Lusin theorem, GLT sequences and matrix computations: An application to the spectral analysis of PDE discretization matrices

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Abstract

We extend previous results on the spectral distribution of discretization matrices arising from B-spline Isogeometric Analysis (IgA) approximations of a general d-dimensional second-order elliptic Partial Differential Equation (PDE) with variable coefficients. First, we provide the spectral symbol of the Galerkin B-spline IgA stiffness matrices, assuming only that the PDE coefficients belong to L^∞. This symbol describes the asymptotic spectral distribution when the fineness parameters tend to zero (so that the matrix-size tends to infinity). Second, we prove the positive semi-definiteness of the $d \times d$ symmetric matrix in the Fourier variables $(\theta_1, \ldots, \theta_d)$, which appears in the expression of the symbol. This matrix is related to the discretization of the (negative) Hessian operator, and its positive semi-definiteness implies the non-negativity of the symbol. The mathematical arguments used in our derivation are based on the Lusin theorem, on the theory of Generalized Locally Toeplitz (GLT) sequences, and on careful Linear Algebra manipulations of matrix determinants. These arguments are very general and can be also applied to other PDE discretization techniques than B-spline IgA.

Keywords: Lusin theorem, Generalized Locally Toeplitz sequences, spectral distribution and symbol, matrix determinant, trace-norm, PDE discretization matrices, Isogeometric Analysis.

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1 Introduction

Consider the following second-order elliptic differential problem with variable coefficients and homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\nabla \cdot K\nabla u + \boldsymbol{\alpha} \cdot \nabla u + \gamma u = \mathrm{f}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^d$ is a bounded open domain with Lipschitz boundary, $K : \Omega \to \mathbb{R}^{d\times d}$ is a Symmetric Positive Definite (SPD) matrix of functions in $L^\infty(\Omega)$, $\boldsymbol{\alpha} : \Omega \to \mathbb{R}^d$ is a vector of functions in $L^\infty(\Omega)$, $\gamma \in L^\infty(\Omega)$, $\gamma \geq 0$ and $\mathrm{f} \in L^2(\Omega)$. Any discretization of the given differential problem for some sequence of stepsizes h tending to zero leads to a sequence of linear systems $A_h\mathbf{x} = \mathbf{b}$, where the size of A_h tends to ∞ for $h \to 0$. For the numerical solution of such linear systems, it is important to understand the spectral properties of the matrices A_h. The spectral distribution of a sequence of matrices is a relevant concept. Roughly speaking, if the sequence of matrices $\{A_h\}$ is distributed like the function f, then the eigenvalues of A_h behave like a sampling of f over an equispaced grid on the domain of f. The function f is called the (spectral) symbol of the sequence.

The spectral distribution of a sequence of discretization matrices arising from B-spline Isogeometric Analysis (IgA) approximations of (1.1) was investigated in a string of recent papers. In particular, Galerkin B-spline

IgA discretizations were addressed in [10, 11, 12]. The symbol of the corresponding stiffness matrices was computed and studied in [10, 11] in the simplified case where K is the identity matrix and $\Omega = (0,1)^d$. Afterwards, in [12], the spectral study was generalized to Partial Differential Equations (PDEs) whose coefficients are continuous functions and to an arbitrary domain Ω using a geometry map $\mathbf{G} : [0,1]^d \to \overline{\Omega}$. A similar spectral study was performed in [9] for a sequence of matrices in the IgA collocation context. In addition, the information contained in the symbol was successfully exploited in the design/analysis of optimal multigrid methods for the numerical solution of the linear systems involved in IgA discretizations [7, 8].

In this paper, we extend the above mentioned results related to the symbol in the following two ways.

- In Section 2, we provide the complete symbol f describing the spectral distribution of a sequence of stiffness matrices related to Galerkin B-spline IgA approximations of (1.1). We only require that the PDE coefficients belong to L^∞. Hence, this result eliminates the continuity hypothesis in [12, Theorem 4.1] and provides a positive answer to the question raised in [12, Remark 4.4]. The argument used in our derivation is based on two main tools:
  - the Lusin theorem [16], to approximate a measurable function by a continuous function;
  - the theory of Generalized Locally Toeplitz (GLT) sequences [19, 20], which stems from Tilli's work [22] and from the theory of classical Toeplitz operators [2].
- In Section 3, we prove the positive semi-definiteness of the $d \times d$ symmetric matrix $H_{\boldsymbol{p}}(\boldsymbol{\theta})$ in the Fourier variables $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_d)$, which appears in the expression of the symbol f. As discussed in [12, Section 4.2], the matrix $H_{\boldsymbol{p}}(\boldsymbol{\theta})$ is related to the discretization of the (negative) Hessian operator, and is sometimes referred to as 'the symbol of the (negative) Hessian operator'. The positive semi-definiteness of $H_{\boldsymbol{p}}(\boldsymbol{\theta})$ implies the non-negativity of f and is a crucial property for the spectral distribution of discretization matrices. This result confirms the conjecture formulated in [12, Remark 5.3]. Moreover, we show that it straightforwardly extends to the IgA collocation case.

The above results show an application of abstract theorems from Functional Analysis and Matrix Theory to the spectral analysis of PDE discretization matrices. It is worth emphasizing that the purpose of this paper is not just to extend the results of [12] (and also [9, 10]). Another, even more important, target is to illustrate some techniques whose applicability is not confined to the particular framework considered herein.

The arguments used in Section 2 are very general, and can be also applied to other PDE discretization techniques than B-spline IgA. For example, the Lusin theorem was already applied in the context of Finite Difference discretizations of PDEs; see [18, Theorem 6.3].

The matrix $H_{\boldsymbol{p}}(\boldsymbol{\theta})$, analyzed in Section 3, is a constitutive part of the symbol describing the asymptotic spectrum of the Galerkin B-spline IgA discretization matrices. If another discretization technique is used, then another symbol is obtained for the resulting discretization matrices. However, the formal structure of the symbol remains the same, as explained, e.g., in [12, p. 2]. In particular, independently of the used (local) approximation technique (Finite Differences [15, 21], Finite Elements [3, 4], Isogeometric Analysis [5], etc.), the corresponding symbol always contains a version of the matrix $H_{\boldsymbol{p}}(\boldsymbol{\theta})$ in its expression. For Finite Differences, see [19, Eqs. (8) and (11)], where the counterpart of $H_{\boldsymbol{p}}(\boldsymbol{\theta})$ is written as $P(s)$; for linear Finite Elements, see [1, Eq. (9)], where the counterpart of $H_{\boldsymbol{p}}(\boldsymbol{\theta})$ shows up implicitly, in dyadic form; and for a general discussion, see [19, Section 2] and [20, Sections 3.1.2 and 3.1.3].

The framework and the notation of this paper are the same as in [12]. Therefore, we keep the presentation as concise as possible and for a better understanding, the reader is recommended to first read the work [12]. In any case, throughout this paper, we will provide the necessary pointers to [12].

2 Lusin theorem and GLT sequences

The main focus of this paper is the spectral analysis of the stiffness matrices coming from Galerkin B-spline IgA approximations of (1.1). We start with defining the concept of spectral distribution for a given sequence of matrices. Let μ_d be the Lebesgue measure in $\mathbb{R}^d$ and let $C_c(\mathbb{C})$ be the space of continuous functions $F : \mathbb{C} \to \mathbb{C}$ with bounded support. If $X \in \mathbb{C}^{m\times m}$, the singular values and the eigenvalues of X are denoted by $\sigma_j(X)$ and $\lambda_j(X)$, respectively, for $j = 1, \ldots, m$.

Definition 2.1. Let $\{X_n\}_n$ be a sequence of matrices, with X_n of size d_n tending to infinity, and let $f : D \to \mathbb{C}$ be a measurable function defined on a measurable set $D \subset \mathbb{R}^d$, with $0 < \mu_d(D) < \infty$. We say that $\{X_n\}_n$ is distributed like f in the sense of the singular values and we write $\{X_n\}_n \sim_\sigma f$, if

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F(\sigma_j(X_n)) = \frac{1}{\mu_d(D)} \int_D F(|f(x_1, \ldots, x_d)|)\, \mathrm{d}x_1 \cdots \mathrm{d}x_d, \quad \forall F \in C_c(\mathbb{C}).$$

In this case, f is referred to as the singular value symbol of $\{X_n\}_n$. Similarly, we say that $\{X_n\}_n$ is distributed like f in the sense of the eigenvalues and we write $\{X_n\}_n \sim_\lambda f$, if

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F(\lambda_j(X_n)) = \frac{1}{\mu_d(D)} \int_D F(f(x_1, \ldots, x_d))\, \mathrm{d}x_1 \cdots \mathrm{d}x_d, \quad \forall F \in C_c(\mathbb{C}).$$

In this case, f is referred to as the eigenvalue symbol (spectral symbol, or just symbol) of $\{X_n\}_n$.

