x_5	x_6	x_4	x_2	x_3	x_1	(σ)
(1,4)(2,6)(3,5)	x_1	x_2	x_3	x_4	x_5	x_6	$\leq_{lex}$	x_4	x_6	x_5	x_1	x_3	x_2	(α_1)
(1,5)(2,4)(3,6)	x_1	x_2	x_3	x_4	x_5	x_6	$\leq_{lex}$	x_5	x_4	x_6	x_2	x_1	x_3	(α_2)
(1,6)(2,5)(3,4)	x_1	x_2	x_3	x_4	x_5	x_6	$\leq_{lex}$	x_6	x_5	x_4	x_3	x_2	x_1	(α_3)
(1,3)(4,6)	x_1	x_2	x_3	x_4	x_5	x_6	$\leq_{lex}$	x_3	x_2	x_1	x_6	x_5	x_4	(c_{13})
(1,2,3)(4,5,6)	x_1	x_2	x_3	x_4	x_5	x_6	$\leq_{lex}$	x_2	x_3	x_1	x_5	x_6	x_4	(c_{123})
(1,3,2)(4,6,5)	x_1	x_2	x_3	x_4	x_5	x_6	$\leq_{lex}$	x_3	x_1	x_2	x_6	x_4	x_5	(c_{132})

5. Improvements, Conjectures, and Directions

Internal Simplifications

Example: (1,3)(4,6) = (1,3)(2)(4,6)(5) induces $[x_1, x_2, x_3, x_4, x_5, x_6] \leq_{lex} [x_3, x_2, x_1, x_6, x_5, x_4]$

$\equiv (x_1 \leq x_3) \wedge (x_1 = x_3 \rightarrow x_2 \leq x_2) \wedge (x_1 = x_3 \wedge x_2 = x_2 \rightarrow x_3 \leq x_1) \wedge (x_1 = x_3 \wedge x_2 = x_2 \wedge x_3 = x_1 \rightarrow x_4 \leq x_6) \wedge \ldots$

$\equiv (x_1 \leq x_3) \wedge (x_1 = x_3 \rightarrow x_4 \leq x_6) \wedge \ldots$

$\equiv [x_1, x_4] \leq_{lex} [x_3, x_6]$

The elements at the positions corresponding to the last indices in each cycle can be deleted!

(1,2)(4,5)	x_1			x_4			$\leq_{lex}$	x_2			x_5			(c_{12})
(2,3)(5,6)		x_2			x_5		$\leq_{lex}$		x_3			x_6		(c_{23})
(1,4)(2,5)(3,6)	x_1	x_2	x_3				$\leq_{lex}$	x_4	x_5	x_6				(r_{12})
(1,6,2,4,3,5)	x_1	x_2	x_3	x_4	x_5		$\leq_{lex}$	x_6	x_4	x_5	x_3	x_1		(δ)
(1,5,3,4,2,6)	x_1	x_2	x_3	x_4	x_5		$\leq_{lex}$	x_5	x_6	x_4	x_2	x_3		(σ)
(1,4)(2,6)(3,5)	x_1	x_2	x_3				$\leq_{lex}$	x_4	x_6	x_5				(α_1)
(1,5)(2,4)(3,6)	x_1	x_2	x_3				$\leq_{lex}$	x_5	x_4	x_6				(α_2)
(1,6)(2,5)(3,4)	x_1	x_2	x_3				$\leq_{lex}$	x_6	x_5	x_4				(α_3)
(1,3)(4,6)	x_1			x_4			$\leq_{lex}$	x_3			x_6			(c_{13})
(1,2,3)(4,5,6)	x_1	x_2		x_4	x_5		$\leq_{lex}$	x_2	x_3		x_5	x_6		(c_{123})
(1,3,2)(4,6,5)	x_1	x_2		x_4	x_5		$\leq_{lex}$	x_3	x_1		x_6	x_4		(c_{132})

Elimination of Logically Implied SBCs

The first two SBCs

(1,2)(4,5)	x_1			x_4			$\leq_{lex}$	x_2			x_5			(c_{12})
(2,3)(5,6)		x_2			x_5		$\leq_{lex}$		x_3			x_6		(c_{23})

logically imply the last three SBCs

(1,3)(4,6)	x_1			x_4			$\leq_{lex}$	x_3			x_6			(c_{13})
(1,2,3)(4,5,6)	x_1	x_2		x_4	x_5		$\leq_{lex}$	x_2	x_3		x_5	x_6		(c_{123})
(1,3,2)(4,6,5)	x_1	x_2		x_4	x_5		$\leq_{lex}$	x_3	x_1		x_6	x_4		(c_{132})

which can thus be eliminated:

- The last three SBCs rule out some permutations of the three columns.
- But $c_{12} \wedge c_{23}$ imposes a particular permutation and also rules out those other permutations.

