On the far-field properties of an acoustic horn

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Abstract

This report presents a derivation of an expression for time harmonic acoustic wave propagation in the far field for two and three space dimensions, and includes detailed descriptions of the numerical evaluation of the far-field pattern in some typical situations. The presentation covers all parts required for computing the far-field properties of acoustical devices, and the report is designed to function as a single reference for these computations.

1 Introduction

When designing a horn loudspeaker it is desired to have high transmission efficiency as well as control of the far-field properties (the directivity) of the horn. The efficiency of an acoustical device can in some cases, such as when the device is connected to a waveguide, be computed in a straightforward way using local measurements. The far field is computed either by solving the wave propagation problem in an unbounded domain or by evaluating an integral of the sound pressure field along a closed surface enclosing the horn.

Numerical design optimization of various acoustical devices, with respect to efficiency and to some extent 'near-field' directivity (using local measurements for a region or point a finite distance from the device), has been performed in the last years by among others Bängtson *et al.* [1], who use boundary-shape optimization to improve the transmission efficiency of the horn. Various results of applying topology optimization for the Helmholtz equations are presented by Sigmund and Jensen [5, 7]. In the manuscript "Topology optimization of an acoustic horn" [9] the author of this report and Berggren present a comprehensive study of the transmission properties of horns by topology optimization. A continuation of this study presents acoustical horns optimized both for efficiency and far-field directivity [10]. Recently, Udawalpola and Berggren [8] present horns optimized with respect to efficiency and directivity obtained using boundary-shape optimization.

The computation of the far-field relies on classical methods from scattering theory. All parts required for the numerical evaluation of the far-field properties are well known, but scattered throughout the literature. To the best of my knowledge, there is no single work in the literature that contains all parts. This report is designed to function as a single reference for far-field computations of acoustical devices, and aims to describe the mathematical theory behind the far-field properties and the numerical evaluation of the far-field pattern in two and three space dimensions. The report is organized as follows. Section 2 introduces the problem. Section 3 contains the mathematical derivation of the far-field pattern, and Section 4 describes how the far-field pattern is computed numerically.

2 Problem Description

The wave propagation is assumed to be governed by the wave equation

$$\frac{\partial^2 P}{\partial t^2} = c^2 \Delta P,$$

where P is the acoustical pressure and c is the speed of sound. Seeking time harmonic solutions, for a single frequency ω and the complex amplitude function p, making use of the ansatz $P = \Re\left\{p(x)\mathrm{e}^{\mathrm{i}\omega t}\right\}$, results in the following Helmholtz equation:

$$c^2 \Delta p + \omega^2 p = 0. \tag{1}$$

Further, it is required that p satisfies the Sommerfeld radiation condition, stipulating that the waves are outgoing in the far field,

$$\lim_{|x| \to +\infty} |x|^{(d-1)/2} \left(\frac{x}{|x|} \cdot \nabla p + \mathrm{i}kp \right) = 0, \tag{2}$$

uniformly for all directions, where d is the number of space dimensions considered.

In the far field, the complex amplitude function is essentially the product of a function of the distance to the origin and a function of the direction. More precisely, let $\rho > 0$, and let $\widehat{x}(\theta)$ be a point on the unit sphere, where θ represents the polar argument(s) of $\widehat{x}$. Then, as we shall see below, for $x_0 = \rho \widehat{x}$,

$$p(x_0) = \frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\rho^{(d-1)/2}} \left\{ p_\infty(\theta) + O\left(\frac{1}{\rho}\right) \right\} \qquad \rho \to +\infty,$$

where $k = \omega/c$ is the wave number. The function $p_\infty(\theta)$ is denoted the *far-field pattern* of the horn and is the subject of this report.

3 Derivation of an expression for the far-field pattern

Assume that the horn is located inside a bounded domain $\widehat{\Omega}$ with sufficiently smooth boundary $\widehat{\Gamma}_{\text{out}}$, and let V be the domain exterior to $\widehat{\Omega}$. The exterior domain V is assumed to be connected; in other words, $\widehat{\Omega}$ consists of one or more simply connected components. The setup is illustrated in Figure 1. Then, for any point $x_0 \in V$, the complex amplitude function p can be evaluated at x_0 using the following theorem.

Theorem 1. *If $p \in \mathcal{C}^2(V)$ satisfies* (1) *in V together with radiation condition* (2)*, then for any point $x_0 \in V$*

$$p(x_0) = \int_{\widehat{\Gamma}_{out}} \left(p \frac{\partial g_d^0}{\partial n} - g_d^0 \frac{\partial p}{\partial n} \right) \, \mathrm{d}\mathcal{H}^{d-1}(x), \tag{3}$$

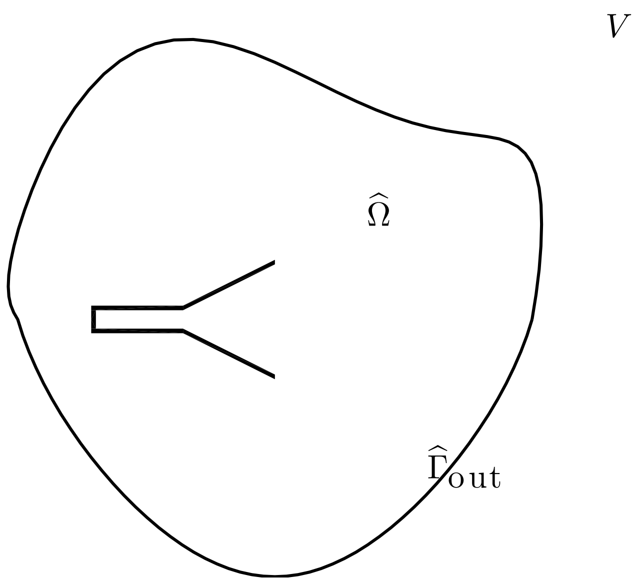

Figure 1: The acoustical horn is located inside the domain $\widehat{\Omega}$.

where n is the outward directed normal of $\widehat{\Gamma}_{out}$, d is the number of space dimensions, and g_d^0 is the fundamental solution to the Helmholtz equation in d dimensions with source located at x_0.

Proof. A proof of this theorem ($d = 2, 3$) is found in Appendix A. □

The fundamental solution of the Helmholtz equation in two dimensions with source located at x_0 is given by

$$g_2^0(x) = \frac{\mathrm{i}}{4} H_0^{(1)}(-k|x - x_0|), \tag{4}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of zero order. The Hankel function of the first kind with order m has the following asymptotic behavior (Appendix B):

$$H_m^{(1)}(-\xi) = (-1)^{m+1} \sqrt{\frac{2}{\pi\xi}} \mathrm{e}^{-\mathrm{i}(\xi - m\pi/2 - \pi/4)} \left\{ 1 + O\left(\frac{1}{\xi}\right) \right\} \quad \xi \to +\infty. \tag{5}$$

In three dimensions, the fundamental solution (with source located at x_0) to Helmholtz equation is given by

$$g_3^0(x) = \frac{\mathrm{e}^{-\mathrm{i}k|x - x_0|}}{4\pi|x - x_0|}. \tag{6}$$

The far-field pattern

Assume that $x_0 \in V$, define $\rho = |x_0|$ and let $\widehat{x}$ be the unit vector such that $x_0 = \widehat{x}\rho$. By letting $\rho \to \infty$ it is now possible to derive an expression for the far-field pattern. For $x \in \widehat{\Gamma}_{\mathrm{out}}$, let

$$r = x_0 - x \text{ and } \widehat{r} = \frac{r}{|r|}. \tag{7}$$

We have that

$$\begin{aligned} |r|^2 = |x_0 - x|^2 = \rho^2 \left| \widehat{x} - \frac{x}{\rho} \right|^2 &= \rho^2 \left(\widehat{x} \cdot \widehat{x} - \frac{2}{\rho} \widehat{x} \cdot x + \frac{x \cdot x}{\rho^2} \right) \\ &= \rho^2 \left(1 - \frac{2}{\rho} \widehat{x} \cdot x + \frac{|x|^2}{\rho^2} \right), \end{aligned}$$

hence, by Taylor expansion,

$$\begin{aligned} |r| = |x_0 - x| &= \rho \sqrt{1 - \frac{2}{\rho} \widehat{x} \cdot x + \frac{|x|^2}{\rho^2}} \\ &= \rho \left(1 + \frac{1}{2} \left(-\frac{2}{\rho} \widehat{x} \cdot x + \frac{|x|^2}{\rho^2} \right) + O\left(\left[-\frac{2}{\rho} \widehat{x} \cdot x + \frac{|x|^2}{\rho^2} \right]^2 \right) \right) \\ &= \rho \left(1 - \frac{\widehat{x} \cdot x}{\rho} + O\left(\frac{1}{\rho^2} \right) \right) = \rho - \widehat{x} \cdot x + O\left(\frac{1}{\rho} \right) \quad \rho \to \infty. \end{aligned} \tag{8}$$