2.1 Problem setting

The Galerkin B-spline IgA approximation of (1.1), described in [12, Section 3], leads to the following (stiffness) matrix

$$A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}} = \Bigg[\int_{[0,1]^d} \big((\nabla N_{\boldsymbol{j}+\mathbf{1},[\boldsymbol{p}]})^T K_{\mathbf{G}} \nabla N_{\boldsymbol{i}+\mathbf{1},[\boldsymbol{p}]} + (\nabla N_{\boldsymbol{j}+\mathbf{1},[\boldsymbol{p}]})^T (J_{\mathbf{G}})^{-1} \boldsymbol{\alpha}(\mathbf{G})\, N_{\boldsymbol{i}+\mathbf{1},[\boldsymbol{p}]} \\ + \gamma(\mathbf{G}) N_{\boldsymbol{j}+\mathbf{1},[\boldsymbol{p}]} N_{\boldsymbol{i}+\mathbf{1},[\boldsymbol{p}]}\big) |\det(J_{\mathbf{G}})|\Bigg]_{\boldsymbol{i},\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{n}+\boldsymbol{p}-\mathbf{2}},$$

where

- $\boldsymbol{p} := (p_1, \ldots, p_d)$ and $\boldsymbol{n} := (n_1, \ldots, n_d)$ are multi-indices in $\mathbb{N}^d$; moreover, $p_i \geq 1$ for all $i = 1, \ldots, d$;
- the indices $\boldsymbol{i}, \boldsymbol{j}$ vary from $\mathbf{1}$ to $\boldsymbol{n}+\boldsymbol{p}-\mathbf{2}$ following the standard lexicographic ordering; see [12, Eq. (2.1)]. In particular, the size of the matrix $A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}$ is $N(\boldsymbol{n}+\boldsymbol{p}-\mathbf{2})$ for which we recall from the multi-index notation described in [12, Section 2.1] that $N(\boldsymbol{m}) := m_1 \cdots m_d$ for any $\boldsymbol{m} := (m_1, \ldots, m_d) \in \mathbb{N}^d$;
- $\mathbf{G} : [0,1]^d \to \overline{\Omega}$ is the geometry map, which is invertible and satisfies $\mathbf{G}(\partial([0,1]^d)) = \partial\overline{\Omega}$; moreover, we assume that $\mathbf{G}$ is regular as in [12, Theorem 4.1];
- $J_{\mathbf{G}}$ is the Jacobian matrix of $\mathbf{G}$ and $K_{\mathbf{G}} = (J_{\mathbf{G}})^{-1} K(\mathbf{G}) (J_{\mathbf{G}})^{-T}$;
- $N_{\boldsymbol{i},[\boldsymbol{p}]} := N_{i_1,[p_1]} \otimes \cdots \otimes N_{i_d,[p_d]} : [0,1]^d \to \mathbb{R}$, $\boldsymbol{i} = \mathbf{2}, \ldots, \boldsymbol{n}+\boldsymbol{p}-\mathbf{1}$, are tensor-product B-splines. If $a_r : E_r \to \mathbb{C}$, $r = 1, \ldots, m$, then the tensor-product function $a_1 \otimes \cdots \otimes a_m : E_1 \times \cdots \times E_m \to \mathbb{C}$ is defined by

  $$(a_1 \otimes \cdots \otimes a_m)(\xi_1, \ldots, \xi_m) = a_1(\xi_1) \cdots a_m(\xi_m), \qquad (\xi_1, \ldots, \xi_m) \in E_1 \times \cdots \times E_m.$$

  The functions $N_{i_k,p_k} : [0,1] \to \mathbb{R}$, $i_k = 2, \ldots, n_k + p_k - 1$, are the B-splines of degree p_k defined on the uniform knot sequence

  $$\{\underbrace{0, \ldots, 0}_{p_k+1}, 1/n_k, 2/n_k, \ldots, \underbrace{1, \ldots, 1}_{p_k+1}\};$$

  for a precise definition we refer the reader to [12, Section 3.2].

The matrix $A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}$ can be decomposed as follows:

$$A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}} = K^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}} + R^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}},$$

where

$$K^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}} = \left[\int_{[0,1]^d} (\nabla N_{\boldsymbol{j}+\mathbf{1},[\boldsymbol{p}]})^T K_{\mathbf{G}} |\det(J_{\mathbf{G}})|\, \nabla N_{\boldsymbol{i}+\mathbf{1},[\boldsymbol{p}]}\right]_{\boldsymbol{i},\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{n}+\boldsymbol{p}-\mathbf{2}} \tag{2.1}$$

is the matrix resulting from the discretization of the diffusive term in (1.1), and

$$R^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}} = \left[\int_{[0,1]^d} \left((\nabla N_{\boldsymbol{j+1},[\boldsymbol{p}]})^T (J_{\mathbf{G}})^{-1} \boldsymbol{\alpha}(\mathbf{G})\, N_{\boldsymbol{i+1},[\boldsymbol{p}]} + \gamma(\mathbf{G}) N_{\boldsymbol{j+1},[\boldsymbol{p}]} N_{\boldsymbol{i+1},[\boldsymbol{p}]} \right) |\det(J_{\mathbf{G}})| \right]_{\boldsymbol{i},\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}+\boldsymbol{p}-\boldsymbol{2}} \tag{2.2}$$

is the matrix resulting from the discretization of the terms in (1.1) with lower order derivatives. The matrix $R^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}$ can be regarded as a 'residual term', because its norm is negligible with respect to the norm of the diffusion matrix $K^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}$ when the discretization parameters $\boldsymbol{n}$ are large; see [12, Section 4.2 (Step 1) and Section 4.3].

Let $\mathbb{Q}^d_+ := \{\boldsymbol{r} := (r_1,\ldots,r_d) \in \mathbb{Q}^d : r_i > 0,\ \forall i = 1,\ldots,d\}$ and fix a vector $\boldsymbol{\nu} := (\nu_1,\ldots,\nu_d) \in \mathbb{Q}^d_+$. From now on, we assume that $n_i = \nu_i n$ for all $i = 1,\ldots,d$, i.e., $\boldsymbol{n} = \boldsymbol{\nu} n$. Moreover, we assume that the discretization parameter n varies in the set of indices such that $\boldsymbol{\nu} n \in \mathbb{N}^d$. In order to describe the (asymptotic) spectral distribution of the sequence of normalized matrices $\{n^{d-2} A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n$, the $d \times d$ symmetric matrix $H_{\boldsymbol{p}} = H_{p_1,\ldots,p_d}$ given by

$$(H_{\boldsymbol{p}})_{ij} := \begin{cases} \left(\bigotimes_{r=1}^{i-1} h_{p_r}\right) \otimes f_{p_i} \otimes \left(\bigotimes_{r=i+1}^{d} h_{p_r}\right), & \text{if } i = j, \\ \left(\bigotimes_{r=1}^{i-1} h_{p_r}\right) \otimes g_{p_i} \otimes \left(\bigotimes_{r=i+1}^{j-1} h_{p_r}\right) \otimes g_{p_j} \otimes \left(\bigotimes_{r=j+1}^{d} h_{p_r}\right), & \text{if } i < j, \\ \left(\bigotimes_{r=1}^{j-1} h_{p_r}\right) \otimes g_{p_j} \otimes \left(\bigotimes_{r=j+1}^{i-1} h_{p_r}\right) \otimes g_{p_i} \otimes \left(\bigotimes_{r=i+1}^{d} h_{p_r}\right), & \text{if } i > j, \end{cases} \tag{2.3}$$

plays an important role. In the expression (2.3), the functions $h_p,\, g_p,\, f_p : [-\pi,\pi] \to \mathbb{R}$ are defined for all $p \geq 1$ by

$$h_p(\theta) := \phi_{[2p+1]}(p+1) + 2\sum_{k=1}^{p} \phi_{[2p+1]}(p+1-k)\cos(k\theta), \tag{2.4}$$

$$g_p(\theta) := -2\sum_{k=1}^{p} \dot{\phi}_{[2p+1]}\,(p+1-k)\sin(k\theta), \tag{2.5}$$

$$f_p(\theta) := -\ddot{\phi}_{[2p+1]}(p+1) - 2\sum_{k=1}^{p} \ddot{\phi}_{[2p+1]}(p+1-k)\cos(k\theta), \tag{2.6}$$

and $\phi_{[p]}$, $\dot{\phi}_{[p]}$, $\ddot{\phi}_{[p]}$ are, respectively, the cardinal B-spline of degree p (see, e.g., in [11, Eqs. (7)–(8)]), its first derivative and its second derivative.

