Well Posed Boundary Conditions for the Navier-Stokes Equations

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Abstract

In this article we propose a general procedure that allows us to determine both the number and type of boundary conditions for time dependent partial differential equations. With those, well posedness can be proven for a general initial-boundary value problem. The procedure is exemplified on the linearised Navier-Stokes equations in two and three space dimensions on a general domain.

1 Introduction

The problem of well posed boundary conditions is an essential question in many areas of physics. In fluid dynamics, characteristic boundary conditions for the Euler equations have long been accepted as one way to impose boundary conditions since the specification of the ingoing variable at a boundary implies well posedness. Often those are used as a guidiance when boundary conditions are chosen for the Navier-Stokes equations as well (see [1, 2, 3, 4, 5]). In [6] characteristic boundary conditions for the one-dimensional linearised Navier-Stokes equations were derived. For the two- and three-dimensional Navier-Stokes equations, the number of boundary conditions implying well posedness can be obtained by using the Laplace transform technique. (See [7] for an introduction of the Laplace transform technique.) Although possible to use, the Laplace transform technique is usually a very complicated procedure for systems of partial differential equations such as the Navier-Stokes equations. However, the exact form of the boundary conditions that leads to a well posed problem is still an open question and will be the issue adressed in this article.

In this paper we assume that we have unlimeted access to accurate boundary data. We do not engage in the elaborate, difficult and stimulating procedure of deriving artificial (or radiation or absorbing) boundary conditions. Examples of extensive research on these matters are given in [8],[9].

We propose a self contained procedure for obtaining both the number and type of boundary conditions for a general time dependent partial differential equation. The procedure is based on the energy method and has substantial similarities to the derivation of characteristic boundary conditions, since it involves a splitting of the boundary terms into ingoing and outgoing parts by a diagonalisation. Compared to the Laplace transform technique our procedure yields a much simpler analysis.

As was already mentioned, boundary conditions for the Navier-Stokes equations has been the subject of many investigations and still there is no theory for the general case. Hence, the linearised and symmetrised Navier-Stokes equations derived in [10] will serve as an example to which our proposed procedure is applied.

Well posedness of the continuous problem is a necessary requirement for all numerical methods. Even for well posed boundary conditions, numerous difficulties arise and virtually all numerical schmemes have their own way of handling the boundary conditions. Hence, we will refrain from numerical calculations for a particular discretisation technique and focus on the mathematical ground work.

The contents of this article are divided as follows. In section 2 a general procedure for determining well posed boundary conditions is presented. Section 3 applies the procedure on the three-dimensional Navier-Stokes equations on a general domain. In section 4 conclusions are drawn.

2 Well Posed Boundary Conditions

Throughout this paper, the analysis will deal with linear constant coefficient equations. Frequently, the equations of interest are not linear constant coefficient equations but rather variable coefficient or even nonlinear equations (such as the Navier-Stokes equations). We will start with a brief discussion on the relevance of analysing the constant coefficient case.

Consider the nonlinear initial-boundary value problem on a domain D with boundary ∂D,

$$\begin{aligned} \tilde{u}_t &= P(\tilde{u})\tilde{u} + F(x,t),\ x \in D,\ t \geq 0 \\ \tilde{u} &= f(x),\ x \in D,\ t = 0 \\ L\tilde{u} &= g(t),\ x \in \partial D, t \geq 0, \end{aligned} \tag{1}$$

where P is the nonlinear differential operator, L a boundary operator. F, f and g are forcing, initial and boundary functions. Let $\tilde{v}(x,t)$ be the solution to (1) with perturbed data $F+\delta F, f+\delta f$ and $g+\delta g$. Then. $\tilde{w}(x,t) = \tilde{u}(x,t) - \tilde{v}(x,t)$ approximately satisfies the linearised equation,

$$
\begin{aligned}
\tilde{w}_t &= P(\tilde{u})\tilde{w} + \delta F(x,t),\ x \in D,\ t \geq 0 \\
\tilde{w} &= \delta f(x),\ x \in D,\ t = 0, \\
L\tilde{w} &= \delta g(t), x \in \partial D, t \geq 0.
\end{aligned}
\tag{2}
$$

By freezing the coefficients in (2) we obtain,

$$
\begin{aligned}
\tilde{w}_t &= P(u)\tilde{w} + \delta F(x,t),\ x \in D,\ t \geq 0 \\
\tilde{w} &= \delta f(x),\ x \in D,\ t = 0 \\
L\tilde{w} &= \delta g(t), x \in \partial D, t \geq 0,
\end{aligned}
\tag{3}
$$

where the dropping of the tilde denotes a constant state.

Definition 2.1 *The problem (3) is well posed if there exist a unique solution bounded by the data,* $\delta F, \delta f$ *and* δg.

Remark There are many definitions of well posedness. Our choice is sometimes referred to as strongly well posed since it involves all types of data (see for example [7]).

Both existence and uniqueness are strongly coupled to the boundedness of the solution. In fact, it suffice to prove that a solution is bounded using a minimal number of boundary conditions, then both existence and uniqueness follows. (See for example [11])

Well posedness of (3) is related to well posedness of (1) via (2). In short, the following principles hold: If (3) is well posed for all values of u, then (2) is well posed. Further, (1) is well posed at $\tilde{u}$, if (2) is well posed for all functions in a neighbourhood of $\tilde{u}$. For more details see [12].

Before considering well posedness of a problem of the type (3), we will briefly state some additional mathematical theory that is the basis of the forthcoming analysis. First we give a definition from [13].

Definition 2.2 *Let A be a Hermitian matrix. The* inertia *of A is the ordered triple*

$$
i(A) = (i_+(A), i_-(A), i_0(A)) \tag{4}
$$

where $i_+(A)$ is the number of positive eigenvalues of A, $i_-(A)$ is the number of negative eigenvalues of A, and $i_0(A)$ is the number of zero eigenvalues of A, counting multiplicities.

We will also need the following theorem from [13] and we refer to that textbook for the proof. The theorem is also known as *Sylvester's law of inertia.*

Theorem 2.3 *Let A, B be Hermitian matrices. There is a nonsingular matrix S such that $A = SBS^*$ if and only if A and B have the same inertia.*

S^* denotes the Hermitian adjoint of S. The following corollary is merely a rephrasing of Theorem 2.3.

Corollary 2.4 *Suppose that R is a nonsingular matrix and A is a real symmetric matrix. Then the number of positive/negative eigenvalues of $R^T AR$ are the same as the number of positive/negative eigenvalues of A.*

Proof Follows immediately from Theorem 2.3 with $B = R^T AR$. ∎

Finally, we state a definition from [13].

Definition 2.5 *If A is a real m by n matrix we set $I(A) = [\mu_{ij}]$, where $\mu_{ij} = 1$ if $a_{ij} \neq 0$ and $\mu_{ij} = 0$ if $a_{ij} = 0$. The matrix $I(A)$ is called the* indicator matrix *of A.*

Now we turn to the main theory of this article. We will give general principles on how to determine boundary conditions such that the constant coefficient problem is well posed. Thus, assuming that linearisation and freezing of coefficients have already been carried out, we consider a linear constant coefficient problem with n space dimensions and $\bar{x} = (x_1, ..., x_n)$,

$$\begin{aligned} \tilde{u}_t + \sum_{i=1}^{n} A_i \tilde{u}_{x_i} &= \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}\tilde{u}_{x_i x_j} + F(\bar{x}, t),\ \bar{x} \in D,\ t \geq 0 \\ \tilde{u}(\bar{x}, 0) &= f(\bar{x}), \bar{x} \in D \\ L\tilde{u}(\bar{x}, t) &= g(t), \bar{x} \in \partial D,\ t \geq 0. \end{aligned} \tag{5}$$

The definition (5) of an initial-boundary value problem, covers hyperbolic, parabolic and incompletely parabolic partial differential equations depending on the rank of the matrices. Let $\|\cdot\|$ denote some norm for functions on D. Our approach of analysing well posedness of (5) is:

1. Symmetrise (5).

2. Apply the energy method. The energy estimate will have the structure,

$$\|\tilde{u}\|_t^2 + c_i \sum_{i=1}^{n} \|\tilde{u}_{x_i}\|^2 + \oint_{\partial D} \tilde{v}^T \mathbf{A} \tilde{v}\, ds \leq 0, \tag{6}$$

   where $c_i \geq 0$, $i = 1, .., n$ are constants and $\tilde{v}$ a vector formed by combinations of $\tilde{u}$ and $\tilde{u}_{x_i}$. Further, $\mathbf{A}$ is reduced to a full rank matrix. The boundedness of $\tilde{u}$ now depends on the possibility of bounding $\tilde{v}^T \mathbf{A} \tilde{v}$ in terms of the boundary data.

3. Find a diagonalising matrix, M, such that $M^T\mathbf{A}M = \mathbf{\Lambda}$ is diagonal. ($\mathbf{A}$ is symmetric due to step 1 above.) Then we also have the coordinate transformation, $M^{-1}\tilde{v} = \tilde{w}$.

4. Split $\mathbf{\Lambda} = \mathbf{\Lambda}^+ + \mathbf{\Lambda}^-$ such that $\mathbf{\Lambda}^+$ is positive semi-definite and $\mathbf{\Lambda}^-$ is negative semi-definite. Also, split $\tilde{w}$ into $\tilde{w} = \tilde{w}^+ + \tilde{w}^-$ corresponding to the nonzero entries of $\mathbf{\Lambda}^{+,-}$. More precisely, let $\tilde{w}^- = I(\mathbf{\Lambda}^-)\tilde{w}$ and $\tilde{w}^+ = \tilde{w} - \tilde{w}^-$.

5. Supply boundary data to the negative part. That is, specify $\tilde{w}^-$ by g.

Remark In Step 4 the number of boundary conditions are given as the number of negative eigenvalues of $\mathbf{A}$ or $\mathbf{\Lambda}$.

This implies boundedness of $\|\tilde{u}\|_t$ and hence $\|\tilde{u}\|$. The difficult part of this scheme is step 3. However, we know that $\mathbf{A}$ is symmetric and we can prove the following proposition regarding the steps 3 to 5.

Proposition 2.6 *Assume that step 1 and 2 are fulfilled, then the matrix* $\mathbf{A}$ *and the vector* $\tilde{v}$ *can be split such that,* $\tilde{v}^T\mathbf{A}\tilde{v} = \tilde{w}^{+T}\mathbf{\Lambda}^+\tilde{w}^+ + \tilde{w}^{-T}\mathbf{\Lambda}^-\tilde{w}^-$ *where* $\mathbf{\Lambda}^+$ *is positive semi-definite,* $\mathbf{\Lambda}^-$ *is negative semi-definite and,* $M^{-1}\tilde{v} = \tilde{w} = \tilde{w}^+ + \tilde{w}^-$ *for some matrix* M^{-1}*. Further, by specifying* $\tilde{w}^-$ *at the boundary, equation (5) is well posed.*

Proof Since $\mathbf{A}$ is symmetric, the eigenvalues are real and there exists a full set of eigenvectors. If Z contains the eigenvectors we have,

$$\tilde{v}^T\mathbf{A}\tilde{v} = \tilde{v}^T ZZ^T\mathbf{A}ZZ^T\tilde{v} = \tilde{w}^T\mathbf{\Lambda_Z}\tilde{w} = \tilde{w}^{+T}\mathbf{\Lambda}^+_{\mathbf{Z}}\tilde{w}^+ + \tilde{w}^{-T}\mathbf{\Lambda}^-_{\mathbf{Z}}\tilde{w}^-, \qquad (7)$$

where $\mathbf{\Lambda}_{\mathbf{Z}}^{-/+}$ are diagonal negative/positive semi-definite. We define $\tilde{w}^- = I(\mathbf{\Lambda}^-)\tilde{w}$ and $\tilde{w}^+ = \tilde{w} - \tilde{w}^-$. This proves the first part of Theorem 2.6.

Another way to prove the first part of Proposition 2.6 is to apply Theorem 2.4, to conclude that any nonsingular matrix R can be used as a transformation, $\mathbf{B} = R^T\mathbf{A}R$, such that $\mathbf{A}$ and $\mathbf{B}$ have the same inertia. By construction $\mathbf{B}$ is symmetric. Then $\mathbf{B}$ may be diagonalised by its eigenvectors and we have another diagonalisation of $\mathbf{A}$. Denote by X the matrix containing the eigenvectors of $\mathbf{B}$ as columns such that,

$$\begin{aligned}\tilde{v}^T\mathbf{A}\tilde{v} = \tilde{v}^T R^{-1,T}R^T\mathbf{A}RR^{-1}\tilde{v} = \tilde{v}^T R^{-1,T}\mathbf{B}R^{-1}\tilde{v} = \\ \tilde{v}^T R^{-1,T}X\mathbf{\Lambda_M}X^T R^{-1}\tilde{v} = \tilde{w}^T\mathbf{\Lambda}^+_{\mathbf{M}}\tilde{w} + \tilde{w}^T\mathbf{\Lambda}^-_{\mathbf{M}}\tilde{w},\end{aligned}$$

or,

$$\tilde{v}^T M^{-1,T}M^T\mathbf{A}MM^{-1}\tilde{v} = \tilde{w}^T\mathbf{\Lambda_M}\tilde{w} = \tilde{w}^{+T}\mathbf{\Lambda}^+_{\mathbf{M}}\tilde{w}^+ + \tilde{w}^{-T}\mathbf{\Lambda}^-_{\mathbf{M}}\tilde{w}^- \qquad (8)$$

where $\tilde{w} = M^{-1}\tilde{v}$, $M = RX$, and $\mathbf{\Lambda}_{\mathbf{M}}^{-/+}$ are diagonal negative/positive semi-definite. Further, $\tilde{w}^- = I(\mathbf{\Lambda}_{\mathbf{M}}^-)\tilde{w}$ and $\tilde{w}^+ = \tilde{w} - \tilde{w}^-$. We conclude that there are several different ways of diagonalising $\mathbf{A}$ but in all cases $\mathbf{\Lambda}_{\mathbf{Z}}$ and $\mathbf{\Lambda}_{\mathbf{M}}$ have the same inertia. The fundamental difference between Z and other diagonalising matrices M, is that M is not orthonormal. We may regard Z as a specific M.

