A Tradeoff Between Data Rate and Regulation Performance in Networked Data Flow Control

Torbjörn Wigren

Department of Information Technology, Uppsala University

Abstract

The report investigates fundamental trade-offs related to the static regulation performance in networked control systems with delay and saturation in the feedback loop. The trade-offs are a consequence of required $\mathcal{L}_2$-stability of the networked system, in the limit where the delay tends to infinity. First a relation between the relative static regulation accuracy and the static gain of the part of the plant that affects the disturbance is derived. Based on this, the special case with one directional networked flow of an arbitrary medium is considered, where the saturation is a consequence of assumed one-directional flow. For this case it is shown that the trade-off becomes one between relative static regulation accuracy and flow capacity. In the wireless case this implies that there is a tradeoff between the relative static regulation performance and the achievable end to end Shannon data rate. A numerical study of a wireless transmit queue data flow controller illustrates and validates the discovered fundamental trade-offs and limitations.

Index Terms

5G; Data Rate; Delay; Flow Control; Fundamental Limitation; $\mathcal{L}_2$ Stability; Networked Control; Regulation; Saturation, Wireless.

I. Introduction

TRADE-OFFS and fundamental limitations have been central in automatic control ever since Bode's discovery of the sensitivity integral [1]. Networked control systems (NCS) [4] is no exception, with the well known data rate theorem stating the minimum data rate needed for stabilization of an unstable plant [3], [17]. The present report studies trade-offs related to the static regulation performance of *linear controllers* applied in NCSs that are subject to saturation and large delay in their feedback loops. The report is inspired by Example 1 of [32], in which a static trade-off between regulation accuracy and data channel capacity was observed for a specific low order lead-lag controller. Applications where this type of single-input-single-output (SISO) NCS occur include different types of one-directional flow control, like fluid flow control [29], hydro-power control [24], traffic flow control [13], and wireless data flow control between internet nodes [33], [35]. In the latter case stability robustness is particularly important due to the varying backhaul interface delay properties, see e.g. [19]. General servo-control problems subject to delay also fall in the category of systems treated in the report since the full actuator range normally needs to be utilized to maximize performance [1], thereby introducing saturation. In the coming years the new 5G cellular standards will evolve to handle so called critical machine type communication (C-MTC) with very low latency in the ms region, and with extremely high reliability [20], [21]. This will enable new feedback control applications ranging from wireless production lines with high bandwidth robots, to haptic control that could e.g. enable remote surgery [20]. In cases where these guaranteed low delays would still be large as compared to the plant dynamics, the results of this report, [32], [33], [34] and [35] provide design constraints and networked controller design methods suitable to address the resulting networked controller design problems.

The present analysis is based on the assumption that an un-modelled disturbance affects the NCS, and that it is required to regulate away the static part of the disturbance to a specified level expressed as a fraction of the absolute value of the amplitude of the disturbance. At the same time it is required that the NCS is globally $\mathcal{L}_2$-stable [31]. The first contribution of the report then proves that if the quotient between the relative static regulation accuracy and the part of the static gain of the plant that is affected by the disturbance is too small, $\mathcal{L}_2$-stability does not follow from the Popov criterion [18], [31]. The second contribution specializes to the case where the regulated plant is a chain of leaky storage reservoirs for the flowing medium, modelled as a chain of leaky integrators. In this case it follows that there is a trade-off between the leakage rates that reduce the achievable flow capacity after the point where the disturbance enters, and the specified relative static regulation accuracy. A structures like this could e.g. provide a basic model of cascaded hydro-power plants in a river. In the wireless data flow control application, this conclusion implies that there is a trade-off between the end to end maximum data rate of a flow controller and the relative static regulation accuracy specified for the corresponding wireless transmit queue, a fact that is important to account for in the ongoing development of 5G wireless standards [20], [21]. In this case the leakage rate of the terminating wireless transmit data queue of the plant node reduces the data rate of the incoming channel, to an effective wireless rate below the Shannon limit [7] of the network interface of the NCS. This constitutes the third contribution of the report. This third contribution is further treated in a numerical example that illustrates and validates the derived trad-offs. The data flow controller of [35] is revisited for this, in a case that is tailored to a study of the stability limit that is the focal point of all contributions of this report.

