A New Analysis of Revenue in the Combinatorial and Simultaneous Auction

Arne Andersson
Computing Science Division
Dept. of Information Technology
Uppsala University *
and Trade Extensions AB
arne@te.se

Jim Wilenius
Computing Science Division
Dept. of Information Technology
Uppsala University
jim.wilenius@it.uu.se

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Abstract

We prove that in many cases, a first-price sealed-bid combinatorial auction gives higher expected revenue than a sealed-bid simultaneous auction. This is the first theoretical evidence that combinatorial auctions indeed generate higher revenue, which has been a common belief for decades.

We use a model with many bidders and items, where bidders are of two types: (i) single-bidders interested in only one item and (ii) synergy-bidders, each interested in one random combination of items. We provide an upper bound on the expected revenue for simultaneous auctions and a lower bound on combinatorial auctions. Our bounds are parameterized on the number of bidders and items, combination size, and synergy.

We derive an asymptotic result, proving that as the number of bidders approach infinity, expected revenue of the combinatorial auction will be higher than that of the simultaneous auction. We also provide concrete examples where the combinatorial auction is revenue-superior.

1 Introduction

It is a common belief that combinatorial auctions provide good solutions to resource-allocation in multi-commodity markets. The idea is that if a bidder has some synergy from winning a specific combination of items, he should be able to express this with one all-or-nothing bid (commonly combinatorial bid) for the entire combination. Allowing combinatorial bids and using a suitable method for winner determination, it seems reasonable that the resulting allocation should be more efficient and result in higher revenue for the auctioneer than if such bids were not allowed. Although these properties are fundamental, no real theoretical evidence with regards to revenue has so far been provided in the literature. In fact, the only known theoretical analysis (by Krishna and Rosenthal [7], and Albano, Germano and Lovo [6]) has been done for the case of two items for sale and indicates the opposite; the combinatorial auction gives a lower revenue.

*Box 337, 75105 Uppsala, Sweden

However, the case of two items is very far from most real cases. In real-world combinatorial auctions we have: (i) many items and bidders, (ii) synergies on independent and unknown combinations of items, (iii) incomplete information amongst bidders, and (iv) a complicated winner determination problem where more than one combination can win and some items may be allocated to single bids.

The model we study in this work, albeit simplified, in essence preserves the important properties of real world combinatorial auctions as mentioned above. Analyzing such complicated situations requires approaches that differ from the common methods of analysis in auction theory. In this work we take a fist step towards understanding these situations.

Completely understanding the optimal behavior in large-scale combinatorial auctions is of course a very hard problem, one which we do not claim to solve. However, our analysis gives valuable insights into key properties of combinatorial and simultaneous auctions. In particular, we provide conditions for which we can show that combinatorial auctions indeed generate higher expected revenue.

A simple example of the type of settings studied here is illustrated in Table 1. In this example, bidders A through C each have an interest in a specific combination of items, and if they win all items in their combination they realize an added value, a synergy. Bidders E, F and G are interested only in a single item each.

Table 1: Bidding scenario

Bidders:	A	B	C	E	F	G
item 1	•		•	•		
item 2	•	•				
item 3		•			•	•
item 4			•			
item 5	•		•			
item 6		•				
Value per item:	0.8	0.5	0.6	0.7	0.8	0.6
Synergy per item:	1.0	1.0	1.0			
Total Combination Value:	5.4	4.5	4.8			

Bidding scenario where bidders A-C have synergies on specific combinations. Bidders E-G are interested in a single item.

In our analysis we focus on two commonly employed protocols: the *sealed-bid simultaneous auction*[1] and the *first-price sealed-bid combinatorial auction.*

First we provide an upper bound on expected revenue in the simultaneous auction (Theorem 4.1). Secondly we show that as the number of bidders goes to infinity, expected revenue in the combinatorial auction is greater than that of the simultaneous auction (Corollary 5.1). Then, we derive a parameterized lower bound on expected revenue in the combinatorial auction. Finally, we provide for several settings, an upper bound on the number of bidders required to achieve expected revenue dominance for the combinatorial auction.

Krishna and Rosenthal [7] analyze a second-price simultaneous auction and although Krishna's results are not directly transferable, it is worth pointing out that a comparison is made between the (second-price) simultaneous auction and a variant of the generalized

[1] We only specify sealed-bid, since our results are general and not sensitive to which payment rule, first-price, second-price etc, is implemented.

Vickrey-Clarke-Groves (VCG) mechanism, with regards to expected revenue. This is done only for the case of two items, with one synergy-bidder and two single-bidders. Krishna shows that allowing combinatorial bids results in significantly lower expected revenue than that of a second-price simultaneous auction (with only single-bids). However, no comparison was made for larger instances.

A comparison of the first-price sealed-bid combinatorial auction to the sealed-bid simultaneous auction appears to be missing in the literature. Focus has been on characterizing equilibrium strategies for various simultaneous auctions, and sequential auctions (see Krishna and Rosenthal [7], Albano, Germano and Lovo [6], and Branco [4]). Some work has also been done in deriving optimal auctions given two items (Levin [9] and Armstrong [1]), and with multiple items (Ledyard [8]). At the time of writing, we have not been able to find any work in the literature that compare the two auctions for more realistically sized problems, that is, many items and combinations of greater size. Most models in the literature seem to concern the 2-items case, and when generalizations are proposed these usually take the form of bidders interested in *all* available items (Krishna and Rosenthal [7] and Levin [9]). Krishna and Rosenthal [7] also describes a pairwise overlapping generalization for more than two items, but where each combination contains two items. A similar model using common values is proposed by Rosenthal and Wang [15]. These generalizations unfortunately are of limited interest from a combinatorial viewpoint since the combination size is still only two items; also the *all-items* generalization in essence reduces the combinatorial auction to a single-item auction since only one combination can win. Ledyard [8] derives an optimal combinatorial auction with single-minded bidders interested in specific combinations given that the auctioneer knows which combinations the bidders are interested in. Ledyard's model is the closest to the one studied here, and also the only of the mentioned models that remain interesting from a combinatorial perspective.