In general:

- An $m \times n$ matrix with total row and column symmetry has $m! \cdot n!$ symmetries.
- *There are (at least) $m! - m + n! - n$ logically implied SBCs, due to the transitivity of $\leq_{lex}$!*
- **Direction:** Try the redundancy detection criteria of ILP, especially [Imbert & Van Hentenryck].

Contextual Simplifications in δ and σ (due to Frisch and Harvey)

(1,2)(4,5)	x_1			x_4			$\leq_{lex}$	x_2			x_5			(c_{12})
(2,3)(5,6)		x_2			x_5		$\leq_{lex}$		x_3			x_6		(c_{23})
(1,4)(2,5)(3,6)	x_1	x_2	x_3				$\leq_{lex}$	x_4	x_5	x_6				(r_{12})
(1,6,2,4,3,5)	x_1	x_2	x_3				$\leq_{lex}$	x_6	x_4	x_5				(δ)
(1,5,3,4,2,6)	x_1	x_2	x_3	x_4			$\leq_{lex}$	x_5	x_6	x_4	x_2			(σ)
(1,4)(2,6)(3,5)	x_1	x_2	x_3				$\leq_{lex}$	x_4	x_6	x_5				(α_1)
(1,5)(2,4)(3,6)	x_1	x_2	x_3				$\leq_{lex}$	x_5	x_4	x_6				(α_2)
(1,6)(2,5)(3,4)	x_1	x_2	x_3				$\leq_{lex}$	x_6	x_5	x_4				(α_3)

Direction: How to mechanise these contextual internal simplifications?

Experimental Results

- Encouraging results even when only using c_{12} , c_{23} , and r_{12} as SBCs, due to the action of the actual problem constraints.

- Nevertheless: When does a *polynomial* number of SBCs suffice to break all / most symmetries?!

Elimination of Domain-Dependent Implied SBCs

The number of implied SBCs grows as the domain size of the decision variables shrinks!

Domain size		c_{12}	c_{23}	r_{12}	δ	σ	α_1	α_2	α_3
2	Implied SBCs				✓	✓	✓	✓	✓
	Minimum set	✗	✗	✗	✗				
	Minimum set	✗	✗	✗			✗		
3	Implied SBCs				✓		✓	✓	✓
	Minimum set	✗	✗	✗	✗	✗			
≥ 4	Implied SBCs								
	Minimum set	✗	✗	✗	✗	✗	✗	✗	✗

Conjecture: *For a domain of size 2, it suffices to add the SBCs induced by the order 2 permutations.*

Experimentally validated up to 6×6 matrices.

Not true for domains of size 3: the constraint σ is necessary, but its permutation is of order 6.

Unfortunately, even the number of order 2 permutations is super-polynomial…

Direction: Will elimination of the implied order 2 SBCs leave a polynomial number of SBCs?

Direction: How to characterise the SBCs necessary for each domain size?

Direction: How to characterise the SBCs that break most of the symmetries?

6. Experimental Results

Enumerating all the 3×3 matrices modulo total row and column symmetry, in the absence of any actual problem constraints:

- 35 SBCs;
- 6 implied SBCs, by transitivity of $\leq_{lex}$;
- 9 further implied SBCs, for domain sizes from 4 to at least 6, which can *all* be eliminated.

Run-times in seconds, under GNU Prolog, on a Sun SPARC Ultra station 10:

		with all the 35 constraints	without 15 implied constraints: before internal simplifications	without 15 implied constraints: after internal simplifications
domain size = 4 (8,240 matrices)	Boolean $\leq_{lex}$	11.0”	5.8”	2.1”
	linear $\leq_{lex}$	8.3”	4.5”	1.6”
domain size = 5 (57,675 matrices)	Boolean $\leq_{lex}$	61.0”	31.8”	12.4”
	linear $\leq_{lex}$	49.6”	26.7”	10.0”
domain size = 6 (289,716 matrices)	Boolean $\leq_{lex}$	269.0”	139.0”	56.1”
	linear $\leq_{lex}$	227.0”	122.6”	46.5”