The analysis is based on the input-output version of the Popov criterion in order to handle the infinite dimensional time delay [22], [31], [38]. A main reason for the use of the Popov criterion is that the analysis produces analytical results that are easy to interpret. The Popov criterion is however not necessary for $\mathcal{L}_2$-stability, which needs to be listed as a limitation of this work. The more modern theory based on integral quadratic constraints (IQC) is likely to provide tighter stability bounds and can handle any delay, also time varying ones [14], [16]. However, bounds, limitations and trade-offs obtained by IQC would likely be based on numerical computations in each special case, and it is unclear if IQC would produce fundamental trade-offs and limitations in analytical form. It would be interesting to investigate this in future work, however for now one alternative would be to apply IQC as a final step of analysis, using guidance obtained by the results derived in this report. As an alternative the framework developed in [30] could be further investigated. Another generalization would be to study the same problem, with the linear controller replaced by non-linear methods, like model predictive control (MPC) [11], [36] or methods based on linear matrix inequalities (LMIs) [6]. A generalized treatment involving also control signal quantization might e.g. be based on some of the NCS techniques of [8], [10] or [12].

As stated above, a trade-off similar to the one of the third contribution of this report was observed in [32], in the special case with classical lead-lag control applied for wireless data flow control as in [33]. Generalizations of the NCS control strategy of [33] are reported in [34] and [35] where H_∞ and LQG state space controllers are applied, however no further analysis of the static regulation accuracy was undertaken in these publications. The starting point of the present report is the low frequency loop gain constraint derived in [34]. A parallel result allowing for feedforward is available in [32]. The general literature on control of systems with delay and systems with saturation is obviously quite extensive. Examples relevant for extensions of the present work may include e.g. [5], [6], [14], [15], and [27]. To the best if the author's knowledge, no results like those contributed here have however been obtained in previous publications. The results are believed to be important mainly since they extend the intuitive understanding of the fundamental coupling between the achievable static regulation accuracy and the flow rate in the case of networked flow control with a large loop delay and a saturation. These quantities can not be independently selected, a fact that is important to have in mind when controllers are designed and tuned. The indication that there may be a need to reduce the static data rate of the total NCS, to less than the Shannon data rate of the networked interface in order to preserve $\mathcal{L}_2$-stability is theoretically interesting. Further studies are needed to investigate this potential link to information theory.

The report is organized as follows. The NCS is defined in section II. Section III applies the $\mathcal{L}_2$-stability framework, to state the conditions under which the NCS is analysed. The trade-off related to static regulation performance is derived in section IV, while the flow control case is treated in section V. The implications for networked wireless data flow capacity are outlined in section VI. Numerical illustrations are given in section VII, followed by a discussion in section VIII and concluding remarks

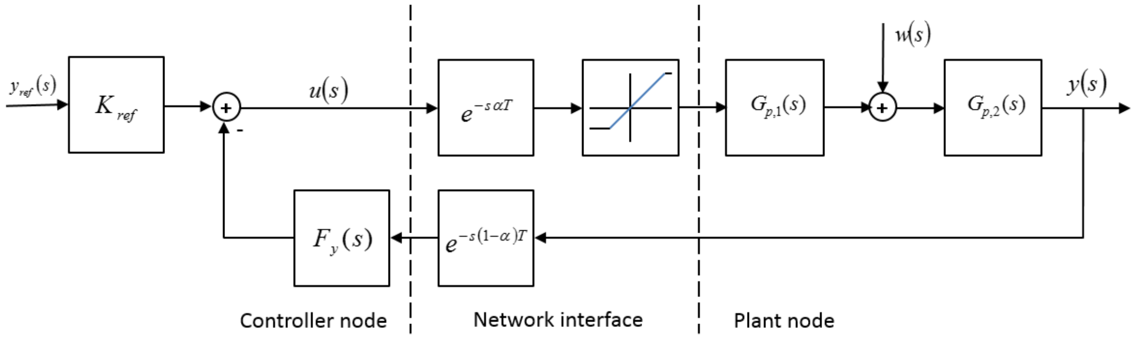

Fig. 1. Block diagram of the analysed networked control problem.