Next, we turn to the proof of the second part of the theorem. Specify $\tilde{w}^- = g$ at the boundary. Equation (6) can be rewritten as,

$$\|\tilde{u}\|_t^2 + \oint_{\partial D} \tilde{w}^{+T}\mathbf{\Lambda}_{\mathbf{M}}^+\tilde{w}^+ ds + c_i \sum_{i=1}^{n} \|\tilde{u}_{x_i}\|^2 = -\oint_{\partial D} g^T \mathbf{\Lambda}_{\mathbf{M}}^- g\, ds, \tag{9}$$

All the terms on the left-hand side of (9) are positive implying that $\|\tilde{u}\|_t$, and hence $\|\tilde{u}\|$, are bounded. ∎

Remark The assumption that step 1 and 2 in Proposition 2.6 can be fullfilled is true for many important partial differential equations. For example, it is true for the Euler, Navier-Stokes and Maxwell's equations.

Remark The procedure of diagonalising $\mathbf{A}$ by using its eigenvectors and bound the negative part is what we mean by charateristic boundary conditions.

For Proposition 2.6 to be practically useful, a crucial point is to find a diagonalising matrix. That is why we gave two examples of diagonalising matrices. In the first example we used the eigenvalues and eigenvectors directly. For a system of equations, the matrix A can be large. (9 by 9 for the Navier-Stokes equations in three dimensions.) The eigenvalues of $\mathbf{A}$ are given as the roots of a polynomial of high degree, to which it in general do not exist roots on closed form.

In the second example, we can proceed in a different way. We will seek a diagonalising matrix to $\mathbf{A}$ that is not orthonormal. By choosing R such that $\mathbf{B}$ has a simpler structure than $\mathbf{A}$ we may be able to find the eigenvalues and eigenvectors to $\mathbf{B}$. In fact, we will show that it is possible for the three-dimensional Navier-Stokes equations on general domains.

Certainly, not all of the points are novel in the above procedure. For example, in [10] a symmetrisation of the linearised Navier-Stokes equations is presented. For the Euler equations the whole procedure has been carried out when deriving the well known characteristic boundary conditions. However, the idea of diagonalising the boundary terms with a non-orthonormal matrix is to the knowledge of the present authors new. Furthermore, it is important to formalise the whole procedure since it should be possible to find well posed boundary conditions to any problem of type (5).

3 The Navier Stokes Equations

3.1 Step 1: Symmetrise the Equations

We will consider the Navier-Stokes equations as an example on how to use the procedure, presented above, to derive well posed boundary conditions. We begin by rescaling the three-dimensional Navier-Stokes equations to nondimensional form. Consider the Navier-Stokes equations in primitive variables $\tilde{V} = [\tilde{\rho}, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{p}]$ as stated in [10],

$$\begin{aligned} \tilde{V}_t + \tilde{A}_1^p \tilde{V}_x + \tilde{A}_2^p \tilde{V}_y + \tilde{A}_3^p \tilde{V}_z = \\ \tilde{B}_{11}^p \tilde{V}_{xx} + \tilde{B}_{22}^p \tilde{V}_{yy} + \tilde{B}_{33}^p \tilde{V}_{zz} + \tilde{B}_{xy}^p \tilde{V}_{xy} + \tilde{B}_{yz}^p \tilde{V}_{yz} + \tilde{B}_{zx}^p \tilde{V}_{zx}, \end{aligned} \tag{10}$$

where the tilde sign emphasizes that the entity depends on the solution. Further, $\tilde{\rho}$ is the density; $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ are the velocities in the x, y and z direction respectively; $\tilde{p}$ is the pressure. We will also use, the ratio between the specific heat capacities, $\gamma = c_p/c_v$, and, the speed of sound c; μ, the dynamic viscosity, λ the bulk viscosity and $\nu = \frac{\mu}{\rho}$ the kinematic viscosity; $Pr = \frac{\nu}{\alpha}$ denotes the Prandtl-number where α is the thermal diffusivity. Let $Re = \frac{\rho_\infty U_\infty L}{\mu_\infty}$ denote the Reynolds-number. The infinity subscript denotes free stream conditions and L is some characteristic length scale.

The equations (10) are nondimensionalised and the coefficients are frozen which corresponds to the linearisation of the Navier-Stokes equations. The tilde signs are dropped on the matrices as they no longer depend on the solution. Using the parabolic symmetriser S_p derived in [10] and letting $\epsilon = \frac{1}{Re}$, yield,

$$\begin{aligned} \tilde{u}_t + A_1 \tilde{u}_x + A_2 \tilde{u}_y + A_3 \tilde{u}_z = \\ \epsilon(B_{11}\tilde{u}_{xx} + B_{22}\tilde{u}_{yy} + B_{33}\tilde{u}_{zz} + B_{xy}\tilde{u}_{xy} + B_{yz}\tilde{u}_{uz} + B_{zx}\tilde{u}_{zx}). \end{aligned} \tag{11}$$

The transformed nondimensionalised variables are,

$$S_p^{-1}\tilde{V} = \begin{pmatrix} \frac{c}{\sqrt{\gamma}\rho} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{c}{\rho\sqrt{\gamma}\sqrt{\gamma-1}} & 0 & 0 & 0 & \sqrt{\frac{\gamma}{\gamma-1}}\frac{1}{\rho c} \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{p} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{c}{\sqrt{\gamma}\rho}\tilde{\rho} \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ -\frac{c}{\sqrt{\gamma}\sqrt{\gamma-1}}\frac{\tilde{\rho}}{\rho} + \sqrt{\frac{\gamma}{\gamma-1}}\frac{1}{\rho c}\tilde{p} \end{pmatrix} = \tilde{u}. \tag{12}$$

The symmetrised matrices are derived and given in [10] and are repeated here for convenience. Let $a = \sqrt{\frac{\gamma-1}{\gamma}}c$ and $b = \frac{c}{\sqrt{\gamma}}$.

$$A_1 = \begin{pmatrix} u_1 & b & 0 & 0 & 0 \\ b & u_1 & 0 & 0 & a \\ 0 & 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 \\ 0 & a & 0 & 0 & u_1 \end{pmatrix}, A_2 = \begin{pmatrix} u_2 & 0 & b & 0 & 0 \\ 0 & u_2 & 0 & 0 & 0 \\ b & 0 & u_2 & 0 & a \\ 0 & 0 & 0 & u_2 & 0 \\ 0 & 0 & a & 0 & u_2 \end{pmatrix}, \tag{13}$$

$$A_3 = \begin{pmatrix} u_3 & 0 & 0 & b & 0 \\ 0 & u_3 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 0 & 0 \\ b & 0 & 0 & u_3 & a \\ 0 & 0 & 0 & a & u_3 \end{pmatrix}, B_{xy} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda+\mu}{\rho} & 0 & 0 \\ 0 & \frac{\lambda+\mu}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{14}$$

$$B_{yz} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda+\mu}{\rho} & 0 \\ 0 & 0 & \frac{\lambda+\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B_{zx} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda+\mu}{\rho} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda+\mu}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{15}$$

$$B_{11} = diag(0, \frac{\lambda+2\mu}{\rho}, \frac{\mu}{\rho}, \frac{\mu}{\rho}, \frac{\gamma\mu}{Pr\rho}), \tag{16}$$

$$B_{22} = diag(0, \frac{\mu}{\rho}, \frac{\lambda+2\mu}{\rho}, \frac{\mu}{\rho}, \frac{\gamma\mu}{Pr\rho}), \tag{17}$$

$$B_{33} = diag(0, \frac{\mu}{\rho}, \frac{\mu}{\rho}, \frac{\lambda+2\mu}{\rho}, \frac{\gamma\mu}{Pr\rho}). \tag{18}$$

3.2 Step 2: Apply the Energy Method

Next, we turn to the analysis of boundary conditions for the Navier-Stokes equations. Consider a general domain D with boundary ∂D in three space dimensions. From equation (11), the symmetrised and nondimensionalised Navier-Stokes equations are,

$$\tilde{u}_t + (A_1\tilde{u} - \epsilon\tilde{F}_v)_x + (A_2\tilde{u} - \epsilon\tilde{G}_v)_y + (A_3\tilde{u} - \epsilon\tilde{H}_v)_z, \tag{19}$$

where,

$$\tilde{F}_v = B_{11}\tilde{u}_x + B_{21}\tilde{u}_y + B_{31}\tilde{u}_z, \tag{20}$$

$$\tilde{G}_v = B_{22}\tilde{u}_y + B_{32}\tilde{u}_z + B_{12}\tilde{u}_x, \tag{21}$$

$$\tilde{H}_v = B_{33}\tilde{u}_z + B_{23}\tilde{u}_y + B_{13}\tilde{u}_x. \tag{22}$$

and,

$$B_{21} = B_{12} = B_{xy}/2, \quad B_{32} = B_{23} = B_{yz}/2, \quad B_{31} = B_{13} = B_{zx}/2.$$

Apply the energy method (step 2),

$$\begin{aligned} \int_D \tilde{u}^T \tilde{u}_t dxdydz + \int_D \frac{\partial}{\partial x}\left(\frac{1}{2}\tilde{u}^T A_1 \tilde{u} - \epsilon \tilde{u}^T \tilde{F}_v\right) + \\ \frac{\partial}{\partial y}\left(\frac{1}{2}\tilde{u}^T A_2 \tilde{u} - \epsilon \tilde{u}^T \tilde{G}_v\right) + \frac{\partial}{\partial z}\left(\frac{1}{2}\tilde{u}^T A_3 \tilde{u} - \epsilon \tilde{u}^T \tilde{H}_v\right) dxdydz = \\ -\epsilon \int_D (\tilde{u}_x^T \tilde{F}_v + \tilde{u}_y^T \tilde{G}_v + \tilde{u}_z^T \tilde{H}_v) dxdydz. \end{aligned} \tag{23}$$

The right-hand side in (23) is negative definite and denoted by $-DI$.

Remark It is easily verified that the last term in (23) is dissipation,

$$DI = \epsilon \int_D \left(\tilde{u}_x^T \, \tilde{u}_y^T \, \tilde{u}_z^T \right) \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \begin{pmatrix} \tilde{u}_x \\ \tilde{u}_y \\ \tilde{u}_z \end{pmatrix} dxdydz.$$

The matrix is symmetric with positive or zero diagonal entries. With $\lambda \leq \mu$ the matrix is diagonally dominant. Thus, it is positive semi-definite.

Denote by $\|\tilde{u}\|^2$ the integral $\int_D \tilde{u}^T \tilde{u} dxdydz$. Using Gauss' theorem we obtain,

$$\begin{aligned} \|\tilde{u}\|_t^2 + \oint_{\partial D} \left(\tilde{u}^T(A_1\tilde{u} - 2\epsilon\tilde{F}_v), \tilde{u}^T(A_2\tilde{u} - 2\epsilon\tilde{G}_v), \tilde{u}^T(A_3\tilde{u} - 2\epsilon\tilde{H}_v)\right) \cdot \hat{\mathbf{n}}\, ds \\ = -2DI, \end{aligned} \tag{24}$$

where $\hat{\mathbf{n}} = (n_1, n_2, n_3)$ is the outward pointing unit normal on the surface ∂D and $ds = \sqrt{dx^2 + dy^2 + dz^2}$. Equation (24) can be rewritten as,

$$\begin{aligned} \|\tilde{u}\|_t^2 + \oint_{\partial D} \begin{pmatrix} \tilde{u} \\ \tilde{F}^V \end{pmatrix}^T \begin{pmatrix} A_1 n_1 + A_2 n_2 + A_3 n_3 & -\epsilon I_5 \\ -\epsilon I_5 & 0_5 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{F}^V \end{pmatrix} ds \\ = -2DI, \end{aligned} \tag{25}$$

where I_n denotes the n by n identity matrix and similarly, 0_n the n by n zero matrix and $\tilde{F}^V = \tilde{F}_v n_1 + \tilde{F}_v n_2 + \tilde{F}_v n_3$.

To prove well posedness we have to split the matrix in the boundary integral into positive definite and negative definite parts. The negative part of the boundary term in (25) caused by,

$$\mathbf{A_1} = \begin{pmatrix} A_1 n_1 + A_2 n_2 + A_3 n_3 & -\epsilon I_5 \\ -\epsilon I_5 & 0_5 \end{pmatrix} \tag{26}$$

has to be supplied with boundary conditions, which in turn bounds the growth of $\|\tilde{u}\|_t^2$ in equation (23).