The techniques used in our proofs differ from standard auction theory techniques, simply because the core of the combinatorial auction problem is fundamentally complex. In a combinatorial auction even determining who wins is a hard maximization problem.

We would like to emphasize that the main point of this work is not to exactly derive equilibrium strategies nor the expected revenue. Solving those problems is simply too hard. The main point is, however, to show that it is possible to find theoretical support for the common belief that combinatorial first-price auctions indeed can give higher expected revenue than the simultaneous single-item auctions.

2 Auctions Considered

Fundamentally and on a high level, there are two main types of auctions, single-item auctions and combinatorial auctions, amongst these there are many different variants but our focus will be on two specific auctions, the sealed-bid simultaneous auction and the first-price sealed-bid combinatorial auction. We will refer to these auctions as simply the simultaneous auction and combinatorial auction when no misunderstanding is possible. The following sections discuss the two auction types in more detail.

2.1 Simultaneous Auction

The simultaneous auction, specifically the sealed-bid simultaneous auction, is actually several simultaneously executed single-item auctions. The winner of each of these auctions is determined by the highest bidder in that particular auction[2]. The winners pay either the amount bid (first-price) or the second highest bid (second-price), but there are other conceivable payment rules as well. Our results regarding the simultaneous auction apply to any reasonable payment-rule, since the upper bound is based on the expected realized synergy, not actual payments by bidders.

The issue of bidding when faced with synergies is not trivial and the bidding strategy naturally depends on the disposition of these synergies and the bidders' view of the competition. A dilemma that a bidder is faced with in this auction is the uncertainty of how many items he will win. This implies an uncertainty regarding whether or not to bid above the single item value, utilizing the potential gain of the synergy in the gamble that all items are won. If he bids above the single item value and not all items are won, he looses money. This is commonly known as the exposure problem. Although some work has been done in the area of simultaneous auctions , unfortunately no work has been found that cover the simultaneous auction for our specific model nor a comparable model.

2.2 Combinatorial Auction

The (first-price sealed-bid) combinatorial auction, is one of many possible combinatorial auctions, perhaps the most straightforward. Bidders submit bids for combinations of items, and if they win, pay the amount bid. Winners are determined by solving the (generally) NP-hard maximization problem known as the winner determination problem, where winners normally consist of the non-colliding bids that maximize the auctioneer's revenue. This combinatorial puzzle is one of the factors that fundamentally separates the combinatorial auction from single item auctions.

Another distinguishing factor is that in a combinatorial auction, a bidder does not have to speculate on how many items he will win, since a bid on a combination of items either wins in its entirety or not at all. This allows bidders to express complex preferences on combinations of items without the risk present in the simultaneous auction. While the exposure problem arises in the simultaneous auction, the combinatorial auction gives rise to a *threshold problem* (free-riding), the fact that a bidder could potentially under-bid and still be a part of the optimal allocation. As a consequence, bidders may be inclined to bid strategically, but despite this, as will be shown later, we manage to construct a useful lower bound on the expected revenue.

The question of optimal bidding strategy, which in the case of one item for sale is well studied, is still an open problem in the first-price sealed-bid combinatorial auction, although some work towards approximating strategies has been done by Wilenius and Andersson [17], and Vorobeychik, Wellman and Singh [16]. Bernheim and Whinston [3] provide an analysis of a first-price menu-auction in a complete information setting. It is not uncommon that work on combinatorial auctions focus on incentive-compatibility, thereby removing the strategy-problem since the optimal strategy is to bid ones true valuation. One of the most commonly addressed incentive-compatible mechanisms are the Vickrey-Clarke-Groves (VCG) family of mechanisms. The VCG has the desirable property that it gives optimally efficient allocations (under certain conditions), however it can be somewhat lacking in revenue, although methods for boosting revenue in the VCG have been investigated by Likhodedov and Sandholm [10]. One significant drawback of the VCG is the computational burden of calculating payments,

[2] In the event of ties, the winner is decided by a lottery.

the winner determination problem must be solved once for each winner. Another problem is the significant burden of communicating exponentially many bids. For more background on combinatorial auctions and the VCG, see for example Cramton, Shoham and Steinberg [5], Pekec and Rothkopf [13], and Milgrom [11].

3 Model – Bidders and Valuations

We focus on a standard model where bidders have independent private valuations, drawn from a uniform distribution[3]. Definition 3.1 specifies these properties and more, and will serve as the basis for comparison of the two auction formats.

We use two types of bidders, *synergy-bidders* and *single-bidders*[4]. A synergy-bidder has a synergy on a specific set of randomly chosen items. A single-bidder is interested in one particular item only.

Definition 3.1. The common model:

(a) Bidders are rational, risk neutral, and symmetric. Only pure symmetric equilibrium is considered.

(b) Valuations are private and independently drawn from a continuous uniform distribution on the interval [0,1].