in section IX. Boldface characters are used for vectors and matrices throughout the report. The fact that the focus is on static properties is highlighted by a zero angular frequency argument in transfer functions, like $G(0)$, in all derivations of the report. Keeping the standard notation for transfer functions in the static case serves the purpose to improve the readability. In addition, it highlights the fact that the analysis is non-trivial and of central importance in engineering practice, despite the restriction to a static case [1].

II. The networked control system

The NCS that is studied is depicted in Fig. 1. The block diagram of the controller node results when state feedback design of any kind is applied, see e.g, [34] section 3.5 and [35] section IV.B for details on the relation between the state feedback controller and the frequency domain filter $F_y(s)$ for H_∞ and LQG control, respectively. In the present report, the delayed feedback signal $e^{-s(1-\alpha)T}y(s)$ is filtered by $F_y(s)$ and subtracted from the reference signal component $K_{ref}y_{ref}(s)$ to give the control signal $u(s)$. The reference signal gain is set by means of the constant K_{ref}, so that the closed loop static gain from the reference signal $y_{ref}(s)$ to the output signal $y(s)$ attains the specified value, usually 1.

The control signal $u(s)$ affects the plant via a network interface that is subject to a delay αT in the downlink to the plant node. Here T denotes the round trip delay, i.e. the delay that affects the loop gain. The constant $\alpha \in [0, 1]$ distributes T between the downlink and the uplink, where the measured feedback signal $y(s)$ is affected by the delay $(1-\alpha)T$. The static saturation $\varphi(\cdot)$ is also assumed to be associated with the network interface in Fig. 1. In communication applications, the upper limit of the saturation could e.g. follow from Shannon's theorem of information theory which states that all bandlimited channels are associated with a maximal data rate, depending on the signal to noise ratio [7]. In e.g. wireless data flow control, the lower limit is typically zero since the data flow control is one-directional [33]. Note that it is by no means necessary to associate the saturation with the interface. Since the saturation is time invariant, it can be moved to appear before the downlink delay operator of Fig. 1, thereby potentially becoming a part of the actuator of the controller. The analysis of the report is valid for both situations. For the same reason the saturation may also be split in two static blocks, appearing directly before and directly after the delay operator.

The dynamics of the plant node of Fig. 1 is divided in the two blocks described by the transfer functions $G_{p,1}(s)$ and $G_{p,2}(s)$, with the disturbance $w(s)$ entering between the two blocks. The output signal which is subject to feedback control is denoted $y(s)$. Note that the disturbance $w(s)$ is not necessarily defined exactly as the one used in [34]. Furthermore, the disturbance $w(s)$ is un-modelled, not used for controller design, and therefore only regulated by the feedback loop. The report is focused on the resulting relative static accuracy of this feedback loop.

The transfer functions of Fig. 1 are assumed to be given by

$$G_{p,1}(s) = G_{o,1}\frac{(s+b_{1,1})...(s+b_{1,nb_1})}{(s+a_{1,1})...(s+a_{1,na_1})} \tag{1}$$

$$G_{p,2}(s) = G_{o,2}\frac{(s+b_{2,1})...(s+b_{2,nb_2})}{(s+a_{2,1})...(s+a_{2,na_2})} \tag{2}$$

$$F_y(s) = F_o\frac{(s+d_1)...(s+d_{nd})}{(s+c_1)...(s+c_{nc})}. \tag{3}$$

Furthermore, the static nonlinearity is assumed to obey,

$$\varphi(u(t)) = \begin{cases} ku_-, & u(t) \le u_- \\ ku(t), & u_- < u(t) < u_+ \\ ku_+, & u(t) \ge u_+, \end{cases} \tag{4}$$

where ku_- and ku_+ are the minimal and maximal control signal values valid outside the interval $[u_-, u_+]$.