We note that the first component of $\tilde{F}^V$ is zero and hence we can reduce the system by omitting that component and denoting the resulting vector $\tilde{G}^V$. By this procedure $\mathbf{A_1}$ is also reduced from a 10×10 matrix to a 9×9 matrix by deleting the 6th row and column. With $\mathbf{u} = (u_1, u_2, u_3)$, we have,

$$\begin{pmatrix} \tilde{u} \\ \tilde{F}^V \end{pmatrix}^T \begin{pmatrix} A_1 n_1 + A_2 n_2 + A_3 n_3 & -\epsilon I_5 \\ -\epsilon I_5 & 0_5 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{F}^V \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{G}^V \end{pmatrix}^T \mathbf{A} \begin{pmatrix} \tilde{u} \\ \tilde{G}^V \end{pmatrix},$$

where,

$$\mathbf{A} = \begin{pmatrix} \mathbf{u}\cdot\hat{\mathbf{n}} & bn_1 & bn_2 & bn_3 & 0 & 0 & 0 & 0 & 0 \\ bn_1 & \mathbf{u}\cdot\hat{\mathbf{n}} & 0 & 0 & an_1 & -\epsilon & 0 & 0 & 0 \\ bn_2 & 0 & \mathbf{u}\cdot\hat{\mathbf{n}} & 0 & an_2 & 0 & -\epsilon & 0 & 0 \\ bn_3 & 0 & 0 & \mathbf{u}\cdot\hat{\mathbf{n}} & an_3 & 0 & 0 & -\epsilon & 0 \\ 0 & an_1 & an_2 & an_3 & \mathbf{u}\cdot\hat{\mathbf{n}} & 0 & 0 & 0 & -\epsilon \\ 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 \end{pmatrix} = \qquad (27)$$

$$\begin{pmatrix} A_{11} & A_{12} & 0_{14} \\ A_{21} & A_{22} & -\epsilon I_4 \\ 0_{41} & -\epsilon I_4 & 0_4 \end{pmatrix},$$

using the notation 0_{nm} for the n by m zero matrix. We will also use the notation $u_n = \mathbf{u}\cdot\hat{\mathbf{n}}$. Since $\hat{\mathbf{n}}$ is the outward pointing normal, $u_n < 0$ implies inflow.

3.3 Step 3: Find a Diagonalising Matrix

Next, we state and prove the following proposition, where $M_n = u_n/c$ is the Mach-number.

Proposition 3.1 *If $|M_n| \neq 1, 0$ and $u_n < 0$, there are 4 positive and 5 negative eigenvalues to* $\mathbf{A}$. *If $|M_n| \neq 1, 0$ and $u_n > 0$, there are 5 positive and 4 negative eigenvalues to* $\mathbf{A}$.

Proposition 3.1 states that an inflow demands 5 and an outflow 4 boundary conditions. The number of boundary conditions can also be derived using the Laplace transform technique, which is shown in Appendix II. However, to prove well posedness of specific boundary conditions using the Laplace transform technique is algebraically very complex as shown in Appendix II. In the proof of Proposition 3.1 we will continue with the procedure outlined in section 2 and find a diagonalising matrix to A (step 3). However, finding the eigenvalues of A corresponds to solving a ninth degree polynomial. Besides the algebraic difficulty of finding roots to ninth degree polynomial it is probable that the roots in this particular case do not exist on closed form. Instead, we will derive another diagonalising matrix.

Proof of Proposition 3.1 Rotate $\mathbf{A}$ by,

$$R^T \mathbf{A} R =$$
$$\begin{pmatrix} 1 & 0_{14} & 0_{14} \\ \bar{\alpha}^T & I_4 & 0_4 \\ \bar{\beta}^T & \bar{\gamma}^T & I_4 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & 0_{14} \\ A_{21} & A_{22} & -\epsilon I_4 \\ 0_{41} & -\epsilon I_4 & 0_4 \end{pmatrix} \begin{pmatrix} 1 & \bar{\alpha} & \bar{\beta} \\ 0_{41} & I_4 & \bar{\gamma} \\ 0_{41} & 0_4 & I_4 \end{pmatrix} = \tag{28}$$
$$\begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} = \mathbf{E},$$

where

$$\begin{aligned}
E_{11} &= A_{11} \\
E_{12} &= A_{11}\bar{\alpha} + A_{12} \\
E_{13} &= A_{11}\bar{\beta} + A_{12}\bar{\gamma} \\
E_{21} &= \bar{\alpha}^T A_{11} + A_{21} \\
E_{22} &= \bar{\alpha}^T (A_{11}\bar{\alpha} + A_{12}) + (A_{21}\bar{\alpha} + A_{22}) \\
E_{23} &= \bar{\alpha}^T (A_{11}\bar{\beta} + A_{12}\bar{\gamma}) + A_{21}\bar{\beta} + A_{22}\bar{\gamma} - \epsilon I_3 \\
E_{31} &= \bar{\beta}^T A_{11} + \bar{\gamma}^T A_{21} \\
E_{32} &= \bar{\beta}^T (A_{11}\bar{\alpha} + A_{12}) + \bar{\gamma}^T (A_{21}\bar{\alpha} + A_{22}) - \epsilon I_3 \\
E_{33} &= \bar{\beta}^T (A_{11}\bar{\beta} + A_{12}\bar{\gamma}) + \bar{\gamma}^T (A_{21}\bar{\beta} + A_{22}\bar{\gamma} - \epsilon I_3) - \epsilon I_3 \bar{\gamma}
\end{aligned}$$

Using $A_{12}^T = A_{21}$ we cancel the off-diagonal blocks and solve for $\bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$. We obtain,

$$\bar{\alpha} = -A_{11}^{-1} A_{12}, \quad \bar{\beta} = \epsilon A_{11}^{-1} A_{12} E_{22}^{-1}, \quad \bar{\gamma} = -\epsilon E_{22}^{-1}, \tag{29}$$

$$\mathbf{E} = \begin{pmatrix} A_{11} & 0_{14} & 0_{14} \\ 0_{41} & E_{22} & 0_4 \\ 0_{41} & 0_4 & -\epsilon^2 E_{22}^{-1} \end{pmatrix}, \quad E_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12} \tag{30}$$

The conditions for this procedure to hold is that $det(A_{11}) \neq 0$ and $det(E_{22}) \neq 0$.

We know from Corollary 2.4 that $i(\mathbf{A}) = i(\mathbf{E})$. Thus, we can as well determine the sign of the eigenvalues of $\mathbf{E}$. Note that, the upper left entry of $\mathbf{E}$ is a scalar and hence an eigenvalue. We denote that by,

$$\lambda_1 = A_{11} = u_n. \tag{31}$$

If $det(E_{22}) \neq 0$ we know that E_{22} has 4 real nonzero eigenvalues, since E_{22} is symmetric by construction. The sign of those do not change as E_{22} is inverted such that from the second and third block there are always 4 negative and 4 positive eigenvalues of $\mathbf{E}$. Including λ_1 we have for $u_n > 0$, 4 negative and 5 positive eigenvalues and, for $u_n < 0$, 5 negative and 4 positive eigenvalues of $\mathbf{E}$ as stated in the proposition (assuming that $det(A_{11}) \neq 0$ and $det(E_{22}) \neq 0$).

We will now show that $det(A_{11}) \neq 0$ and $det(E_{22}) \neq 0$ for $M_n \neq \pm 1, 0$. Since $A_{11} = u_n$, we have $det(A_{11}) \neq 0$ for $M_n \neq 0$. To evaluate the second condition, we compute the eigenvalues of E_{22} explicitly. From (27) and (30), we have,

$$E_{22} = \begin{pmatrix} -\frac{b^2 n_1^2}{u_n} + u_n & -\frac{b^2 n_1 n_2}{u_n} & -\frac{b^2 n_1 n_3}{u_n} & a n_1 \\ -\frac{b^2 n_1 n_2}{u_n} & -\frac{b^2 n_2^2}{u_n} + u_n & -\frac{b^2 n_2 n_3}{u_n} & a n_2 \\ -\frac{b^2 n_1 n_3}{u_n} & -\frac{b^2 n_2 n_3}{u_n} & -\frac{b^2 n_3^2}{u_n} + u_n & a n_3 \\ a n_1 & a n_2 & a n_3 & u_n \end{pmatrix} \tag{32}$$

and the eigenvalues are,

$$\lambda_{2,3} = \frac{-b^2 + 2u_n^2 \pm \sqrt{b^4 + 4a^2 u_n^2}}{2u_n} \tag{33}$$

$$\lambda_4 = \lambda_5 = u_n \tag{34}$$

where $n_1^2 + n_2^2 + n_3^2 = 1$ has been used to simplify the expressions. λ_4 and λ_5 obviously shifts sign at $u_n = 0$. Also, since $\lambda_4 = \lambda_5 = 0$ with $M_n = u_n = 0$ we have that $det(E_{22}) = 0$. Thus, to rotate $\mathbf{A}$ by R we, once more, need $M_n \neq 0$. λ_2 and λ_3 can be expressed as,

$$\lambda_{2,3} = \frac{c}{2\gamma M_n}\left(-1 + 2\gamma M_n^2 \pm \sqrt{1 + 4(\gamma - 1)\gamma M_n^2}\right), \tag{35}$$

Consider, λ_2 and note that, $\gamma \geq 1$. Then $\sqrt{1 - 4\gamma M_n^2 + 4\gamma^2 M_n^2} \geq 1$ such that the sign of λ_2 is the same as the sign of the denominator, i.e. M_n or u_n. This means that $\lambda_2 \neq 0$ for all $M_n \neq 0$ and $\lambda_2 = 0$ for $M_n = 0$.

At last, λ_3 is considered. λ_3 shifts sign when

$$2\gamma M_n^2 - 1 - \sqrt{1 - 4\gamma M_n^2 + 4\gamma^2 M_n^2} = 0.$$

Or, $(2\gamma M_n^2 - 1)^2 = (1 - 4\gamma M_n^2 + 4\gamma^2 M_n^2)$ which has the solutions $M_n = 0, 1, -1$, but $M_n = 0$ is discarded due to the original equality. Thus, $\lambda_3 \neq 0$, and hence $det(E_{22}) \neq 0$, for $|M_n| \neq 1$. Note that, λ_3 is singular for $M_n = 0$. ∎

We have now derived the number of positive and negative eigenvalues of $\mathbf{A}$, and hence the number of boundary conditions, and their dependence on M_n. This was done by calculating the eigenvalues of $\mathbf{E}$ explicitly.

To obtain a set of boundary conditions, we also need the eigenvectors of $\mathbf{E}$. Given the eigenvectors of $\mathbf{E}$, it is a simple task to derive a diagonalising matrix to $\mathbf{A}$. The eigenvectors of E_{22} are possible to derive explicitly since the eigenvalues are explicitly given and they are $Y = (y_2, y_3, y_4, y_5)$, where,

$$\begin{aligned} y_2 &= \left(n_1, n_2, n_3, -\frac{-b^4 - \sqrt{b^2 + 4a^2u_n^2}}{2au_n}\right)^T = \\ &\left(n_1, n_2, n_3, \frac{-\lambda_3 + u_n}{a}\right)^T, \quad (37) \\ y_3 &= \left(n_1, n_2, n_3, -\frac{-b^4 + \sqrt{b^2 + 4a^2u_n^2}}{2au_n}\right)^T = \\ &\left(n_1, n_2, n_3, \frac{-\lambda_2 + u_n}{a}\right)^T, \quad (38) \\ y_4 &= (-n_2, n_1, 0, 0)^T, \quad (39) \\ y_5 &= (-n_3, 0, n_1, 0). \quad (40) \end{aligned}$$

Remark We omit the normalisation of the eigenvectors to keep the expressions (37)-(40) simple.

Now, we can derive a specific diagonalising matrix M and conclude step 3. For convenience, we restate equation (8),

$$\tilde{v}^T M^{-1,T} M^T \mathbf{A} M M^{-1} \tilde{v} = \tilde{w}^T \mathbf{\Lambda_M} \tilde{w}$$

where $M = RX$ and $\tilde{v} = (\tilde{u}^T \, (\tilde{G}^V)^T)^T$. R is given in (28), (29) and (30). Further,

$$X = \begin{pmatrix} 1 & 0_{14} & 0_{14} \\ 0_{41} & Y & 0_4 \\ 0_{41} & 0_4 & Y \end{pmatrix}, \mathbf{\Lambda_M} = \begin{pmatrix} u_n & 0_{14} & 0_{14} \\ 0_{41} & \Lambda & 0_4 \\ 0_{41} & 0_4 & -\epsilon^2\Lambda^{-1} \end{pmatrix},$$

where $\Lambda = diag(\lambda_2, \lambda_3, \lambda_4, \lambda_5)$. Inverting R and M yield,

$$R^{-1} = \begin{pmatrix} 1 & -\bar{\alpha} & 0_{14} \\ 0_{41} & I_4 & -\bar{\gamma} \\ 0_{41} & 0_4 & I_4 \end{pmatrix}, M^{-1} = X^T R^{-1} = \begin{pmatrix} 1 & -\bar{\alpha} & 0_{14} \\ 0_{41} & Y^T & -Y^T\bar{\gamma} \\ 0_{41} & 0_4 & Y^T \end{pmatrix}. \quad (41)$$

To simplify the computation of M^{-1} we use (29), and obtain,

$$-Y^T\gamma = \epsilon Y^T E_{22}^{-1} = \epsilon Y^T Y \Lambda^{-1} Y^T = \epsilon \Lambda^{-1} Y^T = \epsilon \begin{pmatrix} \lambda_2^{-1} y_2^T \\ \lambda_3^{-1} y_3^T \\ \lambda_4^{-1} y_4^T \\ \lambda_5^{-1} y_5^T \end{pmatrix}, \tag{42}$$

yielding,

$$M^{-1} = \begin{pmatrix} 1 & -\bar{\alpha} & 0_{14} \\ 0_{41} & Y^T & \epsilon\Lambda^{-1}Y^T \\ 0_{41} & 0_4 & Y^T \end{pmatrix}, \text{ where } \bar{\alpha} = \left(-\frac{b}{u_n}\hat{\mathbf{n}}, 0\right). \tag{43}$$