(c) There are M items for sale.

(d) A single-bidder is interested in one item.

(e) A synergy-bidder is *weakly single-minded*, that is, interested in one uniformly chosen random combination of size k. He has the same valuation on all k items, and receives a synergy of α per item if all k items are won[5].

(f) There are a total of N synergy-bidders, and for each item there are N single-bidders.

4 Simultaneous Auction

In this section we will illustrate, by providing an upper bound on expected revenue, that the exposure problem is indeed an actual problem that forces synergy-bidders to bid carefully. Simply put, the larger the combination, the smaller the probability of winning all the single-bids on the entire combination will be. Increasing the number of bidders will also decrease the probability of winning all items in the combination (Krishna and Rosenthal [7] similarly concludes that in the second-price simultaneous auction, with increasing number of bidders, bidding becomes less aggressive). We need the following definition.

Definition 4.1. Bidder A is strictly higher than bidder B, if and only if A's lowest bid is greater than B's highest bid. Similarly a set of bidders is strictly ordered if for any pair of bidders, one is strictly higher than the other.

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[3] We choose the uniform distribution as this is a very commonly occurring distribution in examples throughout the literature. Generalizing to other continuous distributions is also likely to be possible but we refrain from doing that at this point since that will not greatly contribute to our main argument.

[4] The terms global and local bidders are used by Krishna et al. [7] and Albano et al. [6]. However, we prefer a more descriptive name since a global bidder could be mistaken for a bidder that bids on all items, which is not the case here.

[5] Krishna et al. [7] and Albano et al. [6] similarly use a public synergy α.

The order in Definition 4.1 orders bidders regardless if they bid on the same items or not.

Lemma 4.1. *The expected sum of all bidders' realized synergies in the simultaneous auction is maximized when bidders are strictly ordered.*

Proof. Assume there is an adversary that wants to maximize the expected sum of realized synergies, he knows the number of bidders and items, but not who bids on what. The adversary is allowed to set the values of all bids. Assume w.l.o.g. that he always assigns unique values to all bids[6].

Consider one set of bids as assigned by the adversary. In this set there is now one bidder, A, that has the highest of all bids. Consider this bidder, there are two outcomes: (i) bidder A wins all his k items, or (ii) bidder A does not win all his items.

If outcome (i) is better for the adversary (i.e generates higher expected realized synergy), then clearly the adversary might as well raise A's bids so that A is strictly higher than all other bidders.

Consider the second case, assuming outcome (ii) is better, then the adversary will do better by lowering A's highest bid, so that A is not the highest on that particular item. To see why this is, consider what happens if A is left as the highest bidder on that one item. Since A does not win all his k items, but does win at least one item, the outcome is limited by the sum expected realized synergy given $M-1$ items, which is less than that of M items.

Lowering A's highest bid however, results in the same situation as we started with, there is one bidder that is highest on some item. Therefore it can not be better that A does not win all his k items.

Given that A wins all his items, disregard bidder A for now. Out of the remaining bidders there is a highest bidder B. Since it is unknown which items each bidder bids on, the best the adversary can achieve is to let B be strictly higher than all the remaining bidders, the reasoning is the same as with bidder A. There is a probability that B will not win all his items, since he may collide with A, but this is equally true for all remaining bidders.

We now have that A is the strictly highest bidder (on all his items), B is the second strictly highest bidder. Applying the same arguments to the remaining bidders gives us a strictly ordered set of bidders. □

Theorem 4.1. *Consider a sealed-bid simultaneous auction with an arbitrary number of single-bidders each with per-item valuation at most* 1*, and an arbitrary number of synergy-bidders each bidding on k items chosen at random (uniformly) and realizing a synergy of αk when all k items are won. Given some number of items $M \geq 2k$ the sum of all bidders' expected realized valuations is less than*

$$M + \frac{\alpha k}{1 - \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}} \quad .$$

Proof. First, we observe that the total value that can be achieved by allocating the M items is at most M plus the total synergy realized by the synergy-bidders.

[6] If the adversary were to choose bids of identical value then chance would determine which bidder wins. If one outcome is better than the other then he can choose this outcome beforehand, if both outcomes are equally good he can arbitrarily choose one beforehand. We can therefore assume w.l.o.g. that the adversary when behaving optimally assigns unique bids.

Given two synergy-bidders, the probability that their combinations do not collide is

$$\prod_{i=0}^{k-1} \frac{M-k-i}{M-i} \quad .$$

Lemma 4.1 states that the expected sum of bidders' realized synergies is greatest when bidders are strictly ordered. Let bidders be strictly ordered and consider the j^{th} highest synergy-bidder. The probability that he will win all his k items, and realize his synergy, is the same as the probability that none of his bids will collide with any of the $j-1$ higher bids, which is

$$\left(\prod_{i=0}^{k-1} \frac{M-k-i}{M-i}\right)^{j-1} \quad .$$

Summing over all j's, the expected total synergy becomes a geometric series, which is less than

$$\frac{\alpha k}{1 - \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}}$$

and adding 1 per item completes the proof. □

Note that when $M < 2k$ only one bidder realizes his synergy and therefore the expected sum of all bidders' realized valuations is less than or equal to $\alpha k + M$.