Moreover,

$$\widehat{r} = \frac{\rho \widehat{x} - x}{|\rho \widehat{x} - x|} = \frac{\widehat{x} - x/\rho}{|\widehat{x} - x/\rho|} = \widehat{x} + O\left(\frac{1}{\rho} \right) \quad \rho \to \infty. \tag{9}$$

Three space dimensions

Rewrite (3), making use of the fundamental solution (6) and relations (7) and (9), as

$$\begin{aligned} p(\rho\widehat{x}) = p(x_0) &= \int_{\widehat{\Gamma}_{\mathrm{out}}} \left(p \frac{\partial g_3^0}{\partial n} - g_3^0 \frac{\partial p}{\partial n} \right) \mathrm{d}\mathcal{H}^2(x) \\ &= \int_{\widehat{\Gamma}_{\mathrm{out}}} \left[p \left(-\mathrm{i}k - \frac{1}{|r|} \right) \frac{\mathrm{e}^{-\mathrm{i}k|r|}}{4\pi|r|} (-\widehat{r} \cdot n) - \frac{\mathrm{e}^{-\mathrm{i}k|r|}}{4\pi|r|} \frac{\partial p}{\partial n} \right] \mathrm{d}\mathcal{H}^2(x) \\ &= \int_{\widehat{\Gamma}_{\mathrm{out}}} \frac{\mathrm{e}^{-\mathrm{i}k|r|}}{4\pi|r|} \left[p \left(\mathrm{i}k + O\left(\frac{1}{\rho} \right) \right) \left(\widehat{x} + O\left(\frac{1}{\rho} \right) \right) \cdot n - \frac{\partial p}{\partial n} \right] \mathrm{d}\mathcal{H}^2(x). \end{aligned}$$

Previous expression is expanded and simplified using (8) as follows:

$$\begin{aligned} p(\rho\widehat{x}) &= \int_{\widehat{\Gamma}_{\text{out}}} \frac{\mathrm{e}^{-\mathrm{i}k(\rho-\widehat{x}\cdot x+O\left(\frac{1}{\rho}\right))}}{\rho-\widehat{x}\cdot x+O\left(\frac{1}{\rho}\right)} \frac{1}{4\pi}\left[p\mathrm{i}k\widehat{x}\cdot n\left(1+O\left(\frac{1}{\rho}\right)\right)-\frac{\partial p}{\partial n}\right]\,\mathrm{d}\mathcal{H}^2(x) \\ &= \frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\rho}\int_{\widehat{\Gamma}_{\text{out}}} \frac{\mathrm{e}^{\mathrm{i}k\widehat{x}\cdot x+O\left(\frac{1}{\rho}\right)}}{1+O\left(\frac{1}{\rho}\right)}\frac{1}{4\pi}\left[p\mathrm{i}k\widehat{x}\cdot n+O\left(\frac{1}{\rho}\right)-\frac{\partial p}{\partial n}\right]\,\mathrm{d}\mathcal{H}^2(x) \\ &= \frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\rho}\frac{1}{4\pi}\int_{\widehat{\Gamma}_{\text{out}}}\left(\mathrm{e}^{\mathrm{i}k\widehat{x}\cdot x}+O\left(\frac{1}{\rho}\right)\right)\left[\mathrm{i}pk\widehat{x}\cdot n-\frac{\partial p}{\partial n}+O\left(\frac{1}{\rho}\right)\right]\,\mathrm{d}\mathcal{H}^2(x). \end{aligned}$$

Thus

$$p(\rho\widehat{x}) = \frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\rho}\left\{p_\infty(\theta)+O\left(\frac{1}{\rho}\right)\right\},$$

where θ are the polar arguments of $\widehat{x}$ and the far-field pattern is given by

$$p_\infty(\theta) = \frac{1}{4\pi}\int_{\widehat{\Gamma}_{\text{out}}}\mathrm{e}^{\mathrm{i}k\widehat{x}\cdot x}\left[\mathrm{i}kp\widehat{x}\cdot n-\frac{\partial p}{\partial n}\right]\,\mathrm{d}\mathcal{H}^2(x).$$

Two space dimensions

Inserting the fundamental solution (4) of the two dimensional Helmholtz equation and relation (7) into (3) yields

$$\begin{aligned} p(\rho\widehat{x}) = p(x_0) &= \int_{\widehat{\Gamma}_{\text{out}}}\left(p\frac{\partial g_2^0}{\partial n}-g_2^0\frac{\partial p}{\partial n}\right)\,\mathrm{d}\mathcal{H}^1(x) \\ &= \int_{\widehat{\Gamma}_{\text{out}}}\left[p\frac{\mathrm{i}k}{4}H_1^{(1)}(-k|r|)(-\widehat{r}\cdot n)-\frac{\mathrm{i}}{4}H_0^{(1)}(-k|r|)\frac{\partial p}{\partial n}\right]\,\mathrm{d}\mathcal{H}^1(x). \end{aligned} \tag{10}$$

Making use of the asymptotics for the Hankel functions given in (5), relation (9), and that $|r| = O(\rho)$, from (8), it is possible expand the integrand in (10)

$$\begin{aligned} &p\frac{-\mathrm{i}k}{4}H_1^{(1)}(-k|r|)\widehat{r}\cdot n-\frac{\mathrm{i}}{4}H_0^{(1)}(-k|r|)\frac{\partial p}{\partial n} \\ &\quad= \frac{1}{4\mathrm{i}}\left[pk\left((-1)^2\sqrt{\frac{2}{\pi k|r|}}\mathrm{e}^{-\mathrm{i}(k|r|-\pi/2-\pi/4)}\left\{1+O\left(\frac{1}{k|r|}\right)\right\}\right)\right. \\ &\qquad\qquad\left(\widehat{x}+O\left(\frac{1}{\rho}\right)\right)\cdot n \\ &\qquad\quad\left.+\left((-1)^1\sqrt{\frac{2}{\pi k|r|}}\mathrm{e}^{-\mathrm{i}(k|r|-\pi/4)}\left\{1+O\left(\frac{1}{k|r|}\right)\right\}\right)\frac{\partial p}{\partial n}\right] \\ &\quad= \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{4\mathrm{i}}\sqrt{\frac{2}{\pi k}}\frac{\mathrm{e}^{-\mathrm{i}k|r|}}{\sqrt{|r|}}\left\{1+O\left(\frac{1}{\rho}\right)\right\}\left(\mathrm{i}pk\widehat{x}\cdot n+O\left(\frac{1}{\rho}\right)-\frac{\partial p}{\partial n}\right). \end{aligned} \tag{11}$$

The first part of previous expression is rewritten using (8) as

$$\begin{aligned}\frac{\mathrm{e}^{\mathrm{i}\pi/4}}{4\mathrm{i}}\sqrt{\frac{2}{\pi k}}\frac{\mathrm{e}^{-\mathrm{i}k|r|}}{\sqrt{|r|}} &= \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{4}\sqrt{\frac{2}{\pi k}}\frac{\mathrm{e}^{-\mathrm{i}k(\rho-\widehat{x}\cdot x+O\left(\frac{1}{\rho}\right))}}{\sqrt{\rho-\widehat{x}\cdot x+O\left(\frac{1}{\rho}\right)}}\\ &= \frac{\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}}{4\sqrt{\pi k}}\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{\rho}}\frac{\mathrm{e}^{\mathrm{i}k\widehat{x}\cdot x+O\left(\frac{1}{\rho}\right))}}{\sqrt{1+O\left(\frac{1}{\rho}\right)}} = \frac{1-\mathrm{i}}{4\sqrt{\pi k}}\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{\rho}}\left(\mathrm{e}^{\mathrm{i}k\widehat{x}\cdot x}+O\left(\frac{1}{\rho}\right)\right).\end{aligned} \tag{12}$$