We are now ready to formulate our main result concerning the (asymptotic) spectral distribution of the sequence of normalized matrices $\{n^{d-2} A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n$. We first recall from [12, Theorem 4.1] the following statement.

Theorem 2.1. *Suppose that the components of K are continuous over $\overline{\Omega}$. Then we have $\{n^{d-2} A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n \sim_\sigma f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$ and $\{n^{d-2} A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n \sim_\lambda f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$, where*

$$f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}} : [0,1]^d \times [-\pi,\pi]^d \to \mathbb{R}, \quad f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}(\hat{\mathbf{x}},\boldsymbol{\theta}) := \frac{\boldsymbol{\nu}\left(|\det(J_{\mathbf{G}}(\hat{\mathbf{x}}))|\, K_{\mathbf{G}}(\hat{\mathbf{x}}) \circ H_{\boldsymbol{p}}(\boldsymbol{\theta})\right) \boldsymbol{\nu}^T}{N(\boldsymbol{\nu})}. \tag{2.7}$$

In the expression (2.7), $\circ$ denotes the componentwise (Hadamard) product of matrices, and $H_{\boldsymbol{p}}$ is the $d \times d$ symmetric matrix of continuous functions defined in (2.3).

However, the continuity hypothesis in Theorem 2.1 is not necessary for the spectral result, and in this paper we show that the theorem also holds when the components of K belong to $L^\infty(\Omega)$, as assumed in the formulation of problem (1.1). This provides a positive answer to the question raised in [12, Remark 4.4]. Hence, the previous theorem can be stated in a stronger form (without any restriction on the components of K except the $L^\infty(\Omega)$ condition, intrinsic to the differential problem) as follows.

Theorem 2.2. *We have $\{n^{d-2} A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n \sim_\sigma f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$ and $\{n^{d-2} A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n \sim_\lambda f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$, where $f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$ is defined in (2.7).*

The proof of Theorem 2.2 is given in Section 2.2. Its main ingredients are the Lusin theorem and the theory of GLT sequences. The latter ingredient was already exploited in the proof of Theorem 2.1; see [12, Theorem 4.1 and Section 4].

We remark that Theorem 2.2 holds even without the assumption that K is positive definite, because this assumption is never used in the proof. Only the symmetry of K is needed. However, the positive definiteness of the diffusion matrix K is necessary for the ellipticity of the differential problem (1.1). A similar statement was already given in [12, Remark 4.2] for Theorem 2.1.

2.2 Proof of Theorem 2.2

Let $\|X\|$ be the spectral (Euclidean) norm of the matrix X, i.e., the maximum singular value of X. We first recall a basic inequality about the trace-norm of a matrix, which is denoted by $\|\cdot\|_1$. The result is proved, e.g., in [14, Appendix A], but for the reader's convenience we include a short and direct proof.

Lemma 2.1. *For any matrix $X \in \mathbb{C}^{m\times m}$, $\|X\|_1 \le \sum_{i,j=1}^m |X_{ij}|$.*

Proof. Let $X = U\Sigma V^*$ be a singular value decomposition of X. Then, the matrix $Q := VU^*$ is unitary and we have

$$\begin{aligned}\|X\|_1 &= \mathrm{trace}(\Sigma) = \mathrm{trace}(U^*XV) = \mathrm{trace}(XQ)\\ &\le \sum_{i=1}^m\sum_{j=1}^m |X_{ij}Q_{ji}| \le \sum_{i=1}^m \max_{j=1,\ldots,m}|Q_{ji}|\sum_{j=1}^m |X_{ij}| \le \sum_{i=1}^m\sum_{j=1}^m |X_{ij}|.\end{aligned}$$

□

The norm $\|X\|_{[1]} := \sum_{i,j=1}^m |X_{ij}|$ is sometimes referred to as the componentwise ℓ^1 norm.

Lemma 2.2. *Let $N_{i,[p]}$, $i = 2, \ldots, n+p-1$, be the B-splines of degree p on the uniform knot sequence*

$$\{t_1, \ldots, t_{n+2p+1}\} = \{\underbrace{0, \ldots, 0}_{p+1}, 1/n, 2/n, \ldots, \underbrace{1, \ldots, 1}_{p+1}\}. \tag{2.8}$$

Then, there exist constants A_p, B_p independent of n such that

$$\sum_{i=2}^{n+p-1} |N_{i,[p]}| \le A_p, \qquad \sum_{i=2}^{n+p-1} |N'_{i,[p]}| \le B_p n.$$

Proof. The first inequality holds with $A_p = 1$ by the positivity and the partition of unity properties of B-splines [6]. To prove the second inequality, we recall from the differentiation properties of B-splines [6] that

$$N'_{i,[p]} = p\left(\frac{N_{i,[p-1]}}{t_{i+p}-t_i} - \frac{N_{i+1,[p-1]}}{t_{i+p+1}-t_{i+1}}\right).$$

Since $t_{i+p} - t_i \ge \frac{1}{n}$ for all $i = 2, \ldots, n+p-1$, by the positivity and partition of unity we get

$$\sum_{i=2}^{n+p-1} |N'_{i,[p]}| \le \frac{p}{n}\sum_{i=2}^{n+p-1}\big(N_{i,[p-1]} + N_{i+1,[p-1]}\big) = 2pn,$$

and the second inequality holds with $B_p = 2p$. □

We now introduce a linear operator $\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(\cdot)$, which is defined on the space $[L^1([0,1]^d)]^{d\times d}$ consisting of the functions $L : [0,1]^d \to \mathbb{R}^{d\times d}$ whose components L_{ij} belong to $L^1([0,1]^d)$; see also [12, Section 4.2 (Step 2)]. This operator is defined as follows

$$\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(\cdot) : [L^1([0,1]^d)]^{d\times d} \to \mathbb{R}^{N(\boldsymbol{n}+\boldsymbol{p}-\boldsymbol{2})\times N(\boldsymbol{n}+\boldsymbol{p}-\boldsymbol{2})}, \quad \mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(L) := \left[\int_{[0,1]^d} (\nabla N_{\boldsymbol{j}+\boldsymbol{1},[\boldsymbol{p}]})^T L\, \nabla N_{\boldsymbol{i}+\boldsymbol{1},[\boldsymbol{p}]}\right]_{\boldsymbol{i},\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}+\boldsymbol{p}-\boldsymbol{2}}. \tag{2.9}$$

Theorem 2.4 highlights a fundamental 'continuity property' of the operator $\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}$. For its proof, we first recall a slightly simplified version of the Lusin theorem [16, Theorem 2.24]. In the following, $C_c(\mathbb{R}^d)$ is the space of continuous functions $F : \mathbb{R}^d \to \mathbb{C}$ with bounded support.

Theorem 2.3 (Lusin). *Let $g : \mathbb{R}^d \to \mathbb{C}$ be any measurable function such that $g = 0$ outside a set A with $\mu_d(A) < \infty$. Then, for any $\epsilon > 0$ there exists a function $g_\epsilon \in C_c(\mathbb{R}^d)$ such that*

$$\sup_{\mathbf{x}\in\mathbb{R}^d} |g_\epsilon(\mathbf{x})| \le \operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{R}^d} |g(\mathbf{x})|$$

and

$$\mu_d(\{g \ne g_\epsilon\}) < \epsilon,$$

where $\{g \ne g_\epsilon\} = \{\mathbf{x} \in \mathbb{R}^d : \ g(\mathbf{x}) \ne g_\epsilon(\mathbf{x})\}$.

Following the notation in [12], we denote by E_{st} the $d\times d$ matrix having 1 in position (s,t) and 0 elsewhere.

Theorem 2.4. *Let $a\in L^\infty([0,1]^d)$ and $\epsilon>0$. Then, there exists $a_\epsilon\in C([0,1]^d)$, depending only on a and ϵ, such that*

$$\|n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a(\hat{\mathbf{x}})E_{st})-n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a_\epsilon(\hat{\mathbf{x}})E_{st})\|_1\le c\,N(\boldsymbol{n}+\boldsymbol{p}-\mathbf{2})\,\epsilon,$$

for all $s,t=1,\ldots,d$, where c is a constant depending only on a, $\boldsymbol{p}$ and $\boldsymbol{\nu}$.