We proceed by computing the variables, $\tilde{w} = X^T R^{-1}\tilde{v} = M^{-1}\tilde{v}$ to which boundary conditions should be applied. Let $\tilde{G}_i^V$ be the ith component of $\tilde{G}^V$. Define $\tilde{v}_{i..j} = (\tilde{v}_i, ..., \tilde{v}_j)^T$, and $\tilde{u}_n = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \cdot \hat{\mathbf{n}}$. For convenience, we restate $\tilde{v}$,

$$\tilde{v} = (\frac{b}{\rho}\tilde{\rho}, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3, -\frac{b}{\sqrt{\gamma - 1}}\frac{\tilde{\rho}}{\rho} + \frac{1}{\rho a}\tilde{p}, \tilde{G}_1^V, \tilde{G}_2^V, \tilde{G}_3^V, \tilde{G}_4^V)^T \tag{44}$$

Then,

$$\tilde{w} = M^{-1}\tilde{v} = \begin{pmatrix} \tilde{v}_1 - \bar{\alpha}\cdot\tilde{v}_{2..5} \\ y_2^T(\tilde{v}_{2..5} - \epsilon\lambda_2^{-1}\tilde{G}^V) \\ y_3^T(\tilde{v}_{2..5} - \epsilon\lambda_3^{-1}\tilde{G}^V) \\ y_4^T(\tilde{v}_{2..5} - \epsilon\lambda_4^{-1}\tilde{G}^V) \\ y_5^T(\tilde{v}_{2..5} - \epsilon\lambda_5^{-1}\tilde{G}^V) \\ y_2^T\tilde{G}^V \\ y_3^T\tilde{G}^V \\ y_4^T\tilde{G}^V \\ y_5^T\tilde{G}^V \end{pmatrix}, \tag{45}$$

by using (37)-(40).

For completeness we also give the reverse transformation. It is $\tilde{v} = RX\tilde{w} = M\tilde{w}$,

$$M = \begin{pmatrix} 1 & \bar{\alpha}Y & \bar{\beta}Y \\ 0_{31} & Y & \bar{\gamma}Y \\ 0_{31} & 0_3 & Y \end{pmatrix} = \begin{pmatrix} 1 & \bar{\alpha}Y & \bar{\alpha}\Lambda^{-1}Y^T \\ 0_{31} & Y & -\epsilon Y\Lambda \\ 0_{31} & 0_3 & Y \end{pmatrix}. \tag{46}$$

The corresponding diagonalising matrices for the two–dimensional case are given in Appendix I.

Remark Note that we have found one of possibly several diagonalising matrices. M is not orthonormal which means that $\mathbf{\Lambda_M}$ does not hold the eigenvalues of $\mathbf{A}$.

eigenvalue	$M_n < -1$	$-1 < M_n < 0$	$0 < M_n < 1$	$M_n > 1$
λ_1	-	-	+	+
λ_2	-	-	+	+
λ_3	-	+	-	+
λ_4	-	-	+	+
λ_5	-	-	+	+
λ_6	+	+	-	-
λ_7	+	-	+	-
λ_8	+	+	-	-
λ_9	+	+	-	-

Table 1: The sign of the eigenvalues for different Mach-numbers.

3.4 Step 4 and 5: Split $\mathbf{\Lambda_M}$ and $\tilde{w}$

In order to know which components of $\tilde{w}$ to bound with boundary conditions we need to investigate the sign of the diagonal entries of $\mathbf{\Lambda_M}$, i.e. the eigenvalues of $\mathbf{E}$. (Step 4)

In the proof of Proposition 3.1, λ_3 given by equation (35) was analysed. It was shown that λ_3 changes sign at $M_n = 0$ and $|M_n| = 1$. The eigenvalues, $\lambda_1, \lambda_2, \lambda_4$, and λ_5 only change signs at $M_n = 0$. Thus, the different cases are; inflow or outflow and sub- or supersonic flow. A consequence is, that sub- or supersonic flow affects what boundary conditions to choose, but not the number of them. In fact, only the boundary condition corresponding to λ_3 (and hence $-\epsilon^2\lambda_3^{-1} \equiv \lambda_7$) changes sign at $|M_n| = 1$. With, $\Lambda = diag(\lambda_2, \lambda_3, \lambda_4, \lambda_5)$, the diagonal form of $\mathbf{E}$ is $\mathbf{\Lambda_M} = diag(\lambda_1, \Lambda, -\epsilon^2\Lambda^{-1})$. In Table 1 the sign of the different eigenvalues are summarised where $\lambda_6, ..., \lambda_9$ denotes the diagonal entries of $-\epsilon^2\Lambda^{-1}$. Those with negative signs has to be supplied with boundary conditions. As mentioned above, since $\hat{\mathbf{n}}$ is the outward pointing normal negative values of M_n indicate inflow and positive values means outflow.

In Table 2 the number of boundary conditions deduced from Table 1 for different flow cases are shown. They are in full agreement with the results from the Laplace transform technique derived in [14] and also in Appendix II. Note that in the Euler limit, i.e. $\epsilon \to 0$, the last 4 eigenvalues will become zero and there are 5 nontrivial eigenvalues. In Table 3 the number of boundary conditions are displayed for the Euler case, $\epsilon \to 0$. The result agrees with the well known theory for the Euler equations.

At last, we can split $\tilde{w}$ given by (45) into $\tilde{w}^+$ and w^- corresponding to the positive and negative eigenvalues and perform step 5, such that well posedness follows.

supersonic inflow	5
subsonic inflow	5
subsonic outflow	4
supersonic outflow	4

Table 2: The number of boundary conditions to be specified at different flow cases for the three-dimensional Navier-Stokes equations.

supersonic inflow	5
subsonic inflow	4
subsonic outflow	1
supersonic outflow	0

Table 3: The number of boundary conditions to be specified at different flow cases for the three-dimensional Euler equations.

3.5 Special case: $u_n = 0$

The above derivation gives a set of boundary conditions that leads to a well posed mathematical problem. However, it is assumed that $u_n \neq 0$ which excludes two cases: tangential flow and the important solid wall condition. We will treat the case $u_n = 0$ separately and redo the steps 3-5. Throughout this paper, we have considered the Navier-Stokes equations linearised around the solution at the boundary, in this case $u_n = 0$. We obtain,

$$\mathbf{A} = \begin{pmatrix} 0 & bn_1 & bn_2 & bn_3 & 0 & 0 & 0 & 0 & 0 \\ bn_1 & 0 & 0 & 0 & an_1 & -\epsilon & 0 & 0 & 0 \\ bn_2 & 0 & 0 & 0 & an_2 & 0 & -\epsilon & 0 & 0 \\ bn_3 & 0 & 0 & 0 & an_2 & 0 & 0 & -\epsilon & 0 \\ 0 & an_1 & an_2 & an_3 & 0 & 0 & 0 & 0 & -\epsilon \\ 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (47)$$

to which the previous rotation does not apply. This leaves us with no other choice but to seek the eigenvalues and eigenvectors of this matrix. It turns out that it is now a simpler task than with $u_n \neq 0$. The result is presented below and the details of the derivation are found in Appendix III.

Define $\mathbf{m_1}$ and $\mathbf{m_2}$ such that, $\hat{\mathbf{n}}^T\mathbf{m_1} = 0$, $\hat{\mathbf{n}}^T\mathbf{m_2} = \mathbf{m_1}^T\mathbf{m_2} = 0$ and,

$$\mu_{1,2} = -\frac{c^2}{2} \pm \sqrt{\frac{c^4}{4} + a^2\epsilon^2}.$$

Then,

$$\begin{aligned}
\lambda_1 &= -\epsilon, & e_1 &= (0, \mathbf{m_1}^T, 0, \mathbf{m_1}^T, 0)^T \\
\lambda_2 &= -\epsilon, & e_2 &= (0, \mathbf{m_2}^T, 0, \mathbf{m_2}^T, 0)^T \\
\lambda_3 &= \epsilon, & e_3 &= (0, \mathbf{m_1}^T, 0, -\mathbf{m_1}^T, 0)^T \\
\lambda_4 &= \epsilon, & e_4 &= (0, \mathbf{m_2}^T, 0, -\mathbf{m_2}^T, 0)^T \\
\lambda_5 &= 0, & e_5 &= (1, 0, 0, 0, 0, \tfrac{b}{\epsilon}\hat{\mathbf{n}}, 0^T) \\
\lambda_6 &= \sqrt{\epsilon^2 - \mu_1}, & e_6 &= (b, \lambda_6\hat{\mathbf{n}}^T, -\tfrac{a\lambda_6^2}{\mu_1^2}, -\epsilon\hat{\mathbf{n}}^T, \tfrac{\epsilon a\lambda_6}{\mu_1^2})^T \\
\lambda_7 &= -\sqrt{\epsilon^2 - \mu_1}, & e_7 &= (b, \lambda_7\hat{\mathbf{n}}^T, -\tfrac{a\lambda_7^2}{\mu_1^2}, -\epsilon\hat{\mathbf{n}}^T, \tfrac{\epsilon a\lambda_7}{\mu_1^2})^T \\
\lambda_8 &= \sqrt{\epsilon^2 - \mu_2}, & e_8 &= (b, \lambda_8\hat{\mathbf{n}}^T, -\tfrac{a\lambda_8^2}{\mu_2^2}, -\epsilon\hat{\mathbf{n}}^T, \tfrac{\epsilon a\lambda_8}{\mu_2^2})^T \\
\lambda_9 &= -\sqrt{\epsilon^2 - \mu_2}, & e_9 &= (b, \lambda_9\hat{\mathbf{n}}^T, -\tfrac{a\lambda_9^2}{\mu_2^2}, -\epsilon\hat{\mathbf{n}}^T, \tfrac{\epsilon a\lambda_9}{\mu_2^2})^T
\end{aligned} \tag{48}$$

Remark With some algebra one can show that $\epsilon^2 \geq \mu_{1,2}$ such that the eigenvalues $\lambda_6, .., \lambda_9$ are real. In fact, since $\mathbf{A}$ is symmetric and the vectors $e_1, .., e_9$ are orthogonal and diagonalises $\mathbf{A}$, $\lambda_1, ...\lambda_9$ have to be real.

Above, step 3 is performed and we turn to step 4. We have,

$$\begin{aligned}
\mathbf{\Lambda}^- &= diag(\lambda_1, \lambda_2, 0, 0, 0, 0, \lambda_7, 0, \lambda_9), \\
\mathbf{\Lambda}^+ &= diag(0, 0, \lambda_3, \lambda_4, 0, \lambda_6, 0, \lambda_8, 0).
\end{aligned}$$

Remark Note that we have 4 negative eigenvalues. This means that a boundary with $u_n = 0$ is classified as an outflow boundary.

Further, $\tilde{w} = X^T v$ where the column vectors of X are the eigenvectors. With $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)^T$, $\tilde{G}^V_{i..j} = (\tilde{G}^V_i, .., \tilde{G}^V_j)^T$ and the ith component of $\tilde{v}$ denoted by $\tilde{v}_i$ we obtain,

$$\tilde{w} = \begin{pmatrix}
\mathbf{m_1}^T(\tilde{\mathbf{u}} + \tilde{G}^V_{1..3}) \\
\mathbf{m_2}^T(\tilde{\mathbf{u}} + \tilde{G}^V_{1..3}) \\
\mathbf{m_1}^T(\tilde{\mathbf{u}} - \tilde{G}^V_{1..3}) \\
\mathbf{m_2}^T(\tilde{\mathbf{u}} - \tilde{G}^V_{1..3}) \\
v_1 + \frac{b}{\epsilon}\hat{\mathbf{n}}^T(\tilde{G}^V)_{1..3} \\
bv_1 + \hat{\mathbf{n}}^T(\lambda_6\tilde{\mathbf{u}} - \epsilon\tilde{G}^V_{1..3}) - \frac{a\lambda_6}{\mu_1^2}(\lambda_6 v_4 - \epsilon\tilde{G}^V_4) \\
bv_1 + \hat{\mathbf{n}}^T(\lambda_7\tilde{\mathbf{u}} - \epsilon\tilde{G}^V_{1..3}) - \frac{a\lambda_7}{\mu_1^2}(\lambda_7 v_4 - \epsilon\tilde{G}^V_4) \\
bv_1 + \hat{\mathbf{n}}^T(\lambda_8\tilde{\mathbf{u}} - \epsilon\tilde{G}^V_{1..3}) - \frac{a\lambda_8}{\mu_2^2}(\lambda_8 v_4 - \epsilon\tilde{G}^V_4) \\
bv_1 + \hat{\mathbf{n}}^T(\lambda_9\tilde{\mathbf{u}} - \epsilon\tilde{G}^V_{1..3}) - \frac{a\lambda_9}{\mu_2^2}(\lambda_9 v_4 - \epsilon\tilde{G}^V_4)
\end{pmatrix}. \tag{49}$$

Finally, we can split $\tilde{w}$ into $\tilde{w}^+$ and $\tilde{w}^-$ as before and perform step 5, i.e. supply $\tilde{w}^-$ with boundary conditions to obtain a well posed system.