Inserting (12) into (11) yields

$$\begin{aligned}&p\frac{-\mathrm{i}k}{4}H_1^{(1)}(-k|r|)\widehat{r}\cdot n-\frac{\mathrm{i}}{4}H_0^{(1)}(-k|r|)\frac{\partial p}{\partial n}\\ &\quad=\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{\rho}}\left\{\frac{1-\mathrm{i}}{4\sqrt{\pi k}}\mathrm{e}^{\mathrm{i}k\widehat{x}\cdot x}\left(\mathrm{i}pk\widehat{x}\cdot n-\frac{\partial p}{\partial n}\right)+O\left(\frac{1}{\rho}\right)\right\}.\end{aligned}$$

Thus expression (10) can be written as

$$p(\rho\widehat{x})=\frac{-\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{\rho}}\left\{p_\infty(\theta)+O\left(\frac{1}{\rho}\right)\right\},$$

where θ is the polar argument of $\widehat{x}$, and the far-field pattern is given by

$$p_\infty(\theta)=\frac{1-\mathrm{i}}{4\sqrt{\pi k}}\int_{\widehat{\Gamma}_{\mathrm{out}}}\mathrm{e}^{\mathrm{i}k\widehat{x}\cdot x}\left(\mathrm{i}pk\widehat{x}\cdot n-\frac{\partial p}{\partial n}\right)\,\mathrm{d}\mathcal{H}^1(x).$$

Hence, in both two and three space dimensions, the complex amplitude function p has the asymptotic behavior

$$p(x_0)=p(\rho\widehat{x}(\theta))=\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\rho^{(d-1)/2}}\left\{p_\infty(\theta)+O\left(\frac{1}{\rho}\right)\right\}\qquad\rho\to\infty,$$

where

$$p_\infty(\theta)=C_d\int_{\widehat{\Gamma}_{\mathrm{out}}}\mathrm{e}^{\mathrm{i}k\widehat{x}\cdot x}\left(\mathrm{i}pk\widehat{x}\cdot n-\frac{\partial p}{\partial n}\right)\,\mathrm{d}\mathcal{H}^{d-1}(x); \tag{13}$$

and where C_d is a constant given by the number of space dimensions. For $d=2$ and $d=3$, C_d is given by

$$C_2=\frac{1-\mathrm{i}}{4\sqrt{\pi k}}\quad\text{and}\quad C_3=\frac{1}{4\pi}.$$

4 Numerical approach

The numerical computation of the far-field pattern is first described in the case where no symmetry properties are taken into account in the computations.

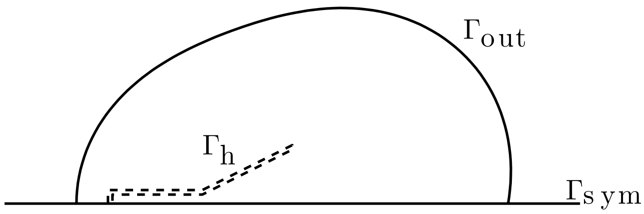

Figure 2: The integration boundary Γ_{out}, the symmetry boundary Γ_{sym}, and the boundary of the horn, Γ_{h}.

Then planar and cylindrical symmetries are addressed. First, recall that the far-field pattern is given by

$$p_\infty(\theta) = C_d \int_{\widehat{\Gamma}_{\mathrm{out}}} \mathrm{e}^{\mathrm{i}k\widehat{x}\cdot x} \left(\mathrm{i}pk\widehat{x}\cdot n - \frac{\partial p}{\partial n} \right) \mathrm{d}\mathcal{H}^{d-1}(x), \tag{14}$$

where C_d is a constant and $\widehat{\Gamma}_{\mathrm{out}}$ is the outer boundary of the domain $\widehat{\Omega}$ containing the horn.

Let Ω be a simply connected (computational) domain, with boundary Γ, and assume that the solution to the wave propagation problem is known in Ω. In the case where no symmetry assumptions are made and the horn is located in Ω, its boundary Γ corresponds to the the evaluation boundary $\widehat{\Gamma}_{\mathrm{out}}$ in (14). The numerical computation of the far-field pattern can be performed by first computing the normal derivative $\partial p/\partial n$ along the integration boundary (Γ); see Section 4 for descriptions on how to compute this normal derivative numerically. Once the normal derivative is computed, the far-field pattern can be approximated by applying quadrature to numerically evaluate the integral.

Using planar symmetry, the boundary Γ, depicted in Figure 2, consists of the parts Γ_{sym}, the symmetry boundary; Γ_{out}, the outer part of the boundary; and—if the horn has been modeled explicitely—Γ_{h}, the boundary of the horn. The evaluation boundary $\widehat{\Gamma}_{\mathrm{out}}$ consists of the union of Γ_{out} and its symmetric twin, here denoted[1] $\overline{\Gamma_{\mathrm{out}}}$. The normal derivative along Γ_{out} can be computed according to the descriptions in Section 4. The expression for far-field pattern using planar symmetry is

$$\begin{aligned} p_\infty(\theta) &= \frac{1-\mathrm{i}}{4\sqrt{\pi k}} \int_{\widehat{\Gamma}_{\mathrm{out}}} \mathrm{e}^{\mathrm{i}k\widehat{x}\cdot x} \left(\mathrm{i}pk\widehat{x}\cdot n - \frac{\partial p}{\partial n} \right) \mathrm{d}\mathcal{H}^1(x) \\ &= \frac{1-\mathrm{i}}{4\sqrt{\pi k}} \int_{\Gamma_{\mathrm{out}}} \left[\mathrm{e}^{\mathrm{i}k\widehat{x}\cdot x} \left(\mathrm{i}pk\widehat{x}\cdot n - \frac{\partial p}{\partial n} \right) + \mathrm{e}^{\mathrm{i}k\widehat{x}\cdot\overline{x}} \left(\mathrm{i}pk\widehat{x}\cdot\overline{n} - \frac{\partial p}{\partial n} \right) \right] \mathrm{d}\mathcal{H}^1(x), \end{aligned}$$

[1]The over-line in $\overline{\Gamma_{\mathrm{out}}}$ should here be viewed as an analogue to the complex conjugate and not the closure of a set.

where $\overline{x}$ and $\overline{n}$ are the mirror images of x and n on the other side of the symmetry boundary. Note that, by symmetry

$$\frac{\partial p}{\partial n}(x) = \frac{\partial p}{\partial \overline{n}}(\overline{x}).$$

Having computed the normal derivative along Γ_{out} the remaining task is to numerically evaluate the integral above.

For cylindrical symmetry it can, without loss of generality, be assumed that the symmetry axis is aligned with the first coordinate axis. The boundary Γ is illustrated in Figure 2 and partitioned as in the planar symmetric case above. Here the integration surface $\widehat{\Gamma}_{\text{out}}$ consists of all points x such that there exist a point $y = (y_1, y_2) \in \Gamma_{\text{out}}$ and an angle $\varphi \in [0, 2\pi]$ such that $x = (y_1, y_2 \cos\varphi, y_2 \sin\varphi)$. The far-field behavior is symmetric with respect to the symmetry axis of the horn. Thus it is sufficient to compute the far field in the plane $x_3 = 0$. Considering $\widehat{x}$ of the form $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, 0)$, the far-field pattern is given by

$$\begin{aligned} p_\infty(\theta) &= \frac{1}{4\pi} \int_{\widehat{\Gamma}_{\text{out}}} \mathrm{e}^{\mathrm{i}k\widehat{x}\cdot x} \left(\mathrm{i}pk\widehat{x} \cdot n(x) - \frac{\partial p}{\partial n(x)}(x) \right) \mathrm{d}\mathcal{H}^2(x) \\ &= \frac{1}{4\pi} \int_{\Gamma_{\text{out}}} \int_0^{2\pi} y_2 \mathrm{e}^{\mathrm{i}k(\widehat{x}_1 y_1 + \widehat{x}_2 y_2 \cos\varphi)} \\ &\qquad \left(\mathrm{i}pk(\widehat{x}_1 n_1(y) + \widehat{x}_2 n_2(y) \cos\varphi) - \frac{\partial p}{\partial n(y)}(y) \right) \mathrm{d}\varphi \, \mathrm{d}\mathcal{H}^1(y), \end{aligned} \tag{15}$$

where n_1 and n_2 are components of the normal on Γ. Rearranging the terms in (15) yields

$$\begin{aligned} p_\infty(\theta) = \frac{1}{4\pi} \int_{\Gamma_{\text{out}}} y_2 \mathrm{e}^{\mathrm{i}k\widehat{x}_1 y_1} \Bigg[\left(\mathrm{i}pk\widehat{x}_1 n_1(y) - \frac{\partial p}{\partial n} \right) \int_0^{2\pi} \mathrm{e}^{\mathrm{i}k\widehat{x}_2 y_2 \cos\varphi} \, \mathrm{d}\varphi \\ + \mathrm{i}pk\widehat{x}_2 n_2(y) \int_0^{2\pi} \mathrm{e}^{\mathrm{i}k\widehat{x}_2 y_2 \cos\varphi} \cos\varphi \, \mathrm{d}\varphi \Bigg] \mathrm{d}\mathcal{H}^1(y). \end{aligned} \tag{16}$$