Proof. By the Lusin theorem, there exists a function $a_\epsilon\in C([0,1]^d)$ such that $\|a_\epsilon\|_{L^\infty([0,1]^d)}\le\|a\|_{L^\infty([0,1]^d)}$ and $\mu_d(\{a\ne a_\epsilon\})\le\epsilon$, where $\{a\ne a_\epsilon\}=\{\hat{\mathbf{x}}\in[0,1]^d:\ a(\hat{\mathbf{x}})\ne a_\epsilon(\hat{\mathbf{x}})\}$. Therefore, for every $\boldsymbol{i},\boldsymbol{j}=\mathbf{2},\ldots,\boldsymbol{n}+\boldsymbol{p}-\mathbf{1}$,

$$\begin{aligned}|(\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a(\hat{\mathbf{x}})E_{st})-\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a_\epsilon(\hat{\mathbf{x}})E_{st}))_{\boldsymbol{i}-\mathbf{1},\boldsymbol{j}-\mathbf{1}}|&=|(\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}((a(\hat{\mathbf{x}})-a_\epsilon(\hat{\mathbf{x}}))E_{st}))_{\boldsymbol{i}-\mathbf{1},\boldsymbol{j}-\mathbf{1}}|\\&=\left|\int_{[0,1]^d}[a(\hat{\mathbf{x}})-a_\epsilon(\hat{\mathbf{x}})]\,\frac{\partial N_{\boldsymbol{j},[\boldsymbol{p}]}}{\partial\hat{x}_s}\frac{\partial N_{\boldsymbol{i},[\boldsymbol{p}]}}{\partial\hat{x}_t}\right|\\&\le C\int_{\{a\ne a_\epsilon\}}\left|\frac{\partial N_{\boldsymbol{j},[\boldsymbol{p}]}}{\partial\hat{x}_s}\right|\left|\frac{\partial N_{\boldsymbol{i},[\boldsymbol{p}]}}{\partial\hat{x}_t}\right|,\end{aligned}$$

where $C:=2\|a\|_{L^\infty([0,1]^d)}$ is a constant depending only on a. Recalling Lemma 2.1 and summing over all multi-indices $\boldsymbol{i},\boldsymbol{j}=\mathbf{2},\ldots,\boldsymbol{n}+\boldsymbol{p}-\mathbf{1}$, we get

$$\begin{aligned}\|\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a(\hat{\mathbf{x}})E_{st})-\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a_\epsilon(\hat{\mathbf{x}})E_{st})\|_1&\le C\sum_{\boldsymbol{i},\boldsymbol{j}=\mathbf{2}}^{\boldsymbol{n}+\boldsymbol{p}-\mathbf{1}}\int_{\{a\ne a_\epsilon\}}\left|\frac{\partial N_{\boldsymbol{j},[\boldsymbol{p}]}}{\partial\hat{x}_s}\right|\left|\frac{\partial N_{\boldsymbol{i},[\boldsymbol{p}]}}{\partial\hat{x}_t}\right|\\&=C\int_{\{a\ne a_\epsilon\}}\sum_{\boldsymbol{j}=\mathbf{2}}^{\boldsymbol{n}+\boldsymbol{p}-\mathbf{1}}\left|\frac{\partial N_{\boldsymbol{j},[\boldsymbol{p}]}}{\partial\hat{x}_s}\right|\sum_{\boldsymbol{i}=\mathbf{2}}^{\boldsymbol{n}+\boldsymbol{p}-\mathbf{1}}\left|\frac{\partial N_{\boldsymbol{i},[\boldsymbol{p}]}}{\partial\hat{x}_t}\right|.\end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}\sum_{\boldsymbol{j}=\mathbf{2}}^{\boldsymbol{n}+\boldsymbol{p}-\mathbf{1}}\left|\frac{\partial N_{\boldsymbol{j},[\boldsymbol{p}]}}{\partial\hat{x}_s}\right|&=\sum_{\boldsymbol{j}=\mathbf{2}}^{\boldsymbol{n}+\boldsymbol{p}-\mathbf{1}}|N_{j_1,[p_1]}|\otimes\cdots\otimes|N_{j_{s-1},[p_{s-1}]}|\otimes|N'_{j_s,[p_s]}|\otimes|N_{j_{s+1},[p_{s+1}]}|\otimes\cdots\otimes|N_{j_d,[p_d]}|\\&\le A_{p_1}\cdots A_{p_{s-1}}B_{p_s}A_{p_{s+1}}\cdots A_{p_d}n_s\le Dn_s,\end{aligned}$$

where $D:=\max_{r=1,\ldots,d}(A_{p_1}\cdots A_{p_{r-1}}B_{p_r}A_{p_{r+1}}\cdots A_{p_d})$ is a constant depending only on $\boldsymbol{p}$. Similarly,

$$\sum_{\boldsymbol{i}=\mathbf{2}}^{\boldsymbol{n}+\boldsymbol{p}-\mathbf{1}}\left|\frac{\partial N_{\boldsymbol{i},[\boldsymbol{p}]}}{\partial\hat{x}_t}\right|\le Dn_t.$$

Thus, recalling $\boldsymbol{n}=\boldsymbol{\nu}n$, we obtain

$$\|\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a(\hat{\mathbf{x}})E_{st})-\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a_\epsilon(\hat{\mathbf{x}})E_{st})\|_1\le CD^2n_sn_t\,\mu_d(\{a\ne a_\epsilon\})\le CD^2\nu_s\nu_tn^2\epsilon,$$

and so there exists a constant c, depending only on a, $\boldsymbol{p}$ and $\boldsymbol{\nu}$, such that

$$\|n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a(\hat{\mathbf{x}})E_{st})-n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a_\epsilon(\hat{\mathbf{x}})E_{st})\|_1\le CD^2\nu_s\nu_tn^d\epsilon\le c\,N(\boldsymbol{n}+\boldsymbol{p}-\mathbf{2})\,\epsilon.$$

□

In the proof of Theorem 2.4, we used the generic constants A_p,B_p instead of the specific values $A_p=1$ and $B_p=2p$ in order to emphasize the generality of the argument. Actually, the proof of Theorem 2.4 can be applied – without any modification – to every basis satisfying the inequalities in Lemma 2.2 for some A_p and B_p independent of n. Many bases used in practical applications satisfy these inequalities, which mainly depend on the fact that the basis functions are locally supported. We recall from [6] that the B-splines $N_{i,[p]}$, $i=2,\ldots,n+p-1$, are indeed locally supported, since $\mathrm{supp}(N_{i,[p]})=[t_i,t_{i+p+1}]$ with the knots $t_1,\ldots,t_{n+2p+1}$ given in (2.8).

We now provide an auxiliary result in the next lemma. Its proof is identical to the one of [13, Lemma 4].

Lemma 2.3. *Let $Z_{n,m}$ be a matrix of size d_n and assume that, for every m and every $n \geq n_m$,*

$$\|Z_{n,m}\|_1 \leq \varepsilon(m) d_n,$$

where $\varepsilon(m)$, n_m depend only on m. Then, for every m and every $n \geq n_m$,

$$Z_{n,m} = R_{n,m} + S_{n,m},$$

where $\mathrm{rank}(R_{n,m}) \leq \sqrt{\varepsilon(m)}\, d_n$ *and* $\|S_{n,m}\| \leq \sqrt{\varepsilon(m)}$.

Using Theorem 2.4 and Lemma 2.3, we show in Lemma 2.4 that, for any $a \in L^\infty([0,1]^d)$ and any sequence $\{\epsilon(m)\}_m$ such that $\epsilon(m) \to 0$, the sequence $\big\{\{n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a_{\epsilon(m)}(\hat{\mathbf{x}})E_{st})\}_n\big\}_m$, with a_ϵ as in Theorem 2.4, is an approximating class of sequences (a.c.s.) for $\{n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a(\hat{\mathbf{x}})E_{st})\}_n$. We recall in the next definition the concept of a.c.s. (see [17, Definition 2.1] or [13, Definition 1]).

Definition 2.2 (approximating class of sequences). Let $\{X_n\}_n$ be a sequence of matrices, with X_n of size d_n tending to infinity. An approximating class of sequences (a.c.s.) for $\{X_n\}_n$ is a sequence of sequences of matrices $\{\{B_{n,m}\}_n\}_m$ with the following property: for every m there exists n_m such that, for all $n \geq n_m$,

$$X_n = B_{n,m} + R_{n,m} + S_{n,m},$$

where

$$\mathrm{rank}(R_{n,m}) \leq \varrho(m) d_n, \qquad \|S_{n,m}\| \leq \omega(m),$$

and the quantities n_m, $\varrho(m)$, $\omega(m)$ depend only on m, and

$$\lim_{m\to\infty} \varrho(m) = \lim_{m\to\infty} \omega(m) = 0.$$

Roughly speaking, $\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for $\{X_n\}_n$ when X_n is equal to $B_{n,m}$ plus a small-rank matrix (with respect to the matrix size d_n) plus a small-norm matrix.