III. $\mathcal{L}_2$-STABILITY AND A LOW FREQUENCY LOOP GAIN LIMITATION

As is evident from Fig. 1 the control loop contains a static nonlinearity, linear dynamics and infinite dimensional delay. The input-output version of the Popov criterion is therefore suitable to address the global $\mathcal{L}_2$-stability of the closed loop NCS. The starting point of the analysis of the report is a result on the low frequency linear loop gain, that is proved below with some modification of the proof of a corresponding case in [34].

To state the result some background results are useful. First note that the linear loop gain of the studied NCS follows by inspection from Fig. 1 as

$$\hat{g}(s) = G_{p,1}(s)G_{p,2}(s)F_y(s)e^{-sT} \tag{5}$$

The following definitions then need to be stated, see e.g. [31] for a detailed background.

Definition 1: For all $p \in [0,\infty)$, $\mathcal{L}_p[0,\infty)$ denotes the set of all measurable functions $f(\cdot):[0,\infty) \rightarrow R$, such that

$$\|f(\cdot)\|_p^p = \int_0^\infty |f(t)|^p dt < \infty.$$

Definition 2: The mapping $A : \mathcal{L}_{pe} \to \mathcal{L}_{pe}$ is $\mathcal{L}_p$-stable if i) $Af \in \mathcal{L}_p$ whenever $f \in \mathcal{L}_p$, and ii) there exist finite constants l, m, such that

$$\|Af\|_p \leq l\|f\|_p + m, \quad \forall f \in \mathcal{L}_p.$$

Definition 3: $\mathcal{A}$ denotes the set of generalized functions of the form

$$f(t) = \begin{cases} 0, & t < 0 \\ \sum_{i=0}^{\infty} f_i \delta(t - t_i) + f_a(t), & t \geq 0 \end{cases}$$

where $\delta(\cdot)$ is the unit delta distribution, t_i are non-negative constant delays, $f_a(t)$ is measurable and

$$\textstyle\sum_{i=0}^{\infty} |f_i| < \infty, \qquad \int_0^{\infty} |f_a(t)|dt < \infty.$$

Definition 4: $\hat{\mathcal{A}}$ denotes the set of all function $\hat{f} : C_+ \to C$ that are Laplace transforms of elements of $\mathcal{A}$.

Here $\mathcal{L}_{pe}$ denotes the extended $\mathcal{L}_p$ space, see [31] for further details.

The analysis below repeatedly refers to the Popov criterion, which is reproduced here for easy reference.

Lemma 1 (Popov Criterion, [31] Theorem 6.7.63): Consider the system of Fig. 2. Assume that the inverse Laplace transform of the transfer function $\hat{g}(s)$ fulfils

$$g(\cdot) \in \mathcal{A}, \qquad \dot{g}(\cdot) \in \mathcal{A},$$

that the time invariant static nonlinearity $\varphi(\cdot)$ fulfils

$$0 \leq \sigma\varphi(\sigma) \leq \tilde{k}\sigma^2,$$

and that $u_1 \in \mathcal{L}_2$, $u_2 \in \mathcal{L}_2$, $\dot{u_2} \in \mathcal{L}_2$. Under these conditions the system is $\mathcal{L}_2$-stable if there exist constants q, δ, such that the Popov plot

$$\omega \in [0, \infty) \to Re[\hat{g}(j\omega)] + j\omega Im[\hat{g}(j\omega)] \in C$$

is entirely to the right of a line through $-1/\tilde{k} + \delta + j0$ with slope $1/q$, for some $q \geq 0$ and some $\delta > 0$.

Proof: See [31], section 6.7.

To state the result corresponding to [34], a number of conditions on the NCS of Fig. 1 are also needed. The conditions are

C1) The static nonlinearity (4) obeys $0 \leq \sigma\varphi(\sigma) \leq \tilde{k}\sigma^2$, $0 < \tilde{k} < \infty$.