The inner integrals in (16) can be expressed as Bessel functions. The Bessel functions J_m of the first kind with order m are given by [6, p.63]

$$J_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}x \sin\varphi} \mathrm{e}^{-\mathrm{i}m\varphi} \, \mathrm{d}\varphi.$$

Applying the substitution $\pi/2 - t = \varphi$ and using that the integrand is 2π-periodic result in

$$\begin{aligned} J_m(x) &= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}x \sin(\pi/2 - t)} \mathrm{e}^{-\mathrm{i}m(\pi/2 - t)} \, \mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}x \cos t} \mathrm{e}^{-\mathrm{i}m\pi/2} \mathrm{e}^{\mathrm{i}mt} \, \mathrm{d}t = \frac{1}{2\pi \mathrm{i}^m} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}x \cos t} \mathrm{e}^{\mathrm{i}mt} \, \mathrm{d}t. \end{aligned} \tag{17}$$

Expression (17) can also be written as

$$J_m(x) = \frac{1}{2\pi \mathrm{i}^m} \left(\int_0^{2\pi} \mathrm{e}^{\mathrm{i}x\cos t} \cos mt \,\mathrm{d}t + \mathrm{i} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}x\cos t} \sin mt \,\mathrm{d}t \right). \tag{18}$$

When the argument x is real, the expression can be simplified by noticing that, since the integrand is odd with respect to the center of the integration interval, the last integral in (18) is zero; therefore

$$2\pi \mathrm{i}^m J_m(x) = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}x\cos t} \cos mt \,\mathrm{d}t. \tag{19}$$

Inserting expression (19) into the far-field expression for cylindrical symmetry (16) results in

$$p_\infty(\theta) = \frac{1}{4\pi} \int_{\Gamma_{\mathrm{out}}} y_2 \mathrm{e}^{\mathrm{i}k\widehat{x}_1 y_1} \Bigg[\left(\mathrm{i}pk\widehat{x}_1 n_1(y) - \frac{\partial p}{\partial n} \right) 2\pi J_0(k\widehat{x}_2 y_2) + \mathrm{i}pk\widehat{x}_2 n_2(y) 2\pi \mathrm{i} J_1(k\widehat{x}_2 y_2) \Bigg] \,\mathrm{d}\mathcal{H}^1(y).$$

Again, the normal derivative of p along Γ_{out} can be computed using the strategies in Section 4, and the integral can be evaluated numerically using quadrature.

Remark 1. If the wave propagation problem has been solved using the finite element method, then the integral

$$\int_\Gamma f(x) \,\mathrm{d}\mathcal{H}^{d-1}(x),$$

can be approximated by expanding f on Γ in the finite element basis, $f \approx f_h = \sum_n f_n \varphi_n$. The integral above is then computed as the sum

$$\int_\Gamma f(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) \approx \int_\Gamma f_h(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) = \sum_n f_n \int_\Gamma \varphi_n(x) \,\mathrm{d}\mathcal{H}^{d-1}(x).$$

This method is usually fast, since the element mass matrix for Γ has been computed during the initial assembly when solving the wave propagation problem.

The normal derivative $\partial p/\partial n$

The numerical evaluation of the far-field pattern according to (14) requires the normal derivative $\partial p/\partial n$ along the integration boundary $\widehat{\Gamma}_{\mathrm{out}}$. In this section the computation of this normal derivative is described in two cases.

Case 1. Assume that there is a direct known relationship between p and $\partial p/\partial n$ on $\widehat{\Gamma}_{\mathrm{out}}$. This is the case when, for example, the outgoing wave property is imposed by a Robin boundary condition, such as the first-order Engquist–Majda boundary condition [3]. Thus, after solving the wave propagation problem for p in $\widehat{\Omega}$, the normal derivative $\partial p/\partial n$ is acquired by employing the relationship between p and $\partial p/\partial n$ on $\widehat{\Gamma}_{\mathrm{out}}$.

Now assume that the wave propagation problem is solved in the planar or cylindrical symmetric case, and that the domain $\widehat{\Omega}$ is a cylinder (planar symmetric case) or a ball (cylindrical symmetric case) with radius R, and that the first-order Engquist–Majda boundary condition is used to impose the outgoing wave property. That is,

$$\left(\mathrm{i}\omega + \frac{c}{2R}\right) p + c\frac{\partial p}{\partial n} = 0 \text{ on } \widehat{\Gamma}_{\mathrm{out}}, \tag{20}$$

for the planar symmetric case, and

$$\left(\mathrm{i}\omega + \frac{c}{R}\right) p + c\frac{\partial p}{\partial n} = 0 \text{ on } \widehat{\Gamma}_{\mathrm{out}}, \tag{21}$$

for cylindrical symmetry. Then the far-field pattern in the planar symmetric case is given by combining (13) and (20),

$$p_\infty = \frac{1-\mathrm{i}}{4\sqrt{\pi k}} \int_{\widehat{\Gamma}_{\mathrm{out}}} \mathrm{e}^{\mathrm{i}k\widehat{x}\cdot x} \left(\frac{\mathrm{i}k\widehat{x}\cdot x}{R} + \mathrm{i}k + \frac{1}{2R}\right) p(x)\,\mathrm{d}\mathcal{H}^1(x). \tag{22}$$

The corresponding expression in the cylindrical symmetric case is given by combining (13) and (21),

$$p_\infty = \frac{1}{4\pi} \int_{\widehat{\Gamma}_{\mathrm{out}}} \mathrm{e}^{\mathrm{i}k\widehat{x}\cdot x} \left(\frac{\mathrm{i}k\widehat{x}\cdot x}{R} + \mathrm{i}k + \frac{1}{R}\right) p(x)\,\mathrm{d}\mathcal{H}^2(x). \tag{23}$$

Case 2. There is no direct known relationship between p and $\partial p/\partial n$ on $\widehat{\Gamma}_{\mathrm{out}}$, so we need to compute a numerical approximation to $\partial p/\partial n$ on $\widetilde{\Gamma}_{\mathrm{out}}$, given an approximation to p in $\widetilde{\Omega}$.

Assume that $p \in H^2(\Omega)$ satisfies (1) with radiation condition (2). Further, assume that $\partial p/\partial n = g$ on a part Γ^p of Γ, note that Γ^p might be empty ($\Gamma^p = \emptyset$). The normal derivative $\partial p/\partial n$ on $\Gamma - \Gamma^p$ satisfies

$$\int_{\Gamma-\Gamma^p} \Psi\frac{\partial p}{\partial n} = \int_\Omega \nabla\Psi\cdot\nabla p - k^2\int_\Omega \Psi p - \int_{\Gamma^p}\Psi g, \quad \forall\,\Psi \in H^1(\Omega). \tag{24}$$

Let $p_h \in \mathcal{V}_h(\Omega^h) \subset H^1(\Omega_h)$, where Ω_h is an approximation of Ω, be a finite element solution to the discretized version of (1) with radiation condition (2). Let Γ^h and Γ_h^p be the approximations of the boundaries Γ and Γ^p respectively. Define the spaces

$$\mathcal{V}_h^0 = \left\{ v_h \in \mathcal{V}_h \,;\, v_h\big|_{\Gamma_h - \Gamma_h^p} = 0 \right\},$$
$$\gamma\mathcal{V}_h = \left\{ f_h \in L^2(\Gamma^h - \Gamma_h^p) \,;\, \exists\, v_h \in \mathcal{V}_h \,:\, f_h = v_h \text{ on } \Gamma^h - \Gamma_h^p \right\},$$

and let M_h be the space such that

$$\mathcal{V}_h = \mathcal{V}_h^0 \oplus M_h;$$

that is, the space $M_h \subset \mathcal{V}_h$ is defined such that each element in $\mathcal{V}_h$ can be written uniquely as a sum of one element from $\mathcal{V}_h^0$ and one element from M_h.