Lemma 2.4. *Let $a \in L^\infty([0,1]^d)$ and, for any $\epsilon > 0$, let a_ϵ be the function specified by Theorem 2.4. Take any sequence of positive numbers $\{\epsilon(m)\}_m$ such that $\epsilon(m) \to 0$ as $m \to \infty$. Then, $\big\{\{n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a_{\epsilon(m)}(\hat{\mathbf{x}})E_{st})\}_n\big\}_m$ is an a.c.s. for $\{n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a(\hat{\mathbf{x}})E_{st})\}_n$, for all $s,t = 1,\ldots,d$.*

Proof. By Theorem 2.4, for every n, m we have

$$\|n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a(\hat{\mathbf{x}})E_{st}) - n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a_{\epsilon(m)}(\hat{\mathbf{x}})E_{st})\|_1 \leq \varepsilon(m) N(\boldsymbol{n}+\boldsymbol{p}-\mathbf{2}),$$

with $\varepsilon(m) = c\,\epsilon(m)$ and c a constant depending only on a, $\boldsymbol{p}$ and $\boldsymbol{\nu}$. Note that $N(\boldsymbol{n}+\boldsymbol{p}-\mathbf{2})$ is the size of the matrix $n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a(\hat{\mathbf{x}})E_{st})$. Using Lemma 2.3, we see that, for every n, m,

$$n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a(\hat{\mathbf{x}})E_{st}) = n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a_{\epsilon(m)}(\hat{\mathbf{x}})E_{st}) + R_{n,m} + S_{n,m},$$

where $\mathrm{rank}(R_{n,m}) \leq \sqrt{\varepsilon(m)}\, N(\boldsymbol{n}+\boldsymbol{p}-\mathbf{2})$ and $\|S_{n,m}\| \leq \sqrt{\varepsilon(m)}$. Since $\varepsilon(m) \to 0$, it follows that $\big\{\{n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a_{\epsilon(m)}(\hat{\mathbf{x}})E_{st})\}_n\big\}_m$ is an a.c.s. for $\{n^{d-2}\mathscr{L}_{\boldsymbol{n}}^{[\boldsymbol{p}]}(a(\hat{\mathbf{x}})E_{st})\}_n$. □

We are now ready to show that $\{n^{d-2}A_{\mathbf{G},\boldsymbol{n}}^{[\boldsymbol{p}]}\}_n$ is a GLT sequence with respect to the symbol $f_{\mathbf{G},\boldsymbol{p}}^{(\boldsymbol{\nu})}$ in (2.7). Despite the importance of sLT and GLT sequences in the following, we do not give the corresponding definitions, since they are rather difficult; we refer the reader to [19, Definition 2.1] for the definition of sLT sequences and to [19, Definition 2.3] for the definition of GLT sequences. After showing that $\{n^{d-2}A_{\mathbf{G},\boldsymbol{n}}^{[\boldsymbol{p}]}\}_n \sim_{\mathrm{GLT}} f_{\mathbf{G},\boldsymbol{p}}^{(\boldsymbol{\nu})}$, we will prove Theorem 2.2.

Let

$$\widehat{K} := K_{\mathbf{G}} |\det(J_{\mathbf{G}})|.$$

In this notation, the symbol $f_{\mathbf{G},\boldsymbol{p}}^{(\boldsymbol{\nu})}$ in (2.7) can be written as

$$f_{\mathbf{G},\boldsymbol{p}}^{(\boldsymbol{\nu})}(\hat{\mathbf{x}},\boldsymbol{\theta}) = \sum_{s,t=1}^{d} \widehat{K}_{st}(\hat{\mathbf{x}}) \frac{\nu_s\nu_t}{N(\boldsymbol{\nu})} (H_{\boldsymbol{p}}(\boldsymbol{\theta}))_{st} = \sum_{s,t=1}^{d} \left(\widehat{K}_{st} \otimes \frac{\nu_s\nu_t}{N(\boldsymbol{\nu})}(H_{\boldsymbol{p}})_{st}\right)(\hat{\mathbf{x}},\boldsymbol{\theta}),$$

where $\widehat{K}_{st}$ is the component of $\widehat{K}$ in position (s,t). We denote by $\widehat{K}_{st,\epsilon}$ the continuous function a_ϵ specified by Theorem 2.4 for $a = \widehat{K}_{st}$. Note that the components of $\widehat{K}$ belong to $L^\infty([0,1]^d)$, because the components of K are assumed to be in $L^\infty(\Omega)$.

Theorem 2.5. *It holds that*

$$\{n^{d-2}A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n \sim_{\mathrm{GLT}} f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}.$$

Moreover, the GLT sequence $\{n^{d-2}A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n$ *has the following properties:*

- *the weight functions* $a_{i,\epsilon}$, $i=1,\dots,N_\epsilon$, $\epsilon>0$, *in the GLT definition [19, Definition 2.3] can be chosen as the functions* $\widehat{K}_{st,\epsilon}$, $s,t=1,\dots,d$, $\epsilon>0$*; hence, we have* d^2 *weight functions for each* $\epsilon>0$*, which are also continuous over* $[0,1]^d$*;*
- *the generating functions* $f_{i,\epsilon}$, $i=1,\dots,N_\epsilon$, $\epsilon>0$, *in the GLT definition can be chosen as the functions* $\frac{\nu_s\nu_t}{N(\boldsymbol{\nu})}(H_{\boldsymbol{p}})_{st}$, $s,t=1,\dots,d$*; hence, we have* d^2 *generating functions for each* $\epsilon>0$*, which, moreover, do not depend on* ϵ*;*
- *the sLT sequences* $\{A_n^{(i,\epsilon)}\}_n$, $i=1,\dots,N_\epsilon$, $\epsilon>0$, *in the GLT definition can be chosen as the sequences* $\{n^{d-2}\mathscr{L}^{[\boldsymbol{p}]}_{\boldsymbol{n}}(\widehat{K}_{st,\epsilon}(\hat{\mathbf{x}})E_{st})\}_n$, $s,t=1,\dots,d$, $\epsilon>0$.

Proof. It is clear that we can decompose $\widehat{K}$ as follows:

$$\widehat{K}=\sum_{s,t=1}^{d}\widehat{K}_{st}E_{st}.$$

Therefore, by the definitions of $K^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}$ and $\mathscr{L}^{[\boldsymbol{p}]}_{\boldsymbol{n}}$, see (2.1) and (2.9), and by the linearity of $\mathscr{L}^{[\boldsymbol{p}]}_{\boldsymbol{n}}$, we obtain

$$n^{d-2}K^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}=n^{d-2}\mathscr{L}^{[\boldsymbol{p}]}_{\boldsymbol{n}}(\widehat{K})=\sum_{s,t=1}^{d}n^{d-2}\mathscr{L}^{[\boldsymbol{p}]}_{\boldsymbol{n}}(\widehat{K}_{st}E_{st}). \tag{2.10}$$

For every $s,t=1,\dots,d$, we know that $\widehat{K}_{st,\epsilon}\to\widehat{K}_{st}$ in measure over $[0,1]^d$ when $\epsilon\to0$. This implies that $\widehat{K}_{st,\epsilon}\otimes\frac{\nu_s\nu_t}{N(\boldsymbol{\nu})}(H_{\boldsymbol{p}})_{st}\to\widehat{K}_{st}\otimes\frac{\nu_s\nu_t}{N(\boldsymbol{\nu})}(H_{\boldsymbol{p}})_{st}$ in measure over $[0,1]^d\times[-\pi,\pi]^d$. Thus,

$$\sum_{s,t=1}^{d}\widehat{K}_{st,\epsilon}\otimes\frac{\nu_s\nu_t}{N(\boldsymbol{\nu})}(H_{\boldsymbol{p}})_{st}\to\sum_{s,t=1}^{d}\widehat{K}_{st}\otimes\frac{\nu_s\nu_t}{N(\boldsymbol{\nu})}(H_{\boldsymbol{p}})_{st}=f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$$

in measure over $[0,1]^d\times[-\pi,\pi]^d$. We also note that $\frac{\nu_s\nu_t}{N(\boldsymbol{\nu})}(H_{\boldsymbol{p}})_{st}$ is a separable trigonometric polynomial; see [12, Eq. (4.1)].