Remark There are two more cases where $u_n \neq 0$. Those are tangential flows with $|M_n| = 1$. To find the eigenvalues of $\mathbf{A}$ directly for $M_n = 1, -1$ is equally difficult as in the general case and we did not find roots on closed form.

3.6 Curvilinear Coordinates

Until now, we have analysed well-posed boundary conditions for the Navier-Stokes equations in a Cartesian coordinate system with a general domain. Considering numerical computations that derivation suffice when using unstructured methods such as finite volume schemes. However, for structured methods, for example finite difference schemes, the Navier-Stokes equations are usually expressed in a curvilinear coordinate system. We have included a breif analysis in Appendix IV showing that the Cartesian results are directly applicable in the curvilinear case through metric transformations.

4 Conclusions

We have proposed a step-by-step procedure to analyse a general time dependent partial differential equation in terms of well posedness including boundary conditions. The procedure applied to the Euler equations results in the well known characteristic boundary conditions.

Applying the procedure to the three-dimensional Navier-Stokes equations on a general domain results in a novel set of well posed boundary conditions.

APPENDIX

I The Two-Dimensional Matrices

With very few comments and leaving most details, we show the differences of the derivation in Section 3 for the two-dimensional case.

With,

$$B_{21} = B_{12} = B_{xy}/2,$$

the symmetrised equations are,

$$\tilde{u}_t + A_1\tilde{u}_x + A_2\tilde{u}_y = \epsilon(B_{11}\tilde{u}_{xx} + B_{22}\tilde{u}_{yy} + B_{12}\tilde{u}_{xy} + B_{21}\tilde{u}_{yx}).$$

The matrices are obtained by deleting the row and column referring to the u_3 component (see [10]). Introduce,

$$\tilde{F}_v = B_{11}\tilde{u}_x + B_{21}\tilde{u}_y, \quad \tilde{G}_v = B_{22}\tilde{u}_y + B_{12}\tilde{u}_x,$$

such that,

$$\frac{1}{2}\|\tilde{u}\|_t^2 + \oint_{\partial D} \frac{1}{2} \begin{pmatrix} \tilde{u} \\ \tilde{F}^V \end{pmatrix}^T \begin{pmatrix} A_1 n_1 + A_2 n_2 & -\epsilon I_4 \\ -\epsilon I_4 & 0_4 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{F}^V \end{pmatrix} = DI,$$

where $\hat{\mathbf{n}} = [n_1, n_2]$, $ds = \sqrt{dx^2 + dy^2}$ and $\tilde{F}^V = \tilde{F}_v n_1 + \tilde{G}_v n_2$.

By deleting the first component of $\tilde{F}^V$ yielding $\tilde{G}^V$, the matrix is reduced from an 8×8 matrix to a 7×7 matrix. With $\mathbf{u} = (u_1, u_2)$, we obtain,

$$\begin{pmatrix} \tilde{u} \\ \tilde{F}^V \end{pmatrix}^T \begin{pmatrix} A_1 n_1 + A_2 n_2 & -\epsilon I_4 \\ -\epsilon I_4 & 0_4 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{F}^V \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{G}^V \end{pmatrix}^T \mathbf{A} \begin{pmatrix} \tilde{u} \\ \tilde{G}^V \end{pmatrix},$$

where,

$$\mathbf{A} = \begin{pmatrix} \mathbf{u}\cdot\hat{\mathbf{n}} & bn_1 & bn_2 & 0 & 0 & 0 & 0 \\ bn_1 & \mathbf{u}\cdot\hat{\mathbf{n}} & 0 & an_1 & -\epsilon & 0 & 0 \\ bn_2 & 0 & \mathbf{u}\cdot\hat{\mathbf{n}} & an_2 & 0 & -\epsilon & 0 \\ 0 & an_1 & an_2 & \mathbf{u}\cdot\hat{\mathbf{n}} & 0 & 0 & -\epsilon \\ 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & 0_{14} \\ A_{21} & A_{22} & -\epsilon I_4 \\ 0_{41} & -\epsilon I_4 & 0_4 \end{pmatrix}.$$

The rotation of $\mathbf{A}$ is precisely similar,

$$\mathbf{R}^T\mathbf{A}\mathbf{R} =$$

$$\begin{pmatrix} 1 & 0_{13} & 0_{13} \\ \bar{\alpha}^T & I_3 & 0_3 \\ \bar{\beta}^T & \bar{\gamma}^T & I_3 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & 0_{13} \\ A_{21} & A_{22} & -\epsilon I_3 \\ 0_{31} & -\epsilon I_3 & 0_3 \end{pmatrix} \begin{pmatrix} 1 & \bar{\alpha} & \bar{\beta} \\ 0_{31} & I_3 & \bar{\gamma} \\ 0_{31} & 0_3 & I_3 \end{pmatrix} =$$

$$\begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} = \mathbf{E}.$$

The same solution is obtained,

$$\bar{\alpha} = -A_{11}^{-1}A_{12}, \quad \bar{\beta} = A_{11}^{-1}A_{12}E_{22}^{-1}, \quad \bar{\gamma} = -\epsilon E_{22}^{-1},$$

$$\mathbf{E} = \begin{pmatrix} A_{11} & 0_{14} & 0_{14} \\ 0_{41} & E_{22} & 0_4 \\ 0_{41} & 0_4 & -\epsilon^2 E_{22}^{-1} \end{pmatrix}, \quad E_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}.$$

The first eigenvlaue of $\mathbf{E}$ is $\lambda_1 = A_{11} = u_n$ and the others are given by the eigenvalues of E_{22},

$$E_{22} = \begin{pmatrix} -\frac{b^2 n_1^2}{u_n} + u_n & -\frac{b^2 n_1 n_2}{u_n} & a n_1 \\ -\frac{b^2 n_1 n_2}{u_n} & -\frac{b^2 n_2^2}{u_n} + u_n & a n_2 \\ a n_1 & a n_2 & u_n \end{pmatrix}$$

$$\lambda_{2,3} = \frac{-b^2 + 2u_n^2 \pm \sqrt{b^4 + 4a^2 u_n^2}}{2u_n}, \quad \lambda_4 = u_n$$

where $n_1^2 + n_2^2 = 1$ and $u_n = \mathbf{u} \cdot \hat{\mathbf{n}}$. These can be simplified similarly as for the three-dimensional case.

The eigenvectors $Y = (y_2, y_3, y_4)$ are,

$$y_2 = \begin{pmatrix} n_1 \\ n_2 \\ \frac{-\lambda_3 + u_n}{a} \end{pmatrix}, y_3 = \begin{pmatrix} n_1 \\ n_2 \\ \frac{-\lambda_2 + u_n}{a} \end{pmatrix}, y_4 = \begin{pmatrix} -n_2, \\ n_1 \\ 0 \end{pmatrix}. \quad (50)$$

Introduce the block matrix, $X = diag(1, Y, Y)$, such that, $X^T\mathbf{E}X = \mathbf{\Lambda}$ where $\mathbf{\Lambda} = diag(u_n, \Lambda, -\epsilon^2\Lambda)$. Let $\tilde{v} = (\tilde{u}^T, (\tilde{G}^V)^T)^T$, then $\tilde{v}^T\mathbf{A}\tilde{v} = \tilde{w}^T\mathbf{\Lambda}\tilde{w}$ where $\tilde{w} = X^T R^{-1}\tilde{v} = M^{-1}\tilde{v}$ and $\mathbf{\Lambda} = M^T\mathbf{A}M$. The matrices are,

$$R^{-1} = \begin{pmatrix} 1 & -\bar{\alpha} & 0_{13} \\ 0_{31} & I_3 & -\bar{\gamma} \\ 0_{31} & 0_3 & I_3 \end{pmatrix}, M^{-1} = \begin{pmatrix} 1 & -\bar{\alpha} & 0_{14} \\ 0_{41} & Y^T & \epsilon\Lambda^{-1}Y^T \\ 0_{41} & 0_4 & Y^T \end{pmatrix}$$

where,

$$\Lambda^{-}Y^{T}=\begin{pmatrix}\lambda_2^{-1}y_2\\ \lambda_3^{-1}y_3\\ \lambda_4^{-1}y_4\end{pmatrix},\ \bar{\alpha}=\left(-\frac{b}{u_n}\mathbf{\hat{n}},0\right),\ M=\begin{pmatrix}1 & \bar{\alpha}Y & \bar{\alpha}\Lambda^{-1}Y^{T}\\ 0_{31} & Y & -\epsilon Y\Lambda\\ 0_{31} & 0_3 & Y\end{pmatrix}.$$

In two dimensions, $\tilde{v}$ is,

$$\tilde{v}=(\frac{b}{\rho}\frac{\tilde{\rho}}{\rho},\tilde{u}_1,\tilde{u}_2,-\frac{b}{\sqrt{\gamma-1}}\tilde{\rho}+\frac{1}{\rho a}\tilde{p},\tilde{G}_1^V,\tilde{G}_2^V,\tilde{G}_3^V).$$

Then,

$$\tilde{w}=M^{-1}\tilde{v}=\begin{pmatrix}\tilde{v}_1-\bar{\alpha}\cdot\tilde{v}_{2..4}\\ y_2^T(\tilde{v}_{2..4}-\epsilon\lambda_2^{-1}\tilde{G}^V)\\ y_3^T(\tilde{v}_{2..4}-\epsilon\lambda_3^{-1}\tilde{G}^V)\\ y_4^T(\tilde{v}_{2..4}-\epsilon\lambda_4^{-1}\tilde{G}^V)\\ y_2^T\tilde{G}^V\\ y_3^T\tilde{G}^V\\ y_4^T\tilde{G}^V\end{pmatrix}. \tag{51}$$

II The Laplace Transform Technique

The Laplace transform technique is a general way to analyse well posedness of linear partial differential equations. (See [7] for a thorough explanation.) A major drawback is that it often leads to very complex algebra, in practice a complete analysis can only be performed on very simple model-like problems.

However, this technique can be used to determine the number of boundary conditions with reasonable complexity. A general analysis of the problem with boundary conditions to the Navier-Stokes equations is not an easy task with this technique.

It can be shown (see [7]) that the full three-dimensional problem is well posed if the quarter space problem is well posed. Thus we only need to consider a problem of the kind,

$$\begin{aligned}u_t &= P(x,y,z,t)u,\ x\geq 0\\ u(x,y,z,0) &= f,\\ L_0^x u(x_0,y,z,t) &= g_0(y,z,t),\\ \|u\| &\leq \infty.\end{aligned} \tag{52}$$

In the y and z directions, we consider wave solutions, i.e. we study u of the form,

$$u(x,y,z,t)=e^{i\omega y+i\zeta z}\phi(x,t). \tag{53}$$

If the differential operator P in (52) is the symmetrised and linearised Navier-Stokes equations, given in (11), equation (53) yields,

$$\begin{aligned}\phi_t = (-i\omega A_2 - i\zeta A_3 - \epsilon\omega^2 B_{22} - \epsilon\zeta^2 B_{33} - \omega\zeta B_{yz})\phi + \\ (-A_1 + i\omega\epsilon B_{xy} + i\zeta B_{zx})\phi_x + \epsilon B_{11}\phi_{xx} \\ x \geq 0,\ \omega, \zeta \in \mathbf{R}.\end{aligned} \tag{54}$$

Equation (54) is Laplace transformed and the anstatz, $\hat{\phi}(x) = e^{\kappa x}\psi$ is used,

$$\begin{aligned}s\psi = (-i\omega A_2 - i\zeta A_3 - \epsilon\omega^2 B_{22} - \epsilon\zeta^2 B_{33} - \omega\zeta B_{yz})\psi + \\ (-A_1 + i\omega\epsilon B_{xy} + i\zeta B_{zx})\kappa\psi + \epsilon B_{11}\kappa^2\psi. \\ x \geq 0,\ \omega, \zeta \in \mathbf{R}.\end{aligned} \tag{55}$$

There are nontrivial solutions if,

$$\begin{aligned}det(sI + i\omega A_2 + i\zeta A_3 + \epsilon\omega^2 B_{22} + \epsilon\zeta^2 B_{33} + \\ (A_1 - i\omega\epsilon B_{xy} - i\zeta B_{zx})\kappa - \epsilon B_{11}\kappa^2) = 0.\end{aligned} \tag{56}$$

Equation (56) is a ninth degree polynomial in $\kappa = \kappa(s)$. In principal we need to solve this equation and deduce which κ:s have positive and negative real parts. Those with positive real parts are discarded due to the bounded norm of the solution and the number of κ:s with negative real part determines the number of boundary conditions. The κ:s with negative real parts for $Re\, s > 0$ are used in the ansatz which is inserted into the Laplace transformed boundary conditions. If there are no eigenvalues with $Re\, s > 0$ the problem is well posed. However, this is usually a nontrivial task algebraically.