Corollary 4.1. *Consider a sealed-bid simultaneous auction with an arbitrary number of single-bidders each with per-item valuation at most* 1*, and an arbitrary number of synergy-bidders each bidding on k items chosen at random (uniformly) and realizing a synergy of αk when all k items are won. Given some number of items $M \geq 2k$ the expected revenue is less than*

$$M + \frac{\alpha k}{1 - \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}} \quad .$$

Proof. A rational (payoff maximizing) bidder will never follow a strategy that gives a negative expected payoff, that is, he can not bid higher than his expected realized valuation. Therefore the sum of the expected winning bids is bounded by the sum of the expected realized valuations which is bounded by Theorem 4.1. □

Further, since the expected efficiency can never exceed the sum of the bidders' expected realized valuations, Theorem 4.1 is implicitly also an upper bound on the expected efficiency.

5 Combinatorial Auction

In this section we determine bounds on the lowest bid in the combinatorial auction, as well as the expected revenue when the number of bidders approach infinity.

Note that, in the combinatorial auction, as a simplification, we do not allow the synergy-bidder to place single-bids as in the simultaneous auction. Each synergy-bidder is thus limited to submitting one bid only, a combinatorial bid.

5.1 Strategies

Ideally we would derive the optimal strategies in equilibrium, however, how to do this is still an unresolved problem. Fortunately, as we show below, it is still possible to determine some basic properties of the bidders' strategies. These properties will in turn allow us to construct a lower bound on expected revenue.

Miltersen and Santillan [12] prove the existence of pure symmetric equilibrium in the first-price combinatorial auction in the same model as we study here, their proof is an extension of the work by Reny [14] and Athey [2].

Lemma 5.1 and Lemma 5.2 state that in pure symmetric equilibrium, the synergy-bidder's strategy is strictly increasing and Lemma 5.3 does the same for the participating single-bidders. We provide these proofs given that a symmetric pure equilibrium does exist. In the literature, it is often (implicitly or explicitly) assumed that strategies are monotone in order to prove symmetric equilibrium. In a combinatorial auction, monotonicity is less obvious and since it is an essential part of our analysis we therefore choose to prove this property, given the existence of pure symmetric equilibrium.

Lemma 5.1. *Consider the combinatorial auction and the model of Definition 3.1. In pure symmetric equilibrium, the following holds for a synergy-bidder's strategy* $\beta : \mathbb{R} \to \mathbb{R}$ *and valuations* v_1 *and* v_2*:* $v_1 < v_2 \Rightarrow \beta(v_1) \leq \beta(v_2)$*.*

Proof. Consider a valuation v and the corresponding optimal pure symmetric equilibrium bid $b = \beta(v)$, and some small constant $\epsilon > 0$. The expected payoff $\Pi(v, b)$ is described by

$$\Pi(v, b) = P(b)(kv + k\alpha - b) \tag{1}$$

where $P(b)$ is the strictly increasing probability of winning with a bid b. In equilibrium, no profitable deviation from b exists, this implies the following inequality

$$\Pi(v, b) \geq \Pi(v, b + \epsilon) \ . \tag{2}$$

We wish to show that for a lower valuation $(v - \delta)$, the bid will not be higher than b. In other words, we wish to show that, for $\delta > 0$, $\epsilon > 0$,

$$\Pi(v - \delta, b) > \Pi(v - \delta, b + \epsilon) \ .$$

We have

$$\begin{aligned}
\Pi(v - \delta, b) &= P(b)(k(v - \delta + \alpha) - b) \\
&= P(b)(kv + k\alpha - b) - P(b)k\delta \\
&\underset{(\text{eq.}2)}{\geq} P(b + \epsilon)(kv + k\alpha - (b + \epsilon)) - P(b)k\delta \\
&> P(b + \epsilon)(kv + k\alpha - (b + \epsilon)) - P(b + \epsilon)k\delta \\
&= \Pi(v - \delta, b + \epsilon)
\end{aligned}$$

where the last inequality follows from the fact that $P(b)$ is strictly increasing. □

Lemma 5.2. *Given the model in Definition 3.1, the pure symmetric equilibrium strategy for a synergy-bidder in the combinatorial auction is a strictly increasing function* $\beta : \mathbb{R} \to \mathbb{R}$*.*

Proof. Pick two arbitrary valuations v_1 and v_2 such that $v_1 < v_2$. Let the pure symmetric equilibrium bid given v_1 be $b = \beta(v_1)$. Assume the pure symmetric equilibrium bid given v_2 is also b. Lemma 5.1 states that all bidders with valuations in the interval $[v_1, v_2]$ will bid b, thus a plateau exists in the strategy. The probability that a bid b on some combination of items wins is $P(b)$. When bidding the plateau bid b,

$$P(b) = P_w(b) \cdot (1 - \delta)$$

where $P_w(b)$ is the probability that any bid on the same items with value b wins, and $0 < \delta < 1$ is some constant representing the probability of winning the lottery amongst all the b-bids on the same items. That is, on the plateau the probability $P(b)$ of winning is the probability of winning the lottery as well as being a part of the winning allocation. The expected payoff given value v_2 and bid b is thus

$$\Pi(v_2, b) = P_w(b) \cdot (1 - \delta) \cdot (kv_2 + k\alpha - b).$$

The probability of winning with a bid $(b + \epsilon)$, that is not on the plateau is $P(b + \epsilon) > P_w(b)$, and the expected payoff given $b + \epsilon$ is

$$\Pi(v_2, b + \epsilon) > P_w(b) \cdot (kv_2 + k\alpha - b - \epsilon).$$

If there exists an $\epsilon > 0$ such that $\Pi(v_2, b + \epsilon) > \Pi(v_2, b)$ a contradiction is reached. Such an ϵ exists since

$$P_w(b) \cdot (kv_2 + k\alpha - b - \epsilon) > P_w(b) \cdot (1 - \delta) \cdot (kv_2 + k\alpha - b)$$

$$\Leftrightarrow$$

$$kv_2 + k\alpha - b - (1 - \delta) \cdot (kv_2 + k\alpha - b) > \epsilon$$

$$\Leftrightarrow$$

$$\delta \cdot (kv_2 + k\alpha - b) > \epsilon$$

that is, $\epsilon > 0$, because $b \leq (kv_1 + k\alpha) < (kv_2 + k\alpha)$ since b is the pure symmetric equilibrium bid given v_1.