Since $\widehat{K}_{st,\epsilon}$ is continuous, it follows from [12, Section 4.2 (Step 4)] that

$$\{n^{d-2}\mathscr{L}^{[\boldsymbol{p}]}_{\boldsymbol{n}}(\widehat{K}_{st,\epsilon}(\hat{\mathbf{x}})E_{st})\}_n\sim_{\mathrm{sLT}}\Big(\widehat{K}_{st,\epsilon},\frac{\nu_s\nu_t}{N(\boldsymbol{\nu})}(H_{\boldsymbol{p}})_{st}\Big).$$

Finally,

$$\left\{\left\{\sum_{s,t=1}^{d}n^{d-2}\mathscr{L}^{[\boldsymbol{p}]}_{\boldsymbol{n}}(\widehat{K}_{st,\epsilon(m)}(\hat{\mathbf{x}})E_{st})\right\}_n\right\}_m,\qquad \epsilon(m)=\frac{1}{m+1}, \tag{2.11}$$

is an a.c.s. for $\{n^{d-2}A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n$. To see this, we first note that

$$\big\{\{n^{d-2}\mathscr{L}^{[\boldsymbol{p}]}_{\boldsymbol{n}}(\widehat{K}_{st,\epsilon(m)}(\hat{\mathbf{x}})E_{st})\}_n\big\}_m,\qquad \epsilon(m)=\frac{1}{m+1}, \tag{2.12}$$

is an a.c.s. for $\{n^{d-2}\mathscr{L}^{[\boldsymbol{p}]}_{\boldsymbol{n}}(\widehat{K}_{st}(\hat{\mathbf{x}})E_{st})\}_n$, by Lemma 2.4. Moreover, it follows directly from Definition 2.2 that $\{\{B_{n,m}+B'_{n,m}\}_n\}_m$ is an a.c.s. for $\{X_n+X'_n\}_n$ whenever $\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for $\{X_n\}_n$ and $\{\{B'_{n,m}\}_n\}_m$ is an a.c.s. for $\{X'_n\}_n$. Therefore, the sum of the a.c.s. in (2.12), that is (2.11), is an a.c.s. for the sum $\big\{\sum_{s,t=1}^{d}n^{d-2}\mathscr{L}^{[\boldsymbol{p}]}_{\boldsymbol{n}}(\widehat{K}_{st}(\hat{\mathbf{x}})E_{st})\big\}_n=\{n^{d-2}K^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n$ (see (2.10)). To conclude the proof, we note that the difference between $n^{d-2}A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}$ and $n^{d-2}K^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}$ is the matrix $n^{d-2}R^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}$ in (2.2), whose spectral norm tends to 0 when $n\to\infty$; see [12, Section 4.3]. Hence, (2.11) is an a.c.s. also for $\{n^{d-2}A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n$, by Definition 2.2.

The thesis now follows from the definition of GLT sequences [19, Definition 2.3]. □

The singular value distribution $\{n^{d-2}A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n \sim_\sigma f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$ in Theorem 2.2 is obtained as a direct consequence of Theorem 2.5 and [19, Theorem 4.5]. In the case where the matrices $A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}$ are symmetric (this happens when $\boldsymbol{\alpha} = \mathbf{0}$), the eigenvalue distribution $\{n^{d-2}A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n \sim_\lambda f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$ in Theorem 2.2 is obtained as a direct consequence of Theorem 2.5 and [19, Theorem 4.8]. In the case where the matrices $A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}$ are not symmetric, the eigenvalue distribution $\{n^{d-2}A^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n \sim_\lambda f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$ is proved in the same way as in [12, Section 4.2 (Step 6)]. This completes the proof of Theorem 2.2.

3 Matrix computations

The $d \times d$ matrix $H_{\boldsymbol{p}}$ defined in (2.3) can be referred to as the 'symbol of the (negative) Hessian operator', because it is related to the Galerkin B-spline IgA discretization of the (negative) Hessian operator. It was conjectured in [12, Remark 5.3] that $H_{\boldsymbol{p}}$ satisfies the properties stated in Theorem 3.1. In the following, we use the abbreviation SPSD for Symmetric Positive Semi-Definite.

Theorem 3.1. *For any $d \geq 1$ and any $\boldsymbol{p} := (p_1, \ldots, p_d) \in \mathbb{N}^d$ such that $p_i \geq 1$ for all $i = 1, \ldots, d$, the $d \times d$ matrix $H_{\boldsymbol{p}}(\boldsymbol{\theta})$ defined in* (2.3) *is SPSD for all $\boldsymbol{\theta} \in [-\pi, \pi]^d$ and SPD for all $\boldsymbol{\theta} \in [-\pi, \pi]^d$ such that $\theta_1 \cdots \theta_d \neq 0$.*

This conjecture was confirmed in [12, Theorem 5.2] for $d = 1, 2, 3$. Here, by using careful manipulations of matrix determinants, we prove the conjecture for any $d \geq 1$. The details of the proof are given in Section 3.1.

An important consequence of Theorem 3.1 is that the properties of the symbol (2.7) provided in [12, Theorem 5.4] are always satisfied, for any $d \geq 1$. In particular, $f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$ is non-negative almost everywhere (a.e.) on its domain $[0, 1]^d \times [-\pi, \pi]^d$.

3.1 Proof of Theorem 3.1

Before going into the details of the proof, let us make a few remarks. First, since the matrix $H_{\boldsymbol{p}}$ is symmetric by definition, we only have to prove the statements in Theorem 3.1 concerning the positive (semi-)definiteness of $H_{\boldsymbol{p}}$. Second, by using the same argument as in [12, Remark 5.3], we see that Theorem 3.1 is equivalent to the following theorem. For the reader's convenience we include a short proof.

Theorem 3.2. *For any $d \geq 1$ and any $\boldsymbol{p} := (p_1, \ldots, p_d) \in \mathbb{N}^d$ such that $p_i \geq 1$ for all $i = 1, \ldots, d$, the determinant of the $d \times d$ matrix $H_{\boldsymbol{p}}(\boldsymbol{\theta})$ defined in* (2.3) *is non-negative for all $\boldsymbol{\theta} \in [-\pi, \pi]^d$ and positive for all $\boldsymbol{\theta} \in [-\pi, \pi]^d$ such that $\theta_1 \cdots \theta_d \neq 0$.*

Proof. To prove the positive definiteness of $H_{\boldsymbol{p}}$, it suffices to prove that the determinant of $H_{\boldsymbol{p}}$ is positive for any d. Indeed, from (2.3) we see that the upper-left $k \times k$ submatrix of $H_{\boldsymbol{p}} = H_{p_1,\ldots,p_d}$ can be written as

$$\big[(H_{p_1,\ldots,p_d})_{i,j}\big]_{i,j=1}^k = \big[(H_{p_1,\ldots,p_k})_{i,j} \otimes \big(\textstyle\bigotimes_{r=k+1}^d h_{p_r}\big)\big]_{i,j=1}^k,$$

and so

$$\det\big(\big[(H_{p_1,\ldots,p_d})_{i,j}\big]_{i,j=1}^k\big) = \det(H_{p_1,\ldots,p_k}) \otimes \big(\textstyle\bigotimes_{r=k+1}^d (h_{p_r})^k\big).$$

From [11, Lemma 7] we know that $h_p > 0$ over $[-\pi, \pi]$. Hence, the k-th leading principal minor of $H_{\boldsymbol{p}}$ is positive if $\det(H_{p_1,\ldots,p_k}) > 0$, and $H_{\boldsymbol{p}}$ is positive definite if

$$\det(H_{p_1,\ldots,p_k}) > 0, \quad k = 1, \ldots, d.$$

The fact that $H_{\boldsymbol{p}}$ is SPD follows by induction from Sylvester's criterion. A continuity argument can be used to show that $H_{\boldsymbol{p}}$ is SPSD over $[-\pi, \pi]^d$. $\square$

In the next theorem we provide an explicit expression for the determinant of $H_{\boldsymbol{p}}$, where

$$e_p : [-\pi, \pi] \to \mathbb{R}, \qquad e_p(\theta) := f_p(\theta)h_p(\theta) - (g_p(\theta))^2.$$

Theorem 3.3. *For any $d \geq 1$, any $\boldsymbol{p} := (p_1, \ldots, p_d) \in \mathbb{N}^d$ such that $p_i \geq 1$ for all $i = 1, \ldots, d$, and any $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_d) \in [-\pi, \pi]^d$, we have*

$$\det(H_{\boldsymbol{p}}(\boldsymbol{\theta})) = \big(h_{p_1}(\theta_1) \cdots h_{p_d}(\theta_d)\big)^{d-2} \bigg(e_{p_1}(\theta_1) \cdots e_{p_d}(\theta_d) + \sum_{\ell=1}^{d} e_{p_1}(\theta_1) \cdots e_{p_{\ell-1}}(\theta_{\ell-1}) \big(g_{p_\ell}(\theta_\ell)\big)^2 e_{p_{\ell+1}}(\theta_{\ell+1}) \cdots e_{p_d}(\theta_d) \bigg). \tag{3.1}$$

Proof. We prove (3.1) by induction on d. The base case ($1 \leq d \leq 2$) can be proved by direct computation, recalling that $H_{\boldsymbol{p}}(\boldsymbol{\theta}) = f_p(\theta)$ for $d = 1$, and

$$H_{\boldsymbol{p}}(\boldsymbol{\theta}) = \begin{bmatrix} f_{p_1}(\theta_1) h_{p_2}(\theta_2) & g_{p_1}(\theta_1) g_{p_2}(\theta_2) \\ g_{p_1}(\theta_1) g_{p_2}(\theta_2) & h_{p_1}(\theta_1) f_{p_2}(\theta_2) \end{bmatrix}$$

for $d = 2$.