Our aim here is to derive the number of boundary conditions and for that reason it suffice to determine the sign of the roots of (56). To simplify this task, we can let $Re\, s$ be arbitrarly large. In fact, if $Re\, s$ is sufficiently large, the terms involving ω and ζ can be neglected since they will not affect the singularity of (56). Hence, we assume that,

$$\frac{\omega}{s}, \frac{\omega^2}{s}, \frac{\zeta}{s}, \frac{\zeta^2}{s} << 1, \tag{57}$$

and obtain the simplified determinant condition,

$$det(sI + A_1\kappa - \epsilon B_{11}\kappa^2) = 0. \tag{58}$$

Using the specific form of A_1 and B_{11}, given in (13) and (16), (58) becomes,

$$\begin{aligned}\left(s + u_1\kappa - \epsilon\tfrac{\mu}{\rho}\kappa^2\right)^2 ((s + u_1\kappa)(s + u_1\kappa - \epsilon\tfrac{\lambda+2\mu}{\rho}\kappa^2)(s + u_1\kappa - \epsilon\tfrac{\gamma\mu}{Pr\rho}\kappa^2) - \\ (s + u_1\kappa)a^2\kappa^2 - b\kappa^2(s + u_1\kappa - \epsilon\tfrac{\gamma\mu}{Pr\rho})) = 0.\end{aligned} \tag{59}$$

The first factor yields four roots, $\kappa_{1,2} = +\sqrt{\frac{\rho s}{\epsilon\mu}} + \mathcal{O}(s^{0.5-\delta})$ and $\kappa_{3,4} = -\sqrt{\frac{\rho s}{\epsilon\mu}} + \mathcal{O}(s^{0.5-\delta})$ for $Re\, s >> 0$ and $\delta > 0$. Next, we will determine the first term of a series expansion of κ as solutions to,

$$\begin{gathered}(s + u_1\kappa)(s + u_1\kappa - \epsilon\tfrac{\lambda+2\mu}{\rho}\kappa^2)(s + u_1\kappa - \epsilon\tfrac{\gamma\mu}{Pr\rho}\kappa^2) - \\ (s + u_1\kappa)a^2\kappa^2 - b\kappa^2(s + u_1\kappa - \epsilon\tfrac{\gamma\mu}{Pr\rho}) = 0,\end{gathered} \tag{60}$$

or showing a few terms from the expansion of the parentheses,

$$\begin{gathered}\frac{\epsilon^2(\lambda + 2\mu)\gamma\mu}{\rho^2 Pr}(u_1\kappa^5 + s\kappa^4) + s^3 - s^2\kappa^2\epsilon\frac{\lambda + 2\mu}{\rho} - s^2\kappa^2\frac{\epsilon\gamma\mu}{Pr\rho}... \\ = p_1p_2(u_1\kappa^5 + s\kappa^4) + s^3 - (p_1 + p_2)s^2\kappa^2... = 0.\end{gathered} \tag{61}$$

Note that $p_1, p_2 > 0$. We make the ansatz, $\kappa = c_1 s^\alpha$. The largest term in (61) is $u\kappa^5 p_1p_2$, if $\alpha \geq 1$, which has to be balanced by $s\kappa^4 p_1p_2$. The solution is, $\alpha = 1$, which is consistent with the assumption, and $c_1 = -1/u_1$. We have $\kappa_5 = -\frac{s}{u_1} + \mathcal{O}(s^{1-\delta})$. Further, $\kappa_{6..9} = \mathcal{O}(s^{1-\delta})$. We proceed with the ansatz $\kappa = c_2 s^\beta$, $\beta < \alpha = 1$. With this ansatz, $s\kappa^4 p_1$ will be the largest term. We obtain the largest possible β if it is balanced by the term s^3 and $-(p_1 + p_2)s^2\kappa^2$. Then $\beta = 0.5$, and $p_1p_2c_2^4 = -1 + c_2^2(p_2 + p_2)$ with the solution,

$$c_2^2 = \begin{cases} \frac{\rho Pr}{\epsilon\mu\gamma} \\ \frac{\rho}{(\lambda+2\mu)\epsilon} \end{cases}, \tag{62}$$

another square root yields two positive and two negative κ:s.

To summarise, we give all the solutions to (56) for $Re\, s >> 1$ and $\delta > 0$,

$$\begin{aligned}
\kappa_{1,2} &= \sqrt{\frac{\rho}{\epsilon\mu}}\sqrt{s} + \mathcal{O}(s^{0.5-\delta}), \\
\kappa_{3,4} &= -\sqrt{\frac{\rho}{\epsilon\mu}}\sqrt{s} + \mathcal{O}(s^{0.5-\delta}), \\
\kappa_5 &= -\frac{s}{u} + \mathcal{O}(s^{1-\delta}), \\
\kappa_6 &= \sqrt{\frac{\rho Pr}{\epsilon\mu\gamma}}\sqrt{s} + \mathcal{O}(s^{0.5-\delta}), \\
\kappa_7 &= -\sqrt{\frac{\rho Pr}{\epsilon\mu\gamma}}\sqrt{s} + \mathcal{O}(s^{0.5-\delta}), \\
\kappa_8 &= \sqrt{\frac{\rho}{(\lambda + 2\mu)\epsilon}}\sqrt{s} + \mathcal{O}(s^{0.5-\delta}), \\
\kappa_9 &= -\sqrt{\frac{\rho}{(\lambda + 2\mu)\epsilon}}\sqrt{s} + \mathcal{O}(s^{0.5-\delta}),
\end{aligned} \tag{63}$$

There are 5 negative κ if $u_1 > 0$, i.e. inflow, and 4 negative if $u_1 < 0$, i.e. outflow.

In the two-dimensional case the derivation is essentially identical. The only difference is the first parenthesis of (59) that appears with a power of 1 instead of 2. That yields 4 boundary conditions at an inflow and 3 at an outflow boundary.

The one-dimensional derivation, gives equation (60) directly such that 3 inflow and 2 outflow boundary conditions need to be specified.

III Diagonalisation with $u_n = 0$

Consider,

$$\mathbf{A}e = \lambda e. \tag{64}$$

where $\mathbf{A}$ is given by (47), repeated here for convenience,

$$\mathbf{A} = \begin{pmatrix} 0 & bn_1 & bn_2 & bn_3 & 0 & 0 & 0 & 0 & 0 \\ bn_1 & 0 & 0 & 0 & an_1 & -\epsilon & 0 & 0 & 0 \\ bn_2 & 0 & 0 & 0 & an_2 & 0 & -\epsilon & 0 & 0 \\ bn_3 & 0 & 0 & 0 & an_2 & 0 & 0 & -\epsilon & 0 \\ 0 & an_1 & an_2 & an_3 & 0 & 0 & 0 & 0 & -\epsilon \\ 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{65}$$

The structure of $\mathbf{A}$ suggests the following ansatz,

$$e_1 = (0, m_1, m_2, m_3, 0, m_1, m_2, m_3, 0)^T, \tag{66}$$

$$e_2 = (0, m_1, m_2, m_3, 0, -m_1, -m_2, -m_3, 0)^T, \tag{67}$$

$$e_3 = (m_4, m_5 n_1, m_5 n_2, m_5 n_3, m_6, m_7 n_1, m_7 n_2, m_7 n_3, m_8). \tag{68}$$

We will use the following notation $\mathbf{m} = (m_1, m_2, m_3)^T$. With (66), equation (64) becomes,

$$\begin{pmatrix} b\hat{\mathbf{n}}^{\mathbf{T}}\mathbf{m} \\ -\epsilon m_1 \\ -\epsilon m_2 \\ -\epsilon m_3 \\ a\hat{\mathbf{n}}^{\mathbf{T}}\mathbf{m} \\ -\epsilon m_1 \\ -\epsilon m_2 \\ -\epsilon m_3 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ m_1 \\ m_2 \\ m_3 \\ 0 \\ m_1 \\ m_2 \\ m_3 \\ 0 \end{pmatrix}. \tag{69}$$

With $\lambda = \lambda_1$ and $\mathbf{m} = \mathbf{m}_1$, the following choice satisfies the above equation, $\hat{\mathbf{n}}^T\mathbf{m_1} = 0$ and $\lambda_1 = -\epsilon$. Further, we may also choose a second solution $\mathbf{m} = \mathbf{m}_2$ and $\lambda_2 = -\epsilon$ such that $\hat{\mathbf{n}}^T\mathbf{m_2} = 0$ and $\mathbf{m_2}^T\mathbf{m_1} = 0$. Similarly, ansatz (67) yields,

$$\lambda_3 = \epsilon, \quad \hat{\mathbf{n}}^T\mathbf{m_3} = 0, \tag{70}$$

$$\lambda_4 = \epsilon, \quad \hat{\mathbf{n}}^T\mathbf{m_4} = 0, \quad \mathbf{m_3}^T\mathbf{m_4} = 0. \tag{71}$$

In fact, we can let $\mathbf{m}_1 = \mathbf{m}_3$ and $\mathbf{m}_2 = \mathbf{m}_4$. It is obvious that the vectors (66) and (67) will be orthogonal and by definition, they are ortogonal to (68). So far, four eigenvalues and eigenvectors out of nine are derived when we turn to the last ansatz (68). In this case equation (64) becomes,

$$\begin{pmatrix} m_5 b \\ (bm_4 + am_6 - \epsilon m_7)n_1 \\ (bm_4 + am_6 - \epsilon m_7)n_2 \\ (bm_4 + am_6 - \epsilon m_7)n_3 \\ am_5 - \epsilon m_8 \\ -\epsilon m_5 n_1 \\ -\epsilon m_5 n_2 \\ -\epsilon m_5 n_3 \\ -\epsilon m_6 \end{pmatrix} = \lambda \begin{pmatrix} m_4 \\ m_5 n_1 \\ m_5 n_2 \\ m_5 n_3 \\ m_6 \\ m_7 n_1 \\ m_7 n_2 \\ m_7 n_3 \\ m_8 \end{pmatrix}, \tag{72}$$

where $n_1^2 + n_2^2 + n_3^3$ has been used. Note that the above system of equations reduces to only 5 equations by the choice of the eigenvector. Further, we have 5 unknowns including λ. (One of the unknowns of the eigenvector drops out since it should only enter as a scaling.) We have,

$$
\begin{aligned}
m_5 b &= \lambda m_4 && (73)\\
b m_4 + a m_6 - \epsilon m_7 &= \lambda m_5 && (74)\\
a m_5 - \epsilon m_8 &= \lambda m_6 && (75)\\
-\epsilon m_5 &= \lambda m_7 && (76)\\
-\epsilon m_6 &= \lambda m_8 && (77)
\end{aligned}
$$

In this case it turns out that the ansatz was satisfactory since 5 solutions to the system (73)-(77) exist.

The case we examine is the marginal case with $u_n = 0$ which leads us to expect one eigenvalue to be zero. Thusly, with $\lambda_5 = 0$ the following eigenvector is obtained,

$$e_5 = (1, 0, 0, 0, 0, \frac{b}{\epsilon} n_1, \frac{b}{\epsilon} n_2, \frac{b}{\epsilon} n_3, 0)^T. \quad (78)$$

Next, we solve full system (73)-(77) without assumptions on the solution. With $\mu = \epsilon^2 - \lambda^2$ a second degree equation in μ is obtained,

$$\mu^2 + (b + a^2)\mu - a^2\epsilon^2 = 0 \quad (79)$$

with the solutions,

$$\mu_{1,2} = -\frac{b + a^2}{2} \pm \sqrt{\frac{(b + a^2)^2}{4} + a^2\epsilon^2} = -\frac{c^2}{2} \pm \sqrt{\frac{c^4}{4} + a^2\epsilon^2}, \quad (80)$$

such that $\lambda_{6,7} = \pm\sqrt{\epsilon^2 - \mu_1}$ and $\lambda_{8,9} = \pm\sqrt{\epsilon^2 - \mu_2}$. For any of these λ:s the eigenvector is given by,

$$e = \begin{pmatrix} b \\ \lambda n_1 \\ \lambda n_2 \\ \lambda n_3 \\ -\frac{a\lambda^2}{\epsilon^2 - \lambda^2} \\ -\epsilon n_1 \\ -\epsilon n_2 \\ -\epsilon n_3 \\ \frac{\epsilon a \lambda}{\epsilon^2 - \lambda^2} \end{pmatrix} \quad (81)$$

Next, we have to show that the different eigenvectors obtained from (81) are orthogonal to each other. We distinguish between two cases: 1) Any of the eigenvalues derived from μ_1 denoted by ξ_1 and another eigenvalue ξ_2 derived from μ_2; 2) Both eigenvalues $\xi_{1,2}$ both derived from the same μ.

The scalar product is,

$$\begin{aligned} e(\xi_1)^T \cdot e(\xi_2) = \\ b + \xi_1\xi_2 + \frac{a^2\xi_1^2\xi_2^2}{(\epsilon^2 - \xi_1^2)(\epsilon^2 - \xi_2^2)} + \epsilon^2 + \frac{\epsilon^2 a^2 \xi_1\xi_2}{(\epsilon^2 - \xi_1^2)(\epsilon^2 - \xi_2^2)}. \end{aligned} \tag{82}$$

Case 1: For a general quadratic equation $x^2 + px + q = 0$ the roots fullfil $x_1x_2 = q$ and $x_1 + x_2 = -p$. When applied to (79) this implies,

$$\mu_1\mu_2 = (\epsilon^2 - \xi_1^2)(\epsilon^2 - \xi_2^2) = -a^2\epsilon^2, \tag{83}$$

$$\mu_1 + \mu_2 = -(b + a^2) \tag{84}$$

Thus, (82) is,

$$\begin{aligned} b + \xi_1\xi_2 - \frac{\xi_1^2\xi_2^2}{\epsilon^2} + \epsilon^2 - \xi_1\xi_2 = \\ b + \epsilon^2 + \frac{(\epsilon^2 - \mu_1)(\epsilon^2 - \mu_2)}{\epsilon^2} = \\ b + \epsilon^2 - (\epsilon^2 - (\mu_1 + \mu_2) - a^2) = \\ b - (b + a^2) + a^2 = 0. \end{aligned} \tag{85}$$

Case 2:

In this case the following relations hold,

$$\lambda^2 = \xi_1^2 = \xi_2^2 \tag{86}$$

$$\lambda = \xi_1 = -\xi_2 \tag{87}$$

$$\begin{aligned} \lambda^2 = -\xi_1\xi_2 = (\mu - \epsilon^2), \\ (\epsilon^2 - \xi_{1,2}^2) = \mu. \end{aligned} \tag{88}$$

Then (82) becomes after multiplying by $(\epsilon^2 - \lambda^2)^2$,

$$\begin{aligned} (\epsilon^2 - \lambda^2)^2(b - \lambda^2 + \epsilon^2) + a^2\lambda^4 - \epsilon^2a^2\lambda^2 = \\ (b - \lambda^2 + \epsilon^2)(\epsilon^2 - \lambda^2)^2 + a^2\lambda^2(\lambda^2 - \epsilon^2) = \\ (\lambda^2 - \epsilon^2)((b + (\epsilon - \lambda^2))(\epsilon^2 - \lambda^2) + a^2\lambda^2) = \\ -\mu((b + \mu)\mu + a^2(\mu - \epsilon^2)) = \\ -\mu(\mu^2 + (b + a^2)\mu - a^2\epsilon^2) = 0, \end{aligned}$$

where the last equality is due to equation (79).