C2) The delay T and the delay division α are constant.

C3) $G_{p,1}(s)$ and $G_{p,2}(s)$ are proper with all poles of $G_{p,1}(s)$, $G_{p,2}(s)$, $G_{p,1}^{-1}(s)$ and $G_{p,2}^{-1}(s)$ strictly in the left half plane.

C4) Complex poles and zeros of $G_{p,1}(s)$ and $G_{p,2}(s)$ appear in complex conjugate pairs and $G_{o,1} > 0$, $G_{o,2} > 0$.

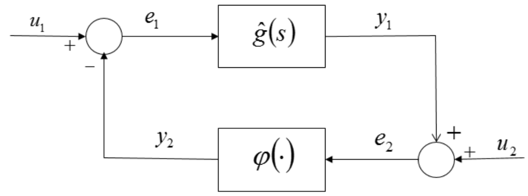

Fig. 2. Block diagram for which the Popov criterion is valid.

C5) $F_y(s)$ is strictly proper and has all poles strictly in the left half plane.

C6) Complex poles and zeros of $F_y(s)$ appear in complex conjugate pairs, $|d_i| > 0$, $i = 1, ..., nd$, $F_o > 0$, and the number of non-minimum phase zeros of $F_y(s)$ is even.

C7) $y_{ref} \in \mathcal{L}_2$ and $\dot{y}_{ref} \in \mathcal{L}_2$.

C8) $w/G_{p,1} \in \mathcal{L}_2$.

C9) The Popov plot $\omega \in [0, \infty) \rightarrow Re[\hat{g}(j\omega)] + j\omega Im[\hat{g}(j\omega)] \in C$ intersects the negative real axis at least once.

The conditions are discussed and explained below, following the proof of:

Lemma 2: Consider the system given by Fig. 1 and (1) - (4). Assume that the conditions C1-C9 hold, and that $T \rightarrow \infty$. Then the Popov criterion of Lemma 1 cannot imply $\mathcal{L}_2$-stability in case

$$\lim_{\omega \rightarrow 0} Re[G_{p,1}(j\omega)G_{p,2}(j\omega)F_y(j\omega)] = G_{p,1}(0)G_{p,2}(0)F_y(0)$$

$$= G_{o,1}G_{o,2}F_o \frac{b_{1,1}...b_{1,nb_1}b_{2,1}...b_{2,nb_2}d_1...d_{nd}}{a_{1,1}...a_{1,na_1}a_{2,1}...a_{2,na_2}c_1...c_{nc}} \geq \frac{1}{k}.$$

Remark 1: As is well known, the Popov criterion implies $\mathcal{L}_2$-stability if there is a line passing through $-1/\tilde{k} + j0$ with arbitrary non-negative slope, such that the Popov plot $\omega \in [0, \infty) \rightarrow Re[\hat{g}(j\omega)] + j\omega Im[\hat{g}(j\omega)] \in C$ is to the right of this line. Lemma 2 evaluates the Popov condition for asymptotically large delay and concludes that the static linear loop gain needs to be bounded by $1/k$ to allow $\mathcal{L}_2$-stability. This follows since when $T \rightarrow \infty$, the first point of the Popov plot that intersects the negative real axis has a gain that approaches the static linear loop gain $G_{p,1}(0)G_{p,2}(0)F_y(0)$, cf. [32].

Proof of Lemma 2: The Lemma follows from Theorem 1 and Theorem 2 of [34] (available on-line), by the following transformations of the block diagram of Fig. 1, making it coincide with the block diagram of Fig. 3 of [34] for which Theorems 1 and 2 of [34] are proved.

First, the delay block $e^{-s\alpha T}$ and the static nonlinearity appear in opposite order in Fig. 3 of [34] and in Fig. 1 of this report. Since the nonlinearity (4) is time invariant, the order of the two blocks of Fig. 1 can however be reversed, regaining the order of Fig. 3 of [34].