By construction, the spaces M_h and $\gamma\mathcal{V}_h$ have the same dimension, in fact, the restriction on $\Gamma_h - \Gamma_h^p$ of functions in M_h spans $\gamma\mathcal{V}_h$. Let g_h correspond to the discretized version of g on Γ_h^p. Choosing $\Psi \in M_h$ in (24), we may define $\lambda_h \in \gamma\mathcal{V}_h$, the approximation of the normal derivative $\partial p/\partial n$ at $\Gamma_h - \Gamma_h^p$, as the solution to the variational problem

Find $\lambda_h \in \gamma\mathcal{V}_h$ such that:

$$\begin{aligned}\int_{\Gamma_h - \Gamma_h^p} \Psi_h \lambda_h = \int_{\Omega_h} \nabla\Psi_h \cdot \nabla p_h \\ - k^2 \int_{\Omega_h} \Psi_h p_h - \int_{\Gamma_h^p} \Psi_h g_h, \quad \forall\, \Phi_h \in M_h.\end{aligned} \tag{25}$$

The variational problem (25) is equivalent to a linear system whose matrix is square, symmetric, and sparse. Computing the approximation of the derivative using this variational approach produces a much more accurate approximation than a direct approximation of element derivatives [4, p. 398].

Acknowledgments

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A Proof of Theorem 1

The proof of Theorem 1 given here covers the two and three dimensional cases in detail. A similar proof in the three dimensional case is given by Colton and Kress [2]. The differences in signs between the proofs are due to the choice

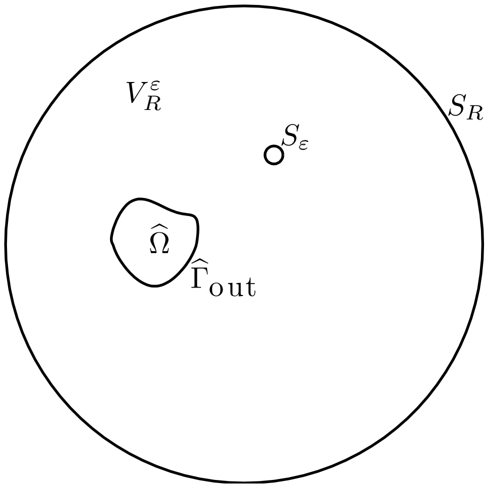

Figure 3: The regions V_R^ε, with boundary $\widehat{\Gamma}_{\text{out}} \cup S_R \cup S_\varepsilon$; and $\widehat{\Omega}$, with boundary $\widehat{\Gamma}_{\text{out}}$.

of sign $\pm$ in the time harmonic ansatz $\left(P = \Re\left\{p(x)\mathrm{e}^{\pm \mathrm{i}\omega t}\right\}\right)$. The proof of the theorem is divided into three lemmas.

Recall that V is the domain exterior to $\overline{\widehat{\Omega}}$, that is $V = \mathbb{R}^d - \overline{\widehat{\Omega}}$, where the overbar denotes closure, and that Theorem 1 is stated as

Theorem 1. *If $p \in \mathcal{C}^2(V)$ satisfies* (1) *in V together with radiation condition* (2)*, then for any point $x_0 \in V$*

$$p(x_0) = \int_{\widehat{\Gamma}_{out}} \left(p \frac{\partial g_d^0}{\partial n} - g_d^0 \frac{\partial p}{\partial n} \right) \mathrm{d}\mathcal{H}^{d-1}(x),$$

where n is the outward directed normal of $\widehat{\Gamma}_{out}$, d is the number of space dimensions, and g_d^0 is the fundamental solution to the Helmholtz equation in d dimensions with source located at x_0.

For $R > |x_0|$, $R > \max_{x \in \widehat{\Omega}} |x|$, $0 < \varepsilon < R - |x_0|$, and $\varepsilon < \min_{x \in \widehat{\Omega}} |x - x_0|$, let $B_R = B(0, R)$ be the open ball of radius R centered at 0 with boundary S_R. Moreover, let $B_\varepsilon = B(x_0, \varepsilon)$ be the open ball of radius ε centered at x_0 with boundary S_ε. Define V_R^ε as the region (Figure 3)

$$V_R^\varepsilon = (V \cap B_R) - \overline{B}_\varepsilon,$$

where the overbar denotes closure.

Under the assumption that p and g_d^0 are sufficiently smooth in V_R^ε, the following version of Green's formula holds:

$$\int_{V_R^\varepsilon} (g_d^0 \Delta p - p \Delta g_d^0)\,\mathrm{d}x = \int_{\partial V_R^\varepsilon} \left(g_d^0 \frac{\partial p}{\partial \tilde{n}} - p \frac{\partial g_d^0}{\partial \tilde{n}} \right) \mathrm{d}\mathcal{H}^{d-1}(x), \tag{26}$$

where $\widetilde{n}$ is the outward directed normal of V_R^ε. Recall that n is the outward directed normal of $\widehat{\Omega}$, hence $\widetilde{n} = -n$ on $\widehat{\Gamma}_{\text{out}}$. Making use of the fact that p and g_d^0 satisfy the Helmholtz equation (1), the left hand side of the previous relation reads

$$\int_{V_R^\varepsilon} (g_d^0 \Delta p - p \Delta g_d^0)\, \mathrm{d}x = \int_{V_R^\varepsilon} (g_d^0(-k^2 p) - p(-k^2 g_d^0))\, \mathrm{d}x = 0,$$

hence (26) simplifies to

$$\int_{\widehat{\Gamma}_{\text{out}}} \left(g_d^0 \frac{\partial p}{\partial \widetilde{n}} - p \frac{\partial g_d^0}{\partial \widetilde{n}} \right) \mathrm{d}\mathcal{H}^{d-1}(x) + \int_{S_R} \left(g_d^0 \frac{\partial p}{\partial \widetilde{n}} - p \frac{\partial g_d^0}{\partial \widetilde{n}} \right) \mathrm{d}\mathcal{H}^{d-1}(x) + \int_{S_\varepsilon} \left(g_d^0 \frac{\partial p}{\partial \widetilde{n}} - p \frac{\partial g_d^0}{\partial \widetilde{n}} \right) \mathrm{d}\mathcal{H}^{d-1}(x) = 0. \tag{27}$$

Lemma 1.

$$\int_{S_R} |p|^2 = O(1) \quad R \to \infty.$$

Proof. A direct calculation shows that

$$|k|^2 \int_{S_R} |p|^2 \,\mathrm{d}\mathcal{H}^{d-1}(x) = \int_{S_R} \left| \frac{\partial p}{\partial \widetilde{n}} + \mathrm{i}kp \right|^2 \mathrm{d}\mathcal{H}^{d-1}(x) - \int_{S_R} \left| \frac{\partial p}{\partial \widetilde{n}} \right|^2 \mathrm{d}\mathcal{H}^{d-1}(x) - 2\Im \left\{ \int_{S_R} \overline{kp} \frac{\partial p}{\partial \widetilde{n}} \,\mathrm{d}\mathcal{H}^{d-1}(x) \right\}.$$

The Sommerfeld condition implies that the first term tends to zero as R tends to infinity, and the second term is negative. Thus given $\varepsilon > 0$, there exists $R_\varepsilon \in \mathbb{R}$ such that the magnitude of the first term is less than ε whenever $R > R_\varepsilon$. Hence, by using integration by parts, equation (1), the fact that $\widetilde{n} = -n$, and stipulating that $R > R_1$, the following holds:

$$\begin{aligned}
|k|^2 \int_{S_R} |p|^2 \,\mathrm{d}\mathcal{H}^{d-1}(x) &\le 1 - 0 - 2\Im \left\{ \int_{S_R} \overline{kp} \frac{\partial p}{\partial \widetilde{n}} \,\mathrm{d}\mathcal{H}^{d-1}(x) \right\} \\
&= 1 - 2\Im \left\{ - \int_{\widehat{\Gamma}_{\text{out}}} \overline{kp} \frac{\partial p}{\partial \widetilde{n}} \,\mathrm{d}\mathcal{H}^{d-1}(x) - \int_{V_R^0} \left(k|kp|^2 - \overline{k} |\nabla p|^2 \right) \mathrm{d}x \right\} \\
&= 1 - 2\Im \left\{ \int_{\widehat{\Gamma}_{\text{out}}} \overline{kp} \frac{\partial p}{\partial n} \,\mathrm{d}\mathcal{H}^{d-1}(x) - \int_{V_R^0} \left(k|kp|^2 - \overline{k} |\nabla p|^2 \right) \mathrm{d}x \right\} \\
&= 1 - 2\Im \left\{ \int_{\widehat{\Gamma}_{\text{out}}} \overline{kp} \frac{\partial p}{\partial n} \,\mathrm{d}\mathcal{H}^{d-1}(x) \right\} + 2\Im\{k\} \int_{V_R^0} \left(|kp|^2 + |\nabla p|^2 \right) \mathrm{d}x.
\end{aligned}$$