For the general case $d \geq 3$, we simplify a little bit the notation and we use h_i, g_i, f_i, e_i instead of $h_{p_i}(\theta_i)$, $g_{p_i}(\theta_i)$, $f_{p_i}(\theta_i)$, $e_{p_i}(\theta_i)$. In this notation, we have

$$(H_{\boldsymbol{p}})_{ij} = \begin{cases} h_1 \cdots h_{i-1}\, f_i\, h_{i+1} \cdots h_d, & \text{if } i = j, \\ h_1 \cdots h_{i-1}\, g_i\, h_{i+1} \cdots h_{j-1}\, g_j\, h_{j+1} \cdots h_d, & \text{if } i < j, \\ h_1 \cdots h_{j-1}\, g_j\, h_{j+1} \cdots h_{i-1}\, g_i\, h_{i+1} \cdots h_d, & \text{if } i > j, \end{cases}$$

and the formula (3.1) becomes

$$\det(H_{\boldsymbol{p}}) = (h_1 \cdots h_d)^{d-2} \bigg(e_1 \cdots e_d + \sum_{\ell=1}^{d} e_1 \cdots e_{\ell-1} (g_\ell)^2 e_{\ell+1} \cdots e_d \bigg). \tag{3.2}$$

Now, let us assume $\theta_2 \neq 0$ (so that $g_2 \neq 0$). We construct the matrix $H_{\boldsymbol{p}}^{(2)}$ starting from $H_{\boldsymbol{p}}$ and replacing the first row with a linear combination of the first two rows and, subsequently, the first column with a linear combination of the first two columns, namely

$$\begin{aligned} (H_{\boldsymbol{p}}^{(1)})_{1,\ell} &:= (H_{\boldsymbol{p}})_{1,\ell} - (H_{\boldsymbol{p}})_{2,\ell} \frac{g_1 h_2}{h_1 g_2}, \qquad \ell = 1, \ldots, d, \\ (H_{\boldsymbol{p}}^{(2)})_{\ell,1} &:= (H_{\boldsymbol{p}}^{(1)})_{\ell,1} - (H_{\boldsymbol{p}}^{(1)})_{\ell,2} \frac{g_1 h_2}{h_1 g_2}, \qquad \ell = 1, \ldots, d. \end{aligned}$$

This results in the entries

$$\begin{aligned} (H_{\boldsymbol{p}}^{(2)})_{1,1} &= \frac{1}{h_1} \left(e_1 + \frac{(g_1)^2 e_2}{(g_2)^2} \right) h_2 \cdots h_d, \\ (H_{\boldsymbol{p}}^{(2)})_{1,2} &= (H_{\boldsymbol{p}}^{(2)})_{2,1} = -\frac{g_1 e_2}{g_2} h_3 \cdots h_d, \\ (H_{\boldsymbol{p}}^{(2)})_{1,\ell} &= (H_{\boldsymbol{p}}^{(2)})_{\ell,1} = 0, \qquad \ell = 3, \ldots, d. \end{aligned}$$

Moreover, it is easy to check that

$$\begin{aligned} [H_{\boldsymbol{p}}^{(2)}]_{i,j=2}^{d} &= [H_{\boldsymbol{p}}]_{i,j=2}^{d} = h_1 H_{p_2,\ldots,p_d}, \\ [H_{\boldsymbol{p}}^{(2)}]_{i,j=3}^{d} &= [H_{\boldsymbol{p}}]_{i,j=3}^{d} = h_1 h_2 H_{p_3,\ldots,p_d}. \end{aligned}$$

Therefore, after two consecutive applications of Laplace's formula for the computation of the determinant, we get

$$\begin{aligned} \det(H_{\boldsymbol{p}}) &= \det(H_{\boldsymbol{p}}^{(2)}) \\ &= \frac{1}{h_1} \left(e_1 + \frac{(g_1)^2 e_2}{(g_2)^2} \right) h_2 \cdots h_d \cdot \det([H_{\boldsymbol{p}}]_{i,j=2}^{d}) - \frac{(g_1)^2 (e_2)^2}{(g_2)^2} (h_3 \cdots h_d)^2 \cdot \det([H_{\boldsymbol{p}}]_{i,j=3}^{d}) \\ &= \frac{1}{h_1} \left(e_1 + \frac{(g_1)^2 e_2}{(g_2)^2} \right) h_2 \cdots h_d \cdot (h_1)^{d-1} \det(H_{p_2,\ldots,p_d}) \\ &\quad - \frac{(g_1)^2 (e_2)^2}{(g_2)^2} (h_3 \cdots h_d)^2 \cdot (h_1 h_2)^{d-2} \det(H_{p_3,\ldots,p_d}). \end{aligned} \tag{3.3}$$

By the induction hypothesis, we have

$$\det(H_{p_2,\ldots,p_d}) = (h_2\cdots h_d)^{d-3}\bigg(e_2\cdots e_d + \sum_{\ell=2}^{d} e_2\cdots e_{\ell-1}(g_\ell)^2 e_{\ell+1}\cdots e_d\bigg),$$
$$\det(H_{p_3,\ldots,p_d}) = (h_3\cdots h_d)^{d-4}\bigg(e_3\cdots e_d + \sum_{\ell=3}^{d} e_3\cdots e_{\ell-1}(g_\ell)^2 e_{\ell+1}\cdots e_d\bigg).$$

After substituting these formulas in (3.3) and performing some manipulations, we obtain (3.2), and so (3.1). We have thus proved (3.1) for all $\boldsymbol{\theta} \in [-\pi,\pi]^d$ such that $\theta_2 \neq 0$, but a continuity argument shows that (3.1) also holds when $\theta_2 = 0$. $\square$

From the formula (3.1) provided in Theorem 3.3, it is easy to show that Theorem 3.2 holds, implying that the equivalent Theorem 3.1 holds as well. To see that (3.1) implies Theorem 3.2, it suffices to recall from [9, Lemmas 3.4, 3.6 and 3.7] the following results: [1]

- $h_p(\theta) > 0$ for all $\theta \in [-\pi,\pi]$;
- $f_p(\theta) \geq 0$ for all $\theta \in [-\pi,\pi]$, $f_p(0) = 0$ and $\theta = 0$ is the unique zero of f_p over $[-\pi,\pi]$;
- $e_p(\theta) \geq 0$ for all $\theta \in [-\pi,\pi]$, $e_p(0) = 0$ and $\theta = 0$ is the unique zero of e_p over $[-\pi,\pi]$.

3.2 An extension to IgA collocation

We now show that Theorem 3.1 can be easily extended to the IgA collocation case addressed in [9] and [10, Section 5].