To conclude, there exists a small $\epsilon > 0$ such that bidding $b+\epsilon$ increases the expected payoff by avoiding the lottery. Therefore $b+\epsilon$ is a better bid than b which contradicts the assumption that b is the pure symmetric equilibrium bid given valuation v_2. Therefore $\beta(v_1) \neq \beta(v_2)$, and Lemma 5.1 gives $\beta(v_2) > \beta(v_1)$.

□

Lemma 5.3. *The pure symmetric equilibrium strategy for a single-bidder in the combinatorial auction is strictly increasing, given the model in Definition 3.1.*

Proof. The lemma follows from applying the same proofs as for Lemma 5.1, and Lemma 5.2 with $\alpha = 0$ and $k = 1$. □

5.2 Asymptotic Expected Revenue

In this section we show that as the number of bidders approach infinity, the expected revenue approaches the theoretical maximum.

Lemma 5.4. *In the first-price combinatorial auction as the number of synergy-bidders approach infinity, the lowest winning bid approaches the maximum combination value* $k \cdot (1+\alpha)$*.*

Proof. Given the number of items and some combination size, there is a fixed number C of possible combinations. Consider an arbitrary constant $\delta > 0$ and a synergy-bidder B with valuation $v_{max} - \frac{\delta}{2}$, where v_{max} is the maximum possible valuation for a combination. His expected payoff approaches zero as the number of synergy-bidders approach infinity, since bids are strictly increasing in the valuation (Lemma 5.2).

Assume that with probability $p > 0$ there exists a winning lowest bid $b < v_{max} - \delta$ for some constant $\delta > 0$. Now consider bidder B; with probability $p \cdot \frac{1}{C}$ the lowest winning bid is on the same combination as B, and thus B's expected payoff from bidding $v_{max} - \delta$ is

$$\frac{p\delta}{2C} \ .$$

Given any $p > 0$ and $\delta > 0$, the number of synergy-bidders can be chosen such that $\frac{p\delta}{2C}$ is greater than the expected payoff, which is a contradiction. Thus, the probability that the lowest winning bid is less than $v_{max} - \delta$, for any $\delta > 0$, approaches zero as the number of synergy-bidders approach infinity. In our specific model, $v_{max} = k \cdot (1 + \alpha)$. □

Theorem 5.1. *In the first-price combinatorial auction as the number of bidders approaches infinity, the expected revenue approaches:*

$$k \cdot (1+\alpha) \cdot \left\lfloor \frac{M}{k} \right\rfloor + (M \bmod k)$$

where M *is the number of items,* k *is the combination size and* α *is the synergy per item.*

Proof. When the number of synergy-bidders approaches infinity, the expected number of bids in the optimal allocation approaches the maximum coverage $\lfloor \frac{M}{k} \rfloor$. Lemma 5.4 states that the lowest winning combinatorial bid will approach the maximum value, $k \cdot (1 + \alpha)$.

As the number of single-bidders approaches infinity, we have that a winning single-bid approaches 1. This follows directly from simple modification of the proof of Lemma 5.4 with $k = 1$, $\alpha = 0$. The expected number of items that will be sold to single-bidders will approach $(M \bmod k)$ as the number of bidders approach infinity. □

Corollary 5.1. *As the number of bidders approach infinity, the expected revenue of the first-price combinatorial auction is greater than that of the simultaneous auction, given* $M \geq 2k$ *and* $k \geq 2$*.*

Proof. From Theorem 4.1 and Theorem 5.1 we have the bounds on the respective auctions. Substituting with $M = ck+b$ where $0 \leq b < k$ and $c > 1$ and $b, c \in \mathbb{N}$, the following inequality must hold for the combinatorial auction revenue to exceed the that of the simultaneous auction

$$k(1+\alpha) \left\lfloor \frac{ck+b}{k} \right\rfloor + \big((ck+b) \bmod k\big) \quad > \quad ck+b+\frac{\alpha k}{1-\prod_{i=0}^{k-1} \frac{ck+b-k-i}{ck+b-i}}$$

$$\Leftrightarrow$$

$$k(1+\alpha)c + \big((ck+b) \bmod k\big) \quad > \quad ck+b+\frac{\alpha k}{1-\prod_{i=0}^{k-1}\frac{ck+b-k-i}{ck+b-i}}$$

$$\Leftrightarrow$$

$$k(1+\alpha)c + b \quad > \quad ck+b+\frac{\alpha k}{1-\prod_{i=0}^{k-1}\frac{(c-1)k+b-i}{ck+b-i}}$$