Recall that the wave number k, given by ω/c, is real,[2] hence

$$0 \leq |k|^2 \int_{S_R} |p|^2 \,\mathrm{d}\mathcal{H}^{d-1}(x) \leq 1 - 2\Im\left\{\int_{\widehat{\Gamma}_{\text{out}}} \overline{kp}\frac{\partial p}{\partial n}\,\mathrm{d}\mathcal{H}^{d-1}(x)\right\} = const.$$

The expression above holds for any $R > R_1$, thus

$$\int_{S_R} |p|^2 \,\mathrm{d}\mathcal{H}^{d-1}(x) = O(1) \quad R \to \infty. \qquad \square$$

Lemma 2. *For any point $x_0 \in V$*

$$\lim_{R\to\infty} \int_{S_R} \left(g_d^0 \frac{\partial p}{\partial \widetilde{n}} - p\frac{\partial g_d^0}{\partial \widetilde{n}} \right) \mathrm{d}\mathcal{H}^{d-1}(x) = 0.$$

Proof.

$$\begin{aligned}
&\int_{S_R} \left(g_d^0 \frac{\partial p}{\partial \widetilde{n}} - p\frac{\partial g_d^0}{\partial \widetilde{n}} \right) \mathrm{d}\mathcal{H}^{d-1}(x) \\
&= \int_{S_R} \left(g_d^0 \frac{\partial p}{\partial \widetilde{n}} - \mathrm{i}k g_d^0 p - p\frac{\partial g_d^0}{\partial \widetilde{n}} + \mathrm{i}kp g_d^0 \right) \mathrm{d}\mathcal{H}^{d-1}(x) \\
&= -\int_{S_R} p \left(\frac{\partial g_d^0}{\partial \widetilde{n}} + \mathrm{i}k g_d^0 \right) \mathrm{d}\mathcal{H}^{d-1}(x) + \int_{S_R} g_d^0 \left(\frac{\partial p}{\partial \widetilde{n}} + \mathrm{i}kp \right) \mathrm{d}\mathcal{H}^{d-1}(x) \\
&= I_1(R) + I_2(R).
\end{aligned}$$

The fact that p satisfies the Sommerfeld radiation condition (2), that is $\frac{\partial p}{\partial \widetilde{n}} + \mathrm{i}kp = o(R^{\frac{d-1}{2}})$, and $g_d^0 = O(R^{\frac{d-1}{2}})$ directly implies that $I_2(R) \to 0$ as $R \to \infty$. Notice that the plus sign in the Sommerfeld condition is due to the choice $\mathrm{e}^{\pm\mathrm{i}\omega t}$ in the time harmonic ansatz.

Further, for $x \in S_R$ the following holds:

$$|x_0 - x| = R\sqrt{1 - \frac{2x_0 \cdot \widetilde{x}}{R} + \frac{|x_0|^2}{R^2}} = O(R) \quad R \to \infty,$$

[2] For the general Helmholtz equation $\Delta p + k^2 p = 0$, where $k^2 \in \mathbb{C}$, choose k with $\Im\{k\} \leq 0$.

where $\widetilde{x} = x/R$ and hence

$$
\begin{aligned}
\frac{\partial}{\partial R}|x_0 - x| &= \frac{\partial}{\partial R} R\sqrt{1 - \frac{2x_0 \cdot \widetilde{x}}{R} + \frac{|x_0|^2}{R^2}} \\
&= \sqrt{1 - \frac{2x_0 \cdot \widetilde{x}}{R} + \frac{|x_0|^2}{R^2}} + R\frac{\frac{2x_0 \cdot \widetilde{x}}{R^2} - \frac{2|x_0|^2}{R^3}}{2\sqrt{1 - \frac{2x_0 \cdot \widetilde{x}}{R} + \frac{|x_0|^2}{R^2}}} \\
&= \frac{2 - \frac{4x_0 \cdot \widetilde{x}}{R} + \frac{2|x_0|^2}{R^2} + \frac{2x_0 \cdot \widetilde{x}}{R} - \frac{2|x_0|^2}{R^2}}{2\sqrt{1 - \frac{2x_0 \cdot \widetilde{x}}{R} + \frac{|x_0|^2}{R^2}}} = \frac{1 - \frac{x_0 \cdot \widetilde{x}}{R}}{\sqrt{1 - \frac{2x_0 \cdot \widetilde{x}}{R} + \frac{|x_0|^2}{R^2}}} \\
&= \left(1 - \frac{x_0 \cdot \widetilde{x}}{R}\right)\left(1 + \frac{x_0 \cdot \widetilde{x}}{R} + O\left(\frac{1}{R^2}\right)\right) \\
&= 1 + O\left(\frac{1}{R^2}\right) \qquad R \to \infty.
\end{aligned}
$$

In the two dimensional case,

$$
\begin{aligned}
\frac{\partial g_2^0}{\partial \widetilde{n}} + \mathrm{i}k g_2^0 &= \frac{\partial}{\partial R}\left(\frac{\mathrm{i}}{4} H_0^{(1)}(-k|x - x_0|)\right) + \mathrm{i}k\frac{\mathrm{i}}{4} H_0^{(1)}(-k|x - x_0|) \\
&= \frac{-\mathrm{i}}{4} H_1^{(1)}(-k|x - x_0|)\left(\frac{\partial}{\partial R}\left\{-k|x - x_0|\right\}\right) + \mathrm{i}k\frac{\mathrm{i}}{4} H_0^{(1)}(-k|x - x_0|) \\
&= \frac{\mathrm{i}k}{4}\left(H_1^{(1)}(-k d_0(x))\left\{1 + O\left(\frac{1}{R^2}\right)\right\} + \mathrm{i}H_0^{(1)}(-k d_0(x))\right),
\end{aligned}
$$

where $d_0(x) = |x - x_0|$. Recall the asymptotics for the Hankel function given in (5),

$$
H_m^{(1)}(-\xi) = (-1)^{m+1}\sqrt{\frac{2}{\pi\xi}}\mathrm{e}^{-\mathrm{i}(\xi - m\pi/2 - \pi/4)}\left\{1 + O\left(\frac{1}{\xi}\right)\right\} \qquad \xi \to +\infty.
$$

Hence

$$\begin{aligned}
\frac{\partial g_2^0}{\partial \widetilde{n}} + \mathrm{i}kg_2^0 &= \frac{\mathrm{i}k}{4}\left(H_1^{(1)}(-kd_0(x))\left\{1+O\left(\frac{1}{R^2}\right)\right\} + \mathrm{i}H_0^{(1)}(-kd_0(x))\right) \\
&= \left[\sqrt{\frac{2}{\pi k d_0(x)}}\mathrm{e}^{-\mathrm{i}(kd_0(x)-\pi/2-\pi/4)}\left\{1+O\left(\frac{1}{R^2}\right)\right\}\left\{1+O\left(\frac{1}{kd_0(x)}\right)\right\}\right. \\
&\quad \left. + \mathrm{i}(-1)\sqrt{\frac{2}{\pi k d_0(x)}}\mathrm{e}^{-\mathrm{i}(kd_0(x)-\pi/4)}\left\{1+O\left(\frac{1}{kd_0(x)}\right)\right\}\right] \\
&= \frac{\mathrm{i}k}{4}\left[\mathrm{e}^{\mathrm{i}\pi/2}\left\{1+O\left(\frac{1}{R^2}\right)\right\}\left\{1+O\left(\frac{1}{R}\right)\right\}\right. \\
&\quad \left. - \mathrm{i}\left\{1+O\left(\frac{1}{R}\right)\right\}\right]\mathrm{e}^{-\mathrm{i}(kd_0(x)-\pi/4)}\sqrt{\frac{2}{\pi k d_0(x)}} \\
&= \frac{\mathrm{i}k}{4}\left[\mathrm{i}\left\{1+O\left(\frac{1}{R}\right)\right\} - \mathrm{i}\left\{1+O\left(\frac{1}{R}\right)\right\}\right]O\left(\frac{1}{\sqrt{R}}\right) \\
&= O(1)O\left(\frac{1}{R}\right)O\left(\frac{1}{\sqrt{R}}\right) = O\left(\frac{1}{R^{3/2}}\right).
\end{aligned} \tag{28}$$