With this modification in place, the delay block $e^{-s(1-\alpha)T}$ of the feedback path can be moved counterclockwise to appear immediately after the delay block $e^{-s\alpha T}$, thereby allowing a merge into the single delay block e^{-sT}. The delay block must also be moved into the disturbance path to give the disturbance $e^{-s(1-\alpha)T}w(s)$. All these transformations follow from linearity of the involved quantities.

Finally, it is noted that the disturbance of Fig. 3 of [34] acts directly on the output $y(s)$. Therefore the disturbance of Fig. 1 of this report is moved to appear additively after $G_{p,2}(s)$, by redefining the disturbance into $e^{-s(1-\alpha)T}G_{p,2}(s)w(s)$.

The end result is that the block diagram of Fig. 1 of the present report has been transformed to the one of Fig. 3 of [34] with the disturbance modified to

$$\tilde{w}(s) = e^{-s(1-\alpha)T}G_{p,2}(s)w(s). \tag{6}$$

It then remains to show that C1 - C9 of the present report imply the conditions A3-A10 of Theorems 1 and 2 of [34], with the disturbance given by (6).

It is immediately noted that A3, A4, A5, A6, A8, A9 and A10 are pairwise equivalent to C2, C3, C5, C7, C9, C4 and C6, respectively, and do not involve the disturbance. It therefore remains to verify that A7 holds for the disturbance $\tilde{w}(s)$ of (6). A7 reads $\tilde{w}(s)/(G_{p,1}(s)G_{p,2}(s)) \in \mathcal{L}_2$. Inserting (6) in this expression and simplifying results in the condition that $e^{s(1-\alpha)T}w(s)/(G_{p,1}(s)) \in \mathcal{L}_2$. Since it follows from Definition 1 that a delayed bounded signal is in $\mathcal{L}_2$ provided that the signal itself is in $\mathcal{L}_2$, C8 implies A7. The condition C1 states the same sector condition that appears in the formulation of Theorem 1 and 2 of [34]. Selecting $\tilde{k} = k$ according to (4) concludes the proof of Lemma 2.

To discuss and explain the conditions C1-C9 it is first noted that the sector condition C1 applied to prove the Popov condition might be relaxed if the more modern IQC theory [14], [16] would be applied. However, an advantage with the use of the Popov criterion is that easily interpretable analytical results are obtained that are valid for a large class of systems. It is not clear wether the numerical search applied in IQC algorithms could provide a similar simplicity. The validity of the constant delay assumption C2 is application dependent. Admittedly, in the wireless internet flow control applications addressed later in this report and in [33] the delays vary significantly. However, the controller design principle of [33] that aimed for stability robustness seems to handle practical internet delay variations very well, as evidenced by the laboratory tests and field trials reported in [33]. The open loop stability of C3 and C5 is a consequence of the treatment of a saturation problem [31], while the inverse stability requirement of C3 is e.g. needed to ensure a positive static gain in the low frequency analysis, and in the filtering of the disturbance, cf. C8. The conditions on non-minimum phase zeros and constants of C6 are also needed to secure a positive static gain. Obviously the treatment is confined to non-complex feedback systems, hence the pairing of complex conjugate poles and zeros. The conditions C7 and C8 on the exciting signals of the feedback loop are there to ensure a sufficiently high frequency roll-off, to avoid amplification of potentially destabilizing high frequency signal components. The condition C9 is obviously necessary to obtain the fundamental limitations of this report, cf. the proof of Theorem 2 of [34]. It is also exactly the distinguishing factor that allows the proof of global unconditional stability in [37].

IV. The static plant gain and regulation performance trade-off

Lemma 2 provides a limit on the static loop gain for a large delay, over which $\mathcal{L}_2$-stability can no longer be guaranteed by the Popov criterion. The next step of the report is to exploit this result and to relate it to the corresponding relative static regulation accuracy, assuming that a nonzero un-modelled disturbance $w(s)$ affects the feedback loop.