One should also normalise these vectors to formally obtain the eigenvectors of the matrix **A**. With this done, we conclude that in the case of neither inflow nor outflow the above derivation gives the eigenvalues and eigenvectors of the linearised Navier-Stokes equations in three dimensions.

IV Curvilinear Coordinates

IV.1 Metric Relations

Let x, y, z denote the usual Cartesian coordiantes. Consider the following coordinate transformation,

$$\xi = \xi(x, y, z), \quad \eta = \eta(x, y, z), \quad \zeta = \zeta(x, y, z).$$

The Jacobian is defined as,

$$\mathbf{J} = \begin{pmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{pmatrix} \tag{89}$$

Let $\bar{x} = (x, y, z) = (x_1, x_2, x_3)$ and $\bar{\xi} = (\xi, \eta, \zeta) = (\xi_1, \xi_2, \xi_3)$. Then we can formally express the Jacobian as $\mathcal{D}_{\bar{\xi}}\bar{x} = \mathbf{J}$. The following relation holds,

$$I = \mathcal{D}_{\bar{x}}\bar{x}(\bar{\xi}) = \mathcal{D}_{\bar{\xi}}\bar{x}\mathcal{D}_{\bar{x}}\bar{\xi}. \tag{90}$$

Hence,

$$\mathbf{J}^{-1} = \mathcal{D}_{\bar{x}}\xi = \begin{pmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{pmatrix}. \tag{91}$$

However, $\mathbf{J}^{-1}$ can also be obtained directly by inverting (89),

$$\mathbf{J}^{-1} = \mathcal{D}_{\bar{x}}\xi = \frac{1}{J}\begin{pmatrix} -(y_\eta z_\zeta - y_\zeta z_\eta) & x_\eta z_\zeta - x_\zeta z_\eta & -(x_\eta y_\zeta - x_\zeta y_\eta) \\ y_\xi z_\zeta - y_\zeta z_\xi & x_\xi z_\zeta - x_\zeta z_\xi & x_\xi y_\zeta - x_\zeta y_\xi \\ -(y_\xi z_\eta - y_\eta z_\xi) & x_\xi z_\eta - x_\eta z_\xi & -(x_\xi y_\eta - x_\eta y_\xi) \end{pmatrix}. \tag{92}$$

where J denotes the determinant of the Jacobian. Then (91) and (92) give relations between the different metric coefficients. For example, we note that,

$$\begin{aligned} &(J\xi_x)_\xi + (J\eta_x)_\eta + (J\zeta_x)_\zeta = \\ -(y_\eta z_\zeta - y_\zeta z_\eta)_\xi + (y_\xi z_\zeta - y_\zeta z_\xi)_\eta &- (y_\xi z_\eta - y_\eta z_\xi)_\zeta = 0, \\ &(J\xi_y)_\xi + (J\eta_y)_\eta + (J\zeta_y)_\zeta = \\ (x_\eta z_\zeta - x_\zeta z_\eta)_\xi + (x_\xi z_\zeta - x_\zeta z_\xi)_\eta &+ (x_\xi z_\eta - x_\eta z_\xi)_\zeta = 0, \\ &(J\xi_z)_\xi + (J\eta_z)_\eta + (J\zeta_z)_\zeta = \\ -(x_\eta y_\zeta - x_\zeta y_\eta)_\xi + (x_\xi y_\zeta - x_\zeta y_\xi)_\eta &- (x_\xi y_\eta - x_\eta y_\xi)_\zeta = 0, \end{aligned} \tag{93}$$

which will be used below.

IV.2 Curvilinear Navier-Stokes Equations

Consider the linearised and symmetrised Navier-Stokes equations (11), restated here for convenience,

$$\begin{aligned}\tilde{u}_t + \quad & (A_1\tilde{u} - \epsilon(B_{11}\tilde{u}_x + B_{12}\tilde{u}_y + B_{13}\tilde{u}_z))_x + \\ & (A_2\tilde{u} - \epsilon(B_{22}\tilde{u}_y + B_{23}\tilde{u}_z + B_{12}\tilde{u}_x))_y + \\ & (A_3\tilde{u} - \epsilon(B_{33}\tilde{u}_z + B_{32}\tilde{u}_y + B_{13}\tilde{u}_x))_z = \quad 0\end{aligned}$$

or,

$$\begin{aligned}\tilde{u}_t + (F^I - \epsilon\tilde{F}_v)_x + (G^I - \epsilon\tilde{G}_v)_y + (H^I - \epsilon\tilde{H}_v)_z = \\ \tilde{u}_t + F_x + G_y + H_z = 0.\end{aligned} \tag{94}$$

Multiply (94) by J and make the change of coordinates,

$$\begin{aligned}0 = (J\tilde{u})_t + JF_x + JG_y + JH_z = \\ (J\tilde{u})_t + J\xi_x F_\xi + J\eta_x F_\eta + J\zeta_x F_\zeta + \\ J\xi_y G_\xi + J\eta_y G_\eta + J\zeta_y G_\zeta + \\ J\xi_z H_\xi + J\eta_z H_\eta + J\zeta_z H_\zeta =\end{aligned} \tag{95}$$

Reformulating (95) yields,

$$\begin{aligned}(J\tilde{u})_t + \quad & (J\xi_x F + J\xi_y G + J\xi_z H)_\xi \quad -R_1 + \\ & (J\eta_x F + J\eta_y G + J\eta_z H)_\eta \quad -R_2 + \\ & (J\zeta_x F + J\zeta_y G + J\zeta_z H)_\zeta \quad -R_3,\end{aligned}$$

where,

$$\begin{aligned}R_1 &= (J\xi_x)_\xi F + (J\xi_y)_\xi G + (J\xi_z)_\xi H \\ R_2 &= (J\eta_x)_\eta F + (J\eta_y)_\eta G + (J\eta_z)_\eta H \\ R_3 &= (J\zeta_x)_\zeta F + (J\zeta_y)_\zeta G + (J\zeta_z)_\zeta H.\end{aligned}$$

By using the metric relations in (93) we obtain,

$$\begin{aligned}R_1 + R_2 + R_3 = \quad & F((J\xi_x)_\xi + (J\eta_x)_\eta + (J\zeta_x)_\zeta) + \\ & G((J\xi_y)_\xi + (J\eta_y)_\eta + (J\zeta_y)_\zeta) + \\ & H((J\xi_z)_\xi + (J\eta_z)_\eta + (J\zeta_z)_\zeta) = \quad 0.\end{aligned}$$

Define,

$$\begin{aligned}\hat{F} &= (J\xi_x F + J\xi_y G + J\xi_z H) \\ \hat{G} &= (J\eta_x F + J\eta_y G + J\eta_z H) \\ \hat{H} &= (J\zeta_x F + J\zeta_y G + J\zeta_z H),\end{aligned}$$

such that,

$$0 = (J\tilde{u})_t + JF_x + JG_y + JH_z = (Ju)_t + \hat{F}_\xi + \hat{G}_\eta + \hat{H}_\zeta. \tag{96}$$

Next, we express the new fluxes in curvilinear coordinates. We obtain,

$$\begin{aligned} \hat{F}^I = & \quad (J\xi_x F^I + J\xi_y G^I + J\xi_z H^I) = & J(\xi_x A_1 + \xi_y A_2 + \xi_z A_3)u \\ \hat{G}^I = & \quad (J\eta_x F^I + J\eta_y G^I + J\eta_z H^I) = & J(\eta_x A_1 + \eta_y A_2 + \eta_z A_3)u \\ \hat{H}^I = & \quad (J\zeta_x F^I + J\zeta_y G^I + J\zeta_z H^I) = & J(\zeta_x A_1 + \zeta_y A_2 + \zeta_z A_3)u. \end{aligned} \tag{97}$$

and,

$$\begin{aligned} \hat{F}_v &= (J\xi_x \tilde{F}_v + J\xi_y \tilde{G}_v + J\xi_z \tilde{H}_v) \\ \hat{G}_v &= (J\eta_x \tilde{F}_v + J\eta_y \tilde{G}_v + J\eta_z \tilde{H}_v) \\ \hat{H}_v &= (J\zeta_x \tilde{F}_v + J\zeta_y \tilde{G}_v + J\zeta_z \tilde{H}_v), \end{aligned} \tag{98}$$

where,

$$\begin{aligned} \tilde{F}_v &= \tilde{B}_{11}\tilde{u}_\xi + \tilde{B}_{12}\tilde{u}_\eta + \tilde{B}_{13}\tilde{u}_\zeta, \\ \tilde{G}_v &= \tilde{B}_{22}\tilde{u}_\eta + \tilde{B}_{23}\tilde{u}_\zeta + \tilde{B}_{12}\tilde{u}_\xi, \\ \tilde{H}_v &= \tilde{B}_{33}\tilde{u}_\zeta + \tilde{B}_{32}\tilde{u}_\eta + \tilde{B}_{13}\tilde{u}_\xi, \end{aligned}$$

and,

$$\begin{aligned} \tilde{B}_{11} &= B_{11}\xi_x + B_{12}\xi_y + B_{13}\xi_z, & \tilde{B}_{12} &= B_{11}\eta_x + B_{12}\eta_y + B_{13}\eta_z, \\ \tilde{B}_{13} &= B_{11}\zeta_x + B_{12}\zeta_y + B_{13}\zeta_z, & \tilde{B}_{22} &= B_{22}\xi_y + B_{23}\xi_z + B_{12}\xi_x, \\ \tilde{B}_{23} &= B_{22}\eta_y + B_{23}\eta_z + B_{12}\eta_x, & \tilde{B}_{21} &= B_{22}\zeta_y + B_{23}\zeta_z + B_{12}\zeta_x, \\ \tilde{B}_{33} &= B_{33}\xi_z + B_{32}\xi_y + B_{13}\xi_x, & \tilde{B}_{32} &= B_{33}\eta_z + B_{32}\eta_y + B_{13}\eta_x, \\ \tilde{B}_{31} &= B_{33}\zeta_z + B_{32}\zeta_y + B_{13}\zeta_x. \end{aligned}$$

IV.3 Energy Estimate

Next, we turn to well-posedness of equation (96). We apply the energy method and derive the boundary terms. Our aim is to relate the boundary terms in curvilinear coordinates to those derived in $\bar{x}$-space.

First we note that,

$$dxdydz = Jd\xi d\eta d\zeta. \tag{99}$$

Further, we use the notation $D_{\bar{\xi}}$ in $\bar{\xi}$-space for the image of the domain $D_{\bar{x}}$ in $\bar{x}$-space.