In the three dimensional case,

$$\begin{aligned}
\frac{\partial g_3^0}{\partial \widetilde{n}} + \mathrm{i}kg_3^0 &= \frac{\partial}{\partial R}\left(\frac{\mathrm{e}^{-\mathrm{i}k|x-x_0|}}{4\pi|x-x_0|}\right) + \mathrm{i}k\frac{\mathrm{e}^{-\mathrm{i}k|x-x_0|}}{4\pi|x-x_0|} \\
&= \frac{\mathrm{e}^{-\mathrm{i}k|x-x_0|}}{4\pi|x-x_0|}\left(-\mathrm{i}k - \frac{1}{|x-x_0|}\right)\left(\frac{\partial}{\partial R}|x-x_0|\right) + \mathrm{i}k\frac{\mathrm{e}^{-\mathrm{i}k|x-x_0|}}{4\pi|x-x_0|} \\
&= \frac{\mathrm{e}^{-\mathrm{i}k|x-x_0|}}{4\pi|x-x_0|}\left(-\mathrm{i}k - \frac{1}{|x-x_0|}\right)\left\{1+O\left(\frac{1}{R^2}\right)\right\} + \mathrm{i}k\frac{\mathrm{e}^{-\mathrm{i}k|x-x_0|}}{4\pi|x-x_0|} \\
&= -\frac{\mathrm{e}^{-\mathrm{i}k|x-x_0|}}{4\pi|x-x_0|}\frac{1}{|x-x_0|}\left\{1+O\left(\frac{1}{R^2}\right)\right\} \\
&= O\left(\frac{1}{R}\right)O\left(\frac{1}{R}\right)O(1) = O\left(\frac{1}{R^2}\right).
\end{aligned} \tag{29}$$

Using Lemma 1 together with expressions (28) and (29) results in

$$\begin{aligned}
|I_1(R)| &= \left|\int_{S_R} p\left(\frac{\partial g_d^0}{\partial \widetilde{n}} + \mathrm{i}kg_d^0\right)\right| \leq \int_{S_R}|p|\left|\frac{\partial g_d^0}{\partial \widetilde{n}} + \mathrm{i}kg_d^0\right| \\
&\leq \left(\int_{S_R}|p|^2\int_{S_R}\left|\frac{\partial g_d^0}{\partial \widetilde{n}} + \mathrm{i}kg_d^0\right|^2\right)^{\frac{1}{2}} = \left(O(1)\int_{S_R}O\left(\frac{1}{R^{d+1}}\right)\right)^{\frac{1}{2}} \\
&= \left(O(1)O\left(\frac{1}{R^2}\right)\right)^{\frac{1}{2}} = O\left(\frac{1}{R}\right) \to 0 \quad \text{as } R\to\infty. \quad \square
\end{aligned}$$

Lemma 3.

$$\lim_{\varepsilon\to 0}\int_{S_\varepsilon}\left(g_d^0\frac{\partial p}{\partial\widetilde{n}}-p\frac{\partial g_d^0}{\partial\widetilde{n}}\right)\,\mathrm{d}\mathcal{H}^{d-1}(x)=-p(x_0).$$

Proof. On S_ε the following holds:

$$g_2^0=\frac{\mathrm{i}}{4}H_0^{(1)}(-k\varepsilon),\quad \frac{\partial g_2^0}{\partial\widetilde{n}}=-\frac{\mathrm{i}}{4}\big(kH_1^{(1)}(-k\varepsilon)\big),$$
$$g_3^0=\frac{\mathrm{e}^{-\mathrm{i}k\varepsilon}}{4\pi\varepsilon},\qquad \frac{\partial g_3^0}{\partial\widetilde{n}}=-\left(-\mathrm{i}k-\frac{1}{\varepsilon}\right)\frac{\mathrm{e}^{-\mathrm{i}k\varepsilon}}{4\pi\varepsilon}=\left(\mathrm{i}k+\frac{1}{\varepsilon}\right)\frac{\mathrm{e}^{-\mathrm{i}k\varepsilon}}{4\pi\varepsilon}.$$

Note the minus sign due to the fact that $\widetilde{n}$ is pointing inward towards x_0. Since x_0 is fixed in the interior of V, we can pick $\tau>0$, such that $\overline{B}(x_0,\tau)\subset V$. Let ν be an arbitrary unit vector, then, since $p\in\mathcal{C}^2(V)$, for all $x\in\overline{B}(x_0,\tau)$

$$\left|\frac{\partial p}{\partial\nu}(x)\right|=|\nu\cdot\nabla p(x)|=|\nabla p(x)|\le\max_{\overline{B}(x_0,\tau)}|\nabla p|=C<\infty,$$

where C is a constant (depending on x_0 and τ). In particular this implies that $|\partial p(x)/\partial\widetilde{n}|\le C$ for $x\in S_\varepsilon$ for $\varepsilon\le\tau$. Hence

$$\begin{aligned}\left|\int_{S_\varepsilon}g_2^0\frac{\partial p}{\partial\widetilde{n}}\,\mathrm{d}\mathcal{H}^1(x)\right|&\le\int_{S_\varepsilon}\left|\frac{\mathrm{i}}{4}H_0^{(1)}(-k\varepsilon)\frac{\partial p}{\partial\widetilde{n}}\right|\,\mathrm{d}\mathcal{H}^{d-1}(x)\\&\le\frac{C}{4}\int_{S_\varepsilon}\left|H_0^{(1)}(-k\varepsilon)\right|\,\mathrm{d}\mathcal{H}^{d-1}(x)=\frac{C}{4}2\pi\varepsilon\left|H_0^{(1)}(-k\varepsilon)\right|\\&=\frac{C\pi}{2}\left|\varepsilon H_0^{(1)}(-k\varepsilon)\right|\to 0\quad\text{as }\varepsilon\to 0,\end{aligned}$$

in two space dimensions, and

$$\begin{aligned}\left|\int_{S_\varepsilon}g_3^0\frac{\partial p}{\partial\widetilde{n}}\,\mathrm{d}\mathcal{H}^2(x)\right|&\le\int_{S_\varepsilon}\left|\frac{\mathrm{e}^{-\mathrm{i}k\varepsilon}}{4\pi\varepsilon}\frac{\partial p}{\partial\widetilde{n}}\right|\,\mathrm{d}\mathcal{H}^{d-1}(x)\\&\le C\int_{S_\varepsilon}\frac{1}{4\pi\varepsilon}\,\mathrm{d}\mathcal{H}^{d-1}(x)=C\varepsilon\to 0\quad\text{as }\varepsilon\to 0,\end{aligned}$$

in three space dimensions.

The second part of the integral over S_ε is rewritten, using the asymptotics near the origin, as

$$\begin{aligned}\int_{S_\varepsilon}p\frac{\partial g}{\partial\widetilde{n}}\,\mathrm{d}\mathcal{H}^{d-1}(x)&=\int_{S_\varepsilon}(p(x_0)+O(\varepsilon))\frac{-\mathrm{i}k}{4}H_1^{(1)}(-k\varepsilon)\,\mathrm{d}\mathcal{H}^{d-1}(x)\\&=2\pi\varepsilon(p(x_0)+O(\varepsilon))\frac{k}{4\mathrm{i}}H_1^{(1)}(-k\varepsilon)\\&=(p(x_0)+O(\varepsilon))\frac{\pi k}{2\mathrm{i}}\varepsilon H_1^{(1)}(-k\varepsilon)\\&\to(p(x_0)+0)\frac{\pi k}{2\mathrm{i}}\frac{2\mathrm{i}}{k\pi}=p(x_0)\quad\text{as }\varepsilon\to 0,\end{aligned}$$