$$\Leftrightarrow$$

$$\frac{c-1}{c} \quad > \quad \prod_{i=0}^{k-1}\frac{(c-1)k+b-i}{ck+b-i}\ .$$

Unfolding the product we get, in the general case,

$$\prod_{i=0}^{k-1}\frac{(c-1)k+b-i}{ck+b-i}$$

$$=$$

$$\frac{(c-1)k+b}{ck+b}\cdots\frac{(c-1)k+b-b}{ck+b-b}\cdots\frac{(c-1)k+b-k+1}{ck+b-k+1}$$

$$=$$

$$C\cdot\frac{c-1}{c}$$

where $0 < C < 1$. Therefore

$$\frac{c-1}{c} \quad > \quad \prod_{i=0}^{k-1}\frac{(c-1)k+b-i}{ck+b-i}$$

which concludes the proof. □

5.3 Parameterized Lower Bound on Expected Revenue

We have shown that the revenue of the combinatorial auction approaches the theoretical maximum as the number of bidders approaches infinity. However we would also like to have more specific information about when the expected revenue of the combinatorial auction exceeds that of the simultaneous auction.

In this section we provide a lower bound on the expected revenue for the combinatorial auction and show for some instances what the requirements are, in terms of the number of bidders, to achieve a higher expected revenue over the simultaneous auction.

In our analysis we assume that at most two combinatorial bids win, this covers all cases when $2k \leq M < 3k$ and it can also be used to derive some interesting bounds in other cases.

We show, for example, that in settings with $M > 3k$ items (see Tables 2, 3 and 4), an auctioneer that implements a combinatorial auction, even with the additional rule that at most two synergy-bidders may win, can expect higher revenue than in a simultaneous auction.

We proceed with a few lemmas that are needed for the proof of Theorem 5.2. First Lemma 5.5 establishes a lower bound on the winning single-bids. Then Lemma 5.6 deals with the probability of the existence of non-colliding bids. Next, Lemma 5.7 deals with the probability of winning, which is used to bound the payoff of a bidder in Lemma 5.8. Finally we have Lemma 5.9 which establishes a lower bound on the lowest winning combinatorial bid in the optimal solution.

Lemma 5.5. *Given the number of single-bidders N and an arbitrarily small constant $\Delta > 0$, then with probability q, a winning single-bid in the combinatorial auction is greater than or equal to:*

$$\left(\frac{1-q}{N}\right)^{\frac{1}{N-1}} - \frac{\left(\frac{1-q}{N}\right)^{\frac{N}{N-1}}}{1-q} - \Delta \ .$$

Proof. Given a single-bidder with valuation u. Since bids are strictly increasing (Lemma 5.3), the probability that he bids higher than all the other single-bidders on the same item is u^{N-1} and the probability that he wins is strictly smaller. Therefore this bidder's expected payoff is less than $u \cdot u^{N-1} = u^N$.

Assume that a winning single-bid is less than $u - \gamma$ with probability $1-q$. A bidder with valuation u bidding $u - \gamma$ would then have an expected revenue of $(1-q)\gamma$. If $(1-q)\gamma > u^N$, or equivalently, if

$$\gamma > \frac{u^N}{1-q} \tag{3}$$

a contradiction is reached and the winning single-bid is greater or equal to $u - \gamma$ with probability q.

We pick a γ satisfying Equation 3 by setting $\gamma = u^N/(1-q) + \Delta$, where $\Delta > 0$ is an arbitrarily small constant. Now, with probability q we have that the winning single-bid is at least $u - \gamma$ which is the same as

$$u - \frac{u^N}{1-q} - \Delta \ . \tag{4}$$

Maximizing Equation (4) as a function of u gives us the first-order condition $1 - N/(1-q) \cdot u^{N-1} = 0$. Note that the second derivative is always negative, so if we find a local optima it is guaranteed to be a local maximum. Solving the first-order condition for u we get:

$$u = \mathrm{e}^{\frac{\ln\left(\frac{1-q}{N}\right)}{N-1}} = \left(\frac{1-q}{N}\right)^{\frac{1}{N-1}} .$$

Therefore, given an arbitrarily small $\Delta > 0$, a winning single-bid is greater or equal to $u - \gamma$ with probability q where

$$u - \gamma = \left(\frac{1-q}{N}\right)^{\frac{1}{N-1}} - \frac{\left(\frac{1-q}{N}\right)^{\frac{N}{N-1}}}{1-q} - \Delta \ .$$

□

Lemma 5.6. *Consider the combinatorial auction and Definition 3.1. Given an arbitrary synergy-bidder B, and the remaining $N-1$ synergy-bidders. A synergy-bidder that does not collide with B, and has a per-item valuation greater than v exists with probability:*

$$1 - \left(1 - (1-v) \cdot \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}\right)^{N-1} .$$

Proof. Choose a random bidder C. Given B's valuation v, the probability that C's valuation is higher, is $1-v$. The probability that C does not collide with B is: $\prod_{i=0}^{k-1}\frac{M-k-i}{M-i}$. Thus, the probability that C has a higher valuation and that he does not collide with B is

$$(1-v)\cdot\prod_{i=0}^{k-1}\frac{M-k-i}{M-i}\ .$$

The complement, that C either collides or has a lower valuation is:

$$1-(1-v)\cdot\prod_{i=0}^{k-1}\frac{M-k-i}{M-i}$$

and the probability that this is the case for all of the remaining $N-1$ bidders is

$$\left(1-(1-v)\cdot\prod_{i=0}^{k-1}\frac{M-k-i}{M-i}\right)^{N-1}\ . \tag{5}$$

Finally, the probability that at least one bidder has a higher valuation and does not collide with B is the complement of Equation 5.