Apply the energy method to (96) to obtain,

$$0 = \int_{D_{\bar{\xi}}} \tilde{u}^T \tilde{u}_t J d\xi d\eta d\zeta + \int_{D_{\bar{\xi}}} \tilde{u}^T (\hat{F}^I_\xi + \hat{G}^I_\eta + \hat{H}^I_\zeta) d\xi d\eta d\zeta -$$

$$\epsilon \int_{D_{\bar{\xi}}} \tilde{u}^T ((\hat{F}_v)_\xi + (\hat{G}_v)_\eta + (\hat{H}_v)_\zeta) d\xi d\eta d\zeta = \int_{D_{\bar{x}}} \tilde{u}^T \tilde{u}_t dx dy dz + I_1 - \epsilon I_2 \quad (100)$$

$$\begin{aligned} I_2 = \int_{D_{\bar{\xi}}} (\tilde{u}^T \hat{F}_v)_\xi + (\tilde{u}^T \hat{G}_v)_\eta + (\tilde{u}^T \hat{H}_v)_\zeta d\xi d\eta d\zeta - \\ \int_{D_{\bar{\xi}}} \tilde{u}^T_\xi (\hat{F}_v)_\xi + \tilde{u}^T_\eta (\hat{G}_v)_\eta + \tilde{u}^T_\zeta (\hat{H}_v)_\zeta d\xi d\eta d\zeta = \\ \int_{D_{\bar{\xi}}} (\tilde{u}^T \hat{F}_v)_\xi + (\tilde{u}^T \hat{G}_v)_\eta + (\tilde{u}^T \hat{H}_v)_\zeta d\xi d\eta d\zeta - DI = \\ \oint_{\Gamma_{\bar{\xi}}} (\tilde{u}^T \hat{F}_v, \tilde{u}^T \hat{G}_v, \tilde{u}^T \hat{H}_v) \cdot \mathbf{n}_{\bar{\xi}} ds_{\bar{\xi}} - DI = \\ \oint_{\Gamma_{\bar{\xi}}} \tilde{u}^T \hat{F}^V ds_{\bar{\xi}} - DI \end{aligned} \quad (101)$$

where $\mathbf{n}_{\bar{\xi}} = (n_\xi, n_\eta, n_\zeta)$ and $ds_{\bar{\xi}}$ denotes the outward pointing normal and surface element in $\bar{\xi}$-space, respectively. Further, $\hat{F}^V = \hat{F}_v n_\xi + \hat{G}_v n_\eta + \hat{H}_v n_\zeta$. DI denotes a dissipative term and is equal to DI defined in subsection 3.2.

$$\begin{aligned} I_1 = \int_{D_{\bar{\xi}}} \tilde{u}^T (\hat{F}^I_\xi + \hat{G}^I_\eta + \hat{H}^I_\zeta) d\xi d\eta d\zeta = \\ \int_{D_{\bar{\xi}}} \tilde{u}^T (J\xi_x A_1 \tilde{u} + J\xi_y A_2 \tilde{u} + J\xi_z A_3 \tilde{u})_\xi + \\ \tilde{u}^T (J\eta_x A_1 \tilde{u} + J\eta_y A_2 \tilde{u} + J\eta_z A_3 \tilde{u})_\eta + \\ \tilde{u}^T (J\zeta_x A_1 \tilde{u} + J\zeta_y A_2 \tilde{u} + J\zeta_z A_3 \tilde{u})_\zeta d\xi d\eta d\zeta \end{aligned}$$

Next, we use relations of the type,

$$\begin{aligned} \tilde{u}^T (J\xi_x A_1 \tilde{u})_\xi = (J\xi_x)_\xi \tilde{u}^T A_1 \tilde{u} + (J\xi_x)(\frac{1}{2}\tilde{u}^T A_1 \tilde{u})_\xi = \\ (J\xi_x)_\xi \tilde{u}^T A_1 \tilde{u} + (J\xi_x \frac{1}{2}\tilde{u}^T A_1 \tilde{u})_\xi - (J\xi_x)_\xi (\frac{1}{2}\tilde{u}^T A_1 \tilde{u}) = \\ (J\xi_x \frac{1}{2}\tilde{u}^T A_1 \tilde{u})_\xi + (J\xi_x)_\xi (\frac{1}{2}\tilde{u}^T A_1 \tilde{u}) \end{aligned}$$

to obtain,

$$
\begin{array}{rl}
I_1 = \int_{D_{\bar{\xi}}} & (J\xi_x \frac{1}{2}\tilde{u}^T A_1 \tilde{u})_\xi + (J\xi_y \frac{1}{2}\tilde{u}^T A_2 \tilde{u})_\xi + (J\xi_z \frac{1}{2}\tilde{u}^T A_3 \tilde{u})_\xi \quad + \\
& (J\eta_x \frac{1}{2}\tilde{u}^T A_1 \tilde{u})_\eta + (J\eta_y \frac{1}{2}\tilde{u}^T A_2 \tilde{u})_\eta + (J\eta_z \frac{1}{2}\tilde{u}^T A_3 \tilde{u})_\eta \quad + \\
& (J\zeta_x \frac{1}{2}\tilde{u}^T A_1 \tilde{u})_\zeta + (J\zeta_y \frac{1}{2}\tilde{u}^T A_2 \tilde{u})_\zeta + (J\zeta_z \frac{1}{2}\tilde{u}^T A_3 \tilde{u})_\zeta \quad + \\
& \frac{1}{2}\tilde{u}^T A_1 \tilde{u}(J\xi_x)_\xi + \frac{1}{2}\tilde{u}^T A_1 \tilde{u}(J\eta_x)_\eta + \frac{1}{2}\tilde{u}^T A_1 \tilde{u}(J\zeta_x)_\zeta \quad + \\
& \frac{1}{2}\tilde{u}^T A_2 \tilde{u}(J\xi_y)_\xi + \frac{1}{2}\tilde{u}^T A_2 \tilde{u}(J\eta_y)_\eta + \frac{1}{2}\tilde{u}^T A_2 \tilde{u}(J\zeta_y)_\zeta \quad + \\
& \frac{1}{2}\tilde{u}^T A_3 \tilde{u}(J\xi_z)_\xi + \frac{1}{2}\tilde{u}^T A_3 \tilde{u}(J\eta_z)_\eta + \frac{1}{2}\tilde{u}^T A_3 \tilde{u}(J\zeta_z)_\zeta \quad d\xi d\eta d\zeta
\end{array} \tag{102}
$$

Hence, by using (93) the last three rows of (102) are identically zero,

$$
I_1 = \oint_{\Gamma_{\bar{\xi}}} \frac{1}{2}(\tilde{u}^T(\hat{A}_1)\tilde{u}, \tilde{u}^T(\hat{A}_2)\tilde{u}, \tilde{u}^T(\hat{A}_3)\tilde{u}) \cdot \mathbf{n}_{\bar{\xi}} ds_{\bar{\xi}} \tag{103}
$$

where,

$$
\begin{aligned}
\hat{A}_1 &= (A_1 J\xi_x + A_2 J\xi_y + A_3 J\xi_z) \\
\hat{A}_2 &= (A_1 J\eta_x + A_2 J\eta_y + A_3 J\eta_z) \\
\hat{A}_3 &= (A_1 J\zeta_x + A_2 J\zeta_y + A_3 J\zeta_z)
\end{aligned}
$$

By inserting (101) and (103) in (100) we obtain,

$$
\begin{aligned}
& 2\int_{D_{\bar{x}}} \tilde{u}^T \tilde{u}_t dxdydz + \\
& \oint_{\Gamma_{\bar{\xi}}} (\tilde{u}^T(\hat{A}_1)\tilde{u}, \tilde{u}^T(\hat{A}_2)\tilde{u}, \tilde{u}^T(\hat{A}_3)\tilde{u}) \cdot \mathbf{n}_{\bar{\xi}} ds_{\bar{\xi}} - \epsilon(\oint_{\Gamma_{\bar{\xi}}} 2\tilde{u}^T \hat{F}^V ds_{\bar{\xi}} - DI) = \\
& \|\tilde{u}\|_t^2 + \oint_{\Gamma_{\bar{\xi}}} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} (\hat{A}_1, \hat{A}_2, \hat{A}_3) \cdot \mathbf{n}_{\bar{\xi}} & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} - DI = \\
& \|\tilde{u}\|_t^2 + \oint_{\Gamma_{\bar{\xi}}} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \hat{\mathbf{A}} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} - DI = 0.
\end{aligned} \tag{104}
$$

The form (104) is completely similar to the one in the $\bar{x}$-system. As mentioned earlier the domain in $\bar{\xi}$-space is a cube. Hence, $\mathbf{n}_{\bar{\xi}}$ is particularly simple. It is unit a vector in the coordinate directions, $\pm e_\xi, \pm e_\eta, \pm e_\zeta$, on the boundary of the computational domain, $0 \leq \xi \leq 1$, $0 \leq \eta \leq 1$, $0 \leq \zeta \leq 1$.

The full formulation for the cube is:

$$
\begin{aligned}
\|\tilde{u}\|_t^2 - & \int_{\xi=0} \begin{pmatrix} \tilde{u} \\ \hat{\mathbf{F}}_{\mathbf{v}} \end{pmatrix} \begin{pmatrix} \hat{A}_1 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} + \\
& \int_{\xi=1} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} \hat{A}_1 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} - \\
& \int_{\eta=0} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} \hat{A}_2 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} + \\
& \int_{\eta=1} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} \hat{A}_2 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} - \\
& \int_{\zeta=0} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} \hat{A}_3 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} + \\
& \int_{\zeta=1} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} \hat{A}_3 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} = \quad DI
\end{aligned}
\tag{105}
$$

Note that $ds_{\bar{\xi}}$ are different in the different coordinate directions. As a last step we will express one of the integrals in (105) in $\bar{x}$-space. Consider, for example,

$$
\begin{aligned}
-\int_{\xi=0} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} \hat{A}_1 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} = \\
\int_{\xi=0} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} -A_1 J\xi_x - A_2 J\xi_y - A_3 J\xi_z & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} = \\
\int_{\xi=0} \begin{pmatrix} \tilde{u} \\ \frac{\hat{F}^V}{JT_1} \end{pmatrix} \begin{pmatrix} -A_1 \frac{\xi_x}{T_1} - A_2 \frac{\xi_y}{T_1} - A_3 \frac{\xi_z}{T_1} & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \frac{\hat{F}^V}{JT_1} \end{pmatrix} JT_1 ds_{\bar{\xi}} = \\
\int_{\xi=0} \begin{pmatrix} \tilde{u} \\ \frac{\hat{F}^V}{JT_1} \end{pmatrix} \begin{pmatrix} A_1 n_1 + A_2 n_2 + A_3 n_3 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \frac{\hat{F}^V}{JT_1} \end{pmatrix} JT_1 ds_{\bar{\xi}}
\end{aligned}
\tag{106}
$$

where $T_1 = \sqrt{(\xi_x)^2 + (\xi_y)^2 + (\xi_z)^2}$ and $n_1^2 + n_2^2 + n_3^2 = 1$. In fact, (n_1, n_2, n_3) is equal to the normal in the $\bar{x}$-system. This is easily seen by the following. Denote by $\mathbf{r} = (x, y, z)$ a position vector in space. The unnormalised normal vector at $\xi = 0$ is,

$$
\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \eta} \times \frac{\partial \mathbf{r}}{\partial \zeta} = (x_\eta, y_\eta, z_\eta) \times (x_\zeta, y_\zeta, z_\zeta) = \\
(y_\eta z_\zeta - z_\eta y_\zeta, -(x_\eta z_\zeta - z_\eta x_\zeta), x_\eta y_\zeta - y_\eta x_\zeta) = JT_1(n_1, n_2, n_3),
\end{aligned}
\tag{107}
$$

where equation (91) and (92) have been used. Hence the matrices appearing in (106) and (25) are equal. Next, we will show that the vectors in (106) and

(25) are also equal. We have,

$$\frac{\hat{F}^V}{JT_1} = \frac{\hat{F}_v \cdot 1 + \hat{G}_v \cdot 0 + \hat{H}_v \cdot 0}{JT_1} = \frac{\hat{F}_v}{JT_1} =$$
$$\frac{(\xi_x \tilde{F}_v + \xi_y \tilde{G}_v + \xi_z \tilde{H}_v)}{T_1} = \tilde{F}_v n_1 + \tilde{G}_v n_2 + \tilde{H}_v n_3 = \tilde{F}^V.$$

At last,

$$ds_{\bar{x}} = |\frac{\partial \mathbf{r}}{\partial \eta} \times \frac{\partial \mathbf{r}}{\partial \zeta}| ds_{\bar{\xi}} = JT_1 ds_{\bar{\xi}}, \tag{108}$$

implying that (106) and (25) equal.

The other boundaries can be treated similarly. To summarise, we have shown that the relations in $\bar{x}$-space are completely equivalent to those in $\bar{\xi}$-space.

References

[1] J.S. Hesthaven and D. Gottlieb. A stable penalty method for the compressible Navier-Stokes equations: I open boundary conditions. *SIAM J.Sci.Comput.*, 1996.

[2] J.S. Hesthaven. A stable penalty method for the compressible Navier-Stokes equations: II One-dimensional domain decomposition schmemes. *SIAM J.Sci.Comput.*, 1997.

[3] J.S. Hesthaven. A stable penalty method for the compressible Navier-Stokes equations: II Multi-dimensional domain decomposition schmemes. *SIAM J.Sci.Comput.*, 1998.

[4] Jan Nordström. The Influence of Open Boundary Conditions on the Convergence to Steady State for the Navier–Stokes Equations. *J. Comput. Phys*, 85, 1989.

[5] B. Gustafsson and A. Sundström. Incompletely parabolic systems in fluid dynamics. *SIAM J. Appl. Math.*, 1978.

[6] Jan Nordström. The Use of Characteristic Boundary Conditions for the Navier-Stokes Equations. *Computers & Fluids*, 24, 1995.

[7] B. Gustafsson, H.-O. Kreiss, and J. Oliger. *Time dependent problems and difference methods.* John Wiley & Sons, Inc., 1995.

[8] T. Hagstrom. Radiation boundary conditions for the numerical simulation of waves. *Acta Numerica*, pages 47–106, 1999.

[9] Semyon V. Tsynkov. Numerical solution of problems on unbounded domains. a review. *Applied Numerical Mathematics*, pages 465–532, 1998.

[10] Saul Abarbanel and David Gottlieb. Optimal Time Splitting for Two- and Three-Dimensional Navier-Stokes Equations with Mixed Derivatives. *J. Comput. Phys.*, 1981.

[11] M. Renardy and R.C. Rogers. *An Introduction to Partial Differential Equations.* Springer Verlag New York, Inc., 1996.

[12] Heinz-Otto Kreiss and Jens Lorenz. *Initial Boundary Value Problems and the Navier–Stokes Equations.* Academic Press, New York, 1989.

[13] R.A. Horn and C.R. Johnson. *Matrix analysis.* Cambridge University Press, 1990.

[14] John C. Strikwerda. Initial boundary value problems for incompletely parabolic systems. *Comm. Pure Applied Math.*, pages 797–822, 1977.