Let us consider the following three functions $\tilde{h}_p,\ \tilde{g}_p,\ \tilde{f}_p : [-\pi,\pi] \to \mathbb{R}$ for all $p \geq 2$,

$$\tilde{h}_p(\theta) := \phi_{[p]}\left(\frac{p+1}{2}\right) + 2\sum_{k=1}^{\lfloor p/2\rfloor} \phi_{[p]}\left(\frac{p+1}{2} - k\right)\cos(k\theta), \tag{3.4}$$

$$\tilde{g}_p(\theta) := -2\sum_{k=1}^{\lfloor p/2\rfloor} \dot{\phi}_{[p]}\left(\frac{p+1}{2} - k\right)\sin(k\theta), \tag{3.5}$$

$$\tilde{f}_p(\theta) := -\ddot{\phi}_{[p]}\left(\frac{p+1}{2}\right) - 2\sum_{k=1}^{\lfloor p/2\rfloor} \ddot{\phi}_{[p]}\left(\frac{p+1}{2} - k\right)\cos(k\theta). \tag{3.6}$$

Let $\boldsymbol{p} := (p_1,\ldots,p_d)$ be a multi-index in $\mathbb{N}^d$ such that $p_i \geq 2$ for all $i = 1,\ldots,d$, and let $\widetilde{H}_{\boldsymbol{p}} = \widetilde{H}_{p_1,\ldots,p_d}$ be the symmetric $d\times d$ matrix defined by

$$(\widetilde{H}_{\boldsymbol{p}})_{ij} := \begin{cases} \left(\bigotimes_{r=1}^{i-1}\tilde{h}_{p_r}\right)\otimes \tilde{f}_{p_i}\otimes\left(\bigotimes_{r=i+1}^{d}\tilde{h}_{p_r}\right), & \text{if } i=j,\\ \left(\bigotimes_{r=1}^{i-1}\tilde{h}_{p_r}\right)\otimes \tilde{g}_{p_i}\otimes\left(\bigotimes_{r=i+1}^{j-1}\tilde{h}_{p_r}\right)\otimes \tilde{g}_{p_j}\otimes\left(\bigotimes_{r=j+1}^{d}\tilde{h}_{p_r}\right), & \text{if } i<j,\\ \left(\bigotimes_{r=1}^{j-1}\tilde{h}_{p_r}\right)\otimes \tilde{g}_{p_j}\otimes\left(\bigotimes_{r=j+1}^{i-1}\tilde{h}_{p_r}\right)\otimes \tilde{g}_{p_i}\otimes\left(\bigotimes_{r=i+1}^{d}\tilde{h}_{p_r}\right), & \text{if } i>j. \end{cases} \tag{3.7}$$

In the case where all the PDE coefficients K, $\boldsymbol{\alpha}$, γ are continuous, we can approximate the elliptic problem (1.1) by means of collocation methods. The isogeometric B-spline collocation approximation of (1.1), described in [9] and [10, Section 5.1], leads to a collocation matrix $\widetilde{A}^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}$. From the analysis in [9] and [10, Section 5], we know that the spectral symbol of the sequence of normalized matrices $\{\frac{1}{n^2}\widetilde{A}^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n$, with $\boldsymbol{n} = \boldsymbol{\nu} n$, is given by

$$\tilde{f}^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}} : [0,1]^d\times[-\pi,\pi]^d \to \mathbb{R},\quad \tilde{f}^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}(\hat{\mathbf{x}},\boldsymbol{\theta}) := \boldsymbol{\nu}(K_{\mathbf{G}}(\hat{\mathbf{x}})\circ \widetilde{H}_{\boldsymbol{p}}(\boldsymbol{\theta}))\boldsymbol{\nu}^T. \tag{3.8}$$

In other words, $\{\frac{1}{n^2}\widetilde{A}^{[\boldsymbol{p}]}_{\mathbf{G},\boldsymbol{n}}\}_n \sim_\lambda \tilde{f}^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$, in the sense of Definition 2.1. By comparing (3.8) with (2.7), we see that $\tilde{f}^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$ is obtained from $f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$ by removing both the determinant factor $|\det(J_{\mathbf{G}})|$ and the denominator $N(\boldsymbol{\nu})$, and by replacing $H_{\boldsymbol{p}}$ with $\widetilde{H}_{\boldsymbol{p}}$.

[1] The functions h_p, g_p, f_p in (2.4)–(2.6) coincide with the functions $\tilde{h}_{2p+1}$, $\tilde{g}_{2p+1}$, $\tilde{f}_{2p+1}$ in (3.4)–(3.6). They are also equal to the functions h_{2p+1}, g_{2p+1}, f_{2p+1} defined in [9, Eqs. (3.7)–(3.9)], which have been deeply studied in [9].

The functions $\tilde{h}_p$, $\tilde{g}_p$, $\tilde{f}_p$ enjoy the same properties as the functions h_p, g_p, f_p; see [9, Section 3] and Footnote 1. Therefore, by using the same line of arguments as in the proof of Theorem 3.1, we deduce that $\widetilde{H}_{\boldsymbol{p}}$ satisfies the following properties.

Theorem 3.4. *For any $d \geq 1$ and any $\boldsymbol{p} := (p_1, \ldots, p_d) \in \mathbb{N}^d$ such that $p_i \geq 2$ for all $i = 1, \ldots, d$, the $d \times d$ matrix $\widetilde{H}_{\boldsymbol{p}}(\boldsymbol{\theta})$ defined in (3.7) is SPSD for all $\boldsymbol{\theta} \in [-\pi, \pi]^d$ and SPD for all $\boldsymbol{\theta} \in [-\pi, \pi]^d$ such that $\theta_1 \cdots \theta_d \neq 0$.*

Note that Theorem 3.4 is an extension of Theorem 3.1. This is due to the fact that the matrix $H_{\boldsymbol{p}}$ coincides with $\widetilde{H}_{\boldsymbol{q}}$ for $\boldsymbol{q} = 2\boldsymbol{p} + \mathbf{1} = (2p_1 + 1, \ldots, 2p_d + 1)$.

Using Theorem 3.4 and following the proof of [12, Theorem 5.4], one can show that $\tilde{f}^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$ possesses properties completely analogous to the ones of $f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$. More precisely,

1. $\tilde{f}^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$ is non-negative on its domain $[0,1]^d \times [-\pi, \pi]^d$;
2. For every $(\hat{\mathbf{x}}, \boldsymbol{\theta}) \in [0,1]^d \times [-\pi, \pi]^d$, we have
$$\tilde{f}^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}(\hat{\mathbf{x}}, \boldsymbol{\theta}) \geq \lambda_{\min}(K_{\mathbf{G}}(\hat{\mathbf{x}}))\tilde{f}^{(\boldsymbol{\nu})}_{\boldsymbol{p}}(\boldsymbol{\theta}),$$
$$\tilde{f}^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}(\hat{\mathbf{x}}, \boldsymbol{\theta}) \leq \lambda_{\max}(K_{\mathbf{G}}(\hat{\mathbf{x}}))\tilde{f}^{(\boldsymbol{\nu})}_{\boldsymbol{p}}(\boldsymbol{\theta}),$$
where
$$\tilde{f}^{(\boldsymbol{\nu})}_{\boldsymbol{p}} := \boldsymbol{\nu}(I \circ \widetilde{H}_{\boldsymbol{p}})\boldsymbol{\nu}^T = \sum_{i=1}^{d} \nu_i^2 \left(\bigotimes_{r=1}^{i-1} h_{p_r}\right) \otimes f_{p_i} \otimes \left(\bigotimes_{r=i+1}^{d} h_{p_r}\right);$$
and as a consequence, if
$$cI \leq K_{\mathbf{G}} \leq CI \quad \text{in } [0,1]^d, \quad \text{for some } c, C > 0,$$
then
$$c\,\tilde{f}^{(\boldsymbol{\nu})}_{\boldsymbol{p}}(\boldsymbol{\theta}) \leq \tilde{f}^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}(\hat{\mathbf{x}}, \boldsymbol{\theta}) \leq C\tilde{f}^{(\boldsymbol{\nu})}_{\boldsymbol{p}}(\boldsymbol{\theta}),$$
for all $\boldsymbol{\theta} \in [-\pi, \pi]^d$ and for almost every $\hat{\mathbf{x}} \in [0,1]^d$.

4 Conclusions

We have extended previous results on the spectral distribution of discretization matrices arising from B-spline IgA approximations of the general elliptic problem (1.1) in two ways.

First, we have computed the spectral symbol $f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$ of the (normalized) Galerkin B-spline IgA stiffness matrices, assuming only that the PDE coefficients belong to L^∞. The argument used in our derivation is based on the Lusin theorem, as well as on the theory of GLT sequences.

Second, we have proved the positive semi-definiteness of the $d \times d$ symmetric matrix $H_{\boldsymbol{p}}(\boldsymbol{\theta})$ appearing in the expression of the symbol $f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$. The positive semi-definiteness of $H_{\boldsymbol{p}}(\boldsymbol{\theta})$ implies the non-negativity of $f^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$. Moreover, we can extend this result to the IgA collocation case, which implies that the spectral symbol $\tilde{f}^{(\boldsymbol{\nu})}_{\mathbf{G},\boldsymbol{p}}$ of the (normalized) B-spline IgA collocation matrices is non-negative as well. Here, the argument of the proof relies on careful Linear Algebra manipulations of matrix determinants and on an induction process over the dimensionality parameter d.

The presented arguments are very general and can also be applied to other PDE discretization techniques than B-spline IgA. Indeed, they can be seen, in combination with the technique shown in [12], as a general apparatus for analyzing the spectral distribution of a sequence of matrices coming from a PDE discretization. A future line of research could be the application of this apparatus to the spectral analysis of PDE discretization matrices arising from high-order Finite Elements or NURBS-based IgA.

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