□

Lemma 5.7. *Given Definition 3.1 and the combinatorial auction, the probability that a bidder with per-item valuation less than or equal to v wins is less than or equal to:*

$$\left(1-(1-v)\cdot\prod_{i=0}^{k-1}\frac{M-k-i}{M-i}\right)^{N-1}\ .$$

Proof. If there exists a feasible solution containing the highest combinatorial bid and the bid of a non-colliding bidder with valuation greater than v then no bidder with valuation $\leq v$ can win. This follows from the monotonicity property proven in Lemma 5.2.

According to Lemma 5.6, such a feasible solution exists with probability

$$1-\left(1-(1-v)\cdot\prod_{i=0}^{k-1}\frac{M-k-i}{M-i}\right)^{N-1}\ .$$

Therefore, a bidder with valuation $\leq v$ can only win in some of the remaining cases.

□

Lemma 5.8. *Given Definition 3.1, in the combinatorial auction the expected payoff of a synergy-bidder with per-item value v is less than or equal to:*

$$k\cdot(v+\alpha)\cdot\left(1-(1-v)\cdot\prod_{i=0}^{k-1}\frac{M-k-i}{M-i}\right)^{N-1}\ .$$

Proof. Lemma 5.7 states that a synergy-bidder with valuation v wins with probability at most $\left(1-(1-v)\cdot\prod_{i=0}^{k-1}\frac{M-k-i}{M-i}\right)^{N-1}$. The maximum possible payoff is $k\cdot(v+\alpha)$ given a valuation of v and synergy α. □

Lemma 5.9. *In the combinatorial auction given Definition 3.1 the lowest combinatorial bid in the optimal solution, in pure symmetric equilibrium, is greater than* $k(v - \delta + \alpha)$ *with probability* p *given* δ *such that*

$$\delta > \frac{k \cdot (v+\alpha) \cdot \left(1 - (1-v) \cdot \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}\right)^{N-1}}{(1-p) \cdot \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}} .$$

Proof. Clearly we are only interested in $\delta < v + \alpha$ since a negative or zero-valued lower bound is not useful. Assume the lowest winning bid is less than $k \cdot (v - \delta + \alpha)$ with a probability of $(1-p)$. Given that this is true, then a bidder with valuation v, (total valuation $k \cdot (v+\alpha)$), can place a bid of $k \cdot (v - \delta + \alpha)$ and win with probability $(1-p) \cdot \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}$, where $\prod_{i=0}^{k-1} \frac{M-k-i}{M-i}$ is the probability that he does not collide with highest winning bid, and thus the bidder's expected payoff is:

$$\delta \cdot (1-p) \cdot \prod_{i=0}^{k-1} \frac{M-k-i}{M-i} .$$

If this is greater than the upper bound established in Lemma 5.8, we have a contradiction. That is, if

$$\delta > \frac{k \cdot (v+\alpha) \cdot \left(1 - (1-v) \cdot \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}\right)^{N-1}}{(1-p) \cdot \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}}$$

then we have a contradiction and the lowest winning bid is greater than $k(v - \delta + \alpha)$ with probability p. □

Theorem 5.2. *In a pure symmetric equilibrium of the first-price sealed-bid combinatorial auction given* N *single-bidders and* N *synergy-bidders, the expected revenue given the model in Definition 3.1 when at most two synergy-bids can win is at least*

$$p \cdot q \cdot \left(2k \cdot (v - \delta + \alpha) + (m - 2k) \cdot \underline{sb}\right)$$

where

$$\underline{sb} = \left(\frac{1-q}{N}\right)^{\frac{1}{N-1}} - \frac{\left(\frac{1-q}{N}\right)^{\frac{N}{N-1}}}{1-q} - \Delta$$

and

$$\delta = \frac{k \cdot (v+\alpha) \cdot \left(1 - (1-v) \cdot \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}\right)^{N-1}}{(1-p) \cdot \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}} + \Delta .$$

where p *is the probability that the lowest combinatorial bid is at least* $k \cdot (v - \delta + \alpha)$*;* q *is the probability that a winning single-bid is at least* $\underline{sb}$ *given an arbitrarily small* $\Delta > 0$.

Proof. Lemma 5.5 provides us with a lower bound, $\underline{sb}$, on the winning single-bid, with probability q. Lemma 5.9 gives us the lowest combinatorial bid in the optimal solution with probability p given a δ that satisfies:

$$\delta > \frac{k \cdot (v+\alpha) \cdot \left(1 - (1-v) \cdot \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}\right)^{N-1}}{(1-p) \cdot \prod_{i=0}^{k-1} \frac{M-k-i}{M-i}} .$$

Setting δ equal to the right hand expression and adding an arbitrarily small constant Δ gives us a δ which satisfies the constraint. Therefore with probability $p \cdot q$, we have a solution with two synergy-bids each of value at least $k \cdot (v - \delta + \alpha)$ and the remaining items are covered by single-bids of value at least $\underline{sb}$. There is of course a possibility that only one combinatorial bid wins, this happens with probability $(1 - p)$, and in this case we count the contribution to the expected revenue as zero. □

Since the expected revenue is a lower bound on the expected efficiency, Theorem 5.2 implicitly provides a lower bound on efficiency, and since Theorem 4.1 is an upper bound on efficiency, we can conclude that our results obtained regarding revenue also apply to efficiency.

6 Comparison – Simultaneous and Combinatorial Auction

In Tables 2, 3 and 4 we compare the bounds of Theorem 4.1 and Theorem 5.1. Each row in the table constitutes an example where the combinatorial auction yields a higher expected revenue than the simultaneous auction.