in the two dimensional case, and

$$\begin{aligned}\int_{S_\varepsilon} p\frac{\partial g}{\partial \widetilde{n}}\,\mathrm{d}\mathcal{H}^{d-1}(x) &= \int_{S_\varepsilon} (p(x_0)+O(\varepsilon))\left(\mathrm{i}k+\frac{1}{\varepsilon}\right)\frac{\mathrm{e}^{-\mathrm{i}k\varepsilon}}{4\pi\varepsilon}\\ &= 4\pi\varepsilon^2(p(x_0)+O(\varepsilon))\left(\mathrm{i}k+\frac{1}{\varepsilon}\right)\frac{\mathrm{e}^{-\mathrm{i}k\varepsilon}}{4\pi\varepsilon}\\ &= (p(x_0)+O(\varepsilon))(\mathrm{i}k\varepsilon+1)\mathrm{e}^{-\mathrm{i}k\varepsilon}\to p(x_0)\quad \text{as } \varepsilon\to 0,\end{aligned}$$

in the three dimensional case. Thus in both two and three space dimensions

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{S_\varepsilon}\left(g_d^0\frac{\partial p}{\partial \widetilde{n}} - p\frac{\partial g_d^0}{\partial \widetilde{n}}\right)\mathrm{d}\mathcal{H}^{d-1}(x)\\ &= \lim_{\varepsilon\to 0}\int_{S_\varepsilon} g_d^0\frac{\partial p}{\partial \widetilde{n}}\,\mathrm{d}\mathcal{H}^{d-1}(x) - \lim_{\varepsilon\to 0}\int_{S_\varepsilon} p\frac{\partial g_d^0}{\partial \widetilde{n}}\,\mathrm{d}\mathcal{H}^{d-1}(x)\\ &= 0 - p(x_0) = -p(x_0). \quad \square\end{aligned}$$

Proof of Theorem 1. Taking the limits $R\to\infty$ and $\varepsilon\to 0$ using the results from Lemma 2 and 3 yields

$$\begin{aligned}&\int_{\widehat{\Gamma}_{\text{out}}}\left(g\frac{\partial p}{\partial \widetilde{n}} - p\frac{\partial g}{\partial \widetilde{n}}\right)\mathrm{d}\mathcal{H}^{d-1}(x)\\ &= -\int_{S_R}\left(g_d^0\frac{\partial p}{\partial \widetilde{n}} - p\frac{\partial g_d^0}{\partial \widetilde{n}}\right)\mathrm{d}\mathcal{H}^{d-1}(x) - \int_{S_\varepsilon}\left(g_d^0\frac{\partial p}{\partial \widetilde{n}} - p\frac{\partial g_d^0}{\partial \widetilde{n}}\right)\mathrm{d}\mathcal{H}^{d-1}(x)\\ &\to -0-(-p(x_0)) = p(x_0),\end{aligned}$$

as $R\to\infty$ and $\varepsilon\to 0$. Hence

$$p(x_0) = \int_{\widehat{\Gamma}_{\text{out}}}\left(g_d^0\frac{\partial p}{\partial \widetilde{n}} - p\frac{\partial g_d^0}{\partial \widetilde{n}}\right)\mathrm{d}\mathcal{H}^{d-1}(x) = \int_{\widehat{\Gamma}_{\text{out}}}\left(p\frac{\partial g_d^0}{\partial n} - g_d^0\frac{\partial p}{\partial n}\right)\mathrm{d}\mathcal{H}^{d-1}(x),$$

where n is the outward directed normal of $\widehat{\Omega}$. □

B Asymptotics for the Bessel functions

The Hankel functions of the first kind are defined as

$$H_m^{(1)}(z) = J_m(z) + \mathrm{i}Y_m(z),$$

where m is the order J_m and Y_m are the Bessel functions of the first and second kind respectively. The asymptotic behavior of the Hankel functions for positive real arguments can be found in books on special functions, see for example the book by Schäfke [6]. The asymptotic behavior for the Hankel functions of the first kind with positive real arguments is given by

$$H_m^{(1)}(\xi) = \sqrt{\frac{2}{\pi\xi}}\mathrm{e}^{\mathrm{i}(\xi - m\pi/2-\pi/4)}\left\{1+O\left(\frac{1}{\xi}\right)\right\}\quad \xi\to+\infty.$$

For $\xi > 0$ the Bessel functions are given by the following expressions [6]:

$$\begin{aligned} J_p(\xi) &= \sum_{k=0}^{\infty} \frac{(-1)^k (\xi/2)^{p+2k}}{k!\Gamma(p+k+1)}, \\ J_{-p}(\xi) &= \sum_{k=0}^{\infty} \frac{(-1)^k (\xi/2)^{2k-p}}{k!\Gamma(k+1-p)}, \\ Y_m(\xi) &= \lim_{p\to m} \frac{J_p(\xi)\cos p\pi - J_{-p}(\xi)}{\sin p\pi}, \end{aligned}$$

where $p \geq 0$ and $m = 0, 1, \ldots$. For J_p we have the following:

$$\begin{aligned} J_p(-\xi) &= \sum_{k=0}^{\infty} \frac{(-1)^k (-\xi/2)^{p+2k}}{k!\Gamma(p+k+1)} \\ &= (-1)^p \sum_{k=0}^{\infty} (-1)^{2k} \frac{(-1)^k (\xi/2)^{p+2k}}{k!\Gamma(p+k+1)} (-1)^p J_p(\xi), \\ J_{-p}(-\xi) &= \sum_{k=0}^{\infty} \frac{(-1)^k (-\xi/2)^{2k-p}}{k!\Gamma(k+1-p)} \\ &= (-1)^{-p} \sum_{k=0}^{\infty} (-1)^{2k} \sum_{k=0}^{\infty} \frac{(-1)^k (\xi/2)^{2k-p}}{k!\Gamma(k+1-p)} = (-1)^{-p} J_{-p}(\xi). \end{aligned}$$

For Y_m we have the following:

$$\begin{aligned} Y_m(-\xi) &= \lim_{p\to m} \frac{J_p(-\xi)\cos p\pi - J_{-p}(-\xi)}{\sin p\pi} \\ &= \lim_{p\to m} \frac{(-1)^p J_p(\xi)\cos p\pi - (-1)^{-p} J_{-p}(\xi)}{\sin p\pi} \\ &= \lim_{p\to m} \left[\frac{(-1)^p J_p(\xi)\cos p\pi - (-1)^p J_{-p}(\xi)}{\sin p\pi} + \frac{\left((-1)^p - (-1)^{-p}\right) J_{-p}(\xi)}{\sin p\pi} \right] \\ &= \lim_{p\to m} \left[(-1)^p \frac{J_p(\xi)\cos p\pi - J_{-p}(\xi)}{\sin p\pi} + 2\mathrm{i} J_{-p}(\xi) \right] \\ &= (-1)^m Y_m(\xi) + 2\mathrm{i} J_{-m}(\xi) = (-1)^m \left[Y_m(\xi) + 2\mathrm{i} J_m(\xi) \right]. \end{aligned}$$

Further, since $\xi > 0$ it is known that the values of the Bessel functions are real, hence

$$\begin{aligned} H_m^{(1)}(-\xi) &= J_m(-\xi) + \mathrm{i} Y_m(-\xi) \\ &= (-1)^m J_m(\xi) + \mathrm{i}(-1)^m \left[Y_m(\xi) + 2\mathrm{i} J_m(\xi) \right] \\ &= (-1)^m J_m(\xi) + \mathrm{i}(-1)^m Y_m(\xi) - 2(-1)^m J_m(\xi) \\ &= (-1)^{m+1} (J_m(\xi) - \mathrm{i} Y_m(\xi)) = (-1)^{m+1} \overline{H_m^{(1)}}(\xi), \end{aligned}$$

where the overline denotes the complex conjugate. Hence, the asymptotic behavior of $H_m^{(1)}$ for large real negative arguments is given by

$$\begin{aligned} H_m^{(1)}(-\xi) &= (-1)^{m+1}\overline{H_m^{(1)}(\xi)} \\ &= (-1)^{m+1}\overline{\sqrt{\frac{2}{\pi\xi}}\mathrm{e}^{\mathrm{i}(\xi-m\pi/2-\pi/4)}\left\{1+O\left(\frac{1}{\xi}\right)\right\}} \\ &= (-1)^{m+1}\sqrt{\frac{2}{\pi\xi}}\mathrm{e}^{-\mathrm{i}(\xi-m\pi/2-\pi/4)}\left\{1+O\left(\frac{1}{\xi}\right)\right\} \quad \xi\to+\infty. \end{aligned}$$

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