We would like to point out the following:

- In Table 3, when the number of items M is greater than 14, it is actually possible that more than two combinatorial bids win, however our analysis is based on the assumption that at most two bids can win.
- Our theoretical bounds are far from tight, therefore examples of when the expected revenue of the combinatorial auction is higher than that of the simultaneous auction are likely to be found for considerably smaller values of the number of bidders N.

Table 2: Example of the application of Theorem 5.2 – varying the problem size.

Items M	Combination Size k	No. Bidders[a] N	Synergy α	Combinatorial Auction Lower Bound[b]	Simultaneous Auction Upper Bound[c]	Verification parameters $q = 0.999$		
						p	v	$\underline{sb}$
8	3	197	1	11.658	11.653	0.987	0.686	0.934
11	4	303	1	15.476	15.475	0.985	0.637	0.955
14	5	502	1	19.341	19.336	0.985	0.607	0.972
17	6	857	1	23.240	23.233	0.985	0.589	0.983
20	7	1483	1	27.160	27.159	0.985	0.577	0.989
23	8	2584	1	31.108	31.107	0.986	0.568	0.993
26	9	4511	1	35.072	35.071	0.986	0.563	0.996
29	10	7889	1	39.047	39.047	0.987	0.558	0.997

[a] This is purely an existence proof, the number of bidders n is therefore an upper bound. [b] The combinatorial auction lower bound is rounded down. [c] The simultaneous auction upper bound is rounded up.

7 Conclusions

In this work, we have studied some fundamental properties of auctions in a complex valuation setting, with many items and bidders, synergies for random sets of items, and incomplete knowledge among bidders.

Table 3: Example of the application of Theorem 5.2, contd. – varying the number of items.

Items M	Combination Size k	No. Bidders[a] N	Synergy α	Combinatorial Auction Lower Bound[b]	Simultaneous Auction Upper Bound[c]	Verification parameters $q = 0.999$ p	v	$\underline{sb}$
10	5	8667	1.0	15.020	15.020	0.982	0.56	0.998
11	5	2479	1.0	16.066	16.066	0.982	0.569	0.993
12	5	1180	1.0	17.137	17.137	0.983	0.579	0.987
13	5	716	1.0	18.228	18.228	0.984	0.592	0.979
14	5	502	1.0	19.341	19.336	0.985	0.607	0.972
17	5	279	1.0	22.737	22.734	0.989	0.661	0.952
20	5	228	1.0	26.208	26.202	0.992	0.724	0.942
23	5	227	1.0	29.713	29.709	0.995	0.781	0.942
26	5	257	1.0	33.240	33.240	0.996	0.836	0.948
29	5	321	1.0	36.789	36.788	0.998	0.878	0.958
32	5	450	1.0	40.347	40.346	0.999	0.916	0.969
35	5	759	1.0	43.936	43.913	0.999	0.953	0.981
36	5	963	1.0	45.103	45.103	0.999	0.963	0.984
37	5	1294	1.0	46.294	46.294	0.999	0.973	0.988
38	5	1928	1.0	47.486	47.485	0.999	0.982	0.992
39	5	4048	1.0	48.691	48.677	0.999	0.991	0.996

[a] This is purely an existence proof, the number of bidders n is therefore an upper bound. [b] The combinatorial auction lower bound is rounded down. [c] The simultaneous auction upper bound is rounded up.

Table 4: Example of the application of Theorem 5.2, contd. – varying the synergy α.

Items M	Combination Size k	No. Bidders[a] N	Synergy α	Combinatorial Auction Lower Bound[b]	Simultaneous Auction Upper Bound[c]	Verification parameters $q = 0.999$ p	v	$\underline{sb}$
11	4	681	0.50	13.241	13.238	0.992	0.817	0.979
11	4	425	0.75	14.359	14.356	0.988	0.727	0.967
11	4	303	1.00	15.476	15.475	0.985	0.637	0.955
11	4	233	1.25	16.603	16.594	0.982	0.549	0.944
11	4	190	1.50	17.722	17.712	0.972	0.500	0.932
11	4	132	2.00	19.963	19.950	0.976	0.282	0.907
11	4	80	3.00	24.444	24.424	0.960	0.001	0.855
11	4	65	4.00	29.027	28.899	0.918	0.001	0.828
11	4	58	5.00	33.610	33.373	0.890	0.001	0.810
11	4	54	6.00	38.479	37.848	0.874	0.001	0.799
11	4	50	7.00	42.492	42.323	0.854	0.001	0.785
11	4	48	8.00	47.491	46.797	0.847	0.001	0.778

[a] This is purely an existence proof, the number of bidders n is therefore an upper bound. [b] The combinatorial auction lower bound is rounded down. [c] The simultaneous auction upper bound is rounded up.

Our analysis clearly indicates that the exposure problem is a real problem in the simultaneous auction, while the threshold problem for the combinatorial auction does not affect bidder's strategies as much. Furthermore, we derive the first theoretical support for the common belief, that combinatorial auctions yield higher expected revenues than simultaneous auctions.

Our analysis is in itself interesting as a contribution to auction theory. It should also serve as a theoretical motivation for employing combinatorial auctions in real life more frequently. We leave as an open problem to further tighten the upper and lower bounds presented here, and to provide even more detailed understanding of the fundamental relation between the expected revenues of simultaneous and combinatorial auctions.

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