A first problem is then that the static loop gain of Lemma 2 may define a stability boundary, meaning that closed loop stability cannot be assumed a priori in the limit when $T \to \infty$. Further conditions and assumptions are therefore needed to secure a static closed loop signal solution. Note therefore that the main property required for Lemma 2 to hold is that the static linear loop gain $\hat{g}(0)$ obtained from (5) fulfills the inequality involving the maximum gain k of the nonlinearity. The distribution of gain between the components of the loop gain is of no consequence, and neither is the gain for $\omega > 0$. A solution to the problem is therefore to consider a sequence of loop gains $\hat{g}_i(s) = G_{p,1,i}(s)G_{p,2,i}(s)F_{y,i}(s)e^{-sT}$, $i = 1, ...$, such that $\hat{g}_i(s) \to \hat{g}(s) = G_{p,1}(s)G_{p,2}(s)F_y(s)$ as $i \to \infty$, and such that the NCS of Fig. 1 obeys the Popov criterion of Lemma 1 with $\hat{g}(s)$ replaced by $\hat{g}_i(s)$, $i = 1, ...$ The static characteristics of the NCS of Fig. 1 can then be evaluated for $\hat{g}_i(s)$, after which the end result is obtained for $i \to \infty$. It is easy to see that the sequence $\hat{g}_i(s)$ always exists. It is e.g. possible to take the scaled down sequence $\hat{g}_i(s) = i\hat{g}(s)/(i+1)$, $i = 1, ...$, with $\hat{g}(s)$ designed to meet the Popov criterion. The systematic design of a loop gain that obeys the Popov-criterion for the NCS at hand is treated in detail e.g. in the publications [33], [34] and [35]. With this approach a NCS is designed to meet the Popov criterion for finite T after which the loop gain is scaled down using the index i. This scaled down loop gain $\hat{g}_i(s)$ is then also stable according to the Popov criterion. This leads to the following additional technical assumptions relating to Fig, 1:

C10) $\{\hat{g}_i(s)\}_{i=1}^{\infty}$ given by $\hat{g}_i(s) = G_{p,1,i}(s)G_{p,2,i}(s)F_{y,i}(s)e^{-sT}$ fulfil the Popov criterion of Lemma 1, $\forall\ 0 < i < \infty$.

C11) $\lim_{i\to\infty} G_{p,1,i}(s) = G_{p,1}(s)$.

C12) $\lim_{i\to\infty} G_{p,2,i}(s) = G_{p,2}(s)$.

C13) $\lim_{i\to\infty} F_{y,i}(s) = F_y(s)$.

To quantify the relative static regulation accuracy in response to the disturbance $w(s)$, the regulation error $e(s)$ and the corresponding relative static regulation accuracy Δ_{we} are defined as follows, referring to Fig. 1

$$e(s) = y_{ref}(s) - y(s), \tag{7}$$

$$\Delta_{we} = \frac{e(0)}{w(0)}. \tag{8}$$

Note that the independent variable is here the Laplace variable s and not the time t, with (0) defining the static case. The index i is used to denote quantities where C10-C13 are used. Note also that the external signals $y_{ref}(s)$ and $w(s)$ are not dependent on i.

The static case is then considered for an index i of C10-C13. Due to C10 a static signal solution exists $\forall\ 0 < i < \infty$. Moreover, the static nonlinearity can be exactly handled by a division into cases. It follows from Fig. 1 that

$$e_i(0) = y_{ref}(0) - y_i(0)$$

$$= y_{ref}(0) - G_{p,2,i}(0)w(0) - G_{p,1,i}(0)G_{p,2,i}(0)\varphi(u_i(0)), \tag{9}$$

$$u_i(0) = K_{ref,i}y_{ref}(0) - F_{y,i}(0)y_i(0) \tag{10}$$

A multiplication of (9) with $F_{y,i}(0)$, followed by a replacement of $F_{y,i}(0)y_i(0)$ with the value following from (10) results in the static equation

$$u_i(0) + F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)\varphi(u_i(0))$$

$$= K_{ref,i}y_{ref}(0) - F_{y,i}(0)G_{p,2,i}(0)w(0). \tag{11}$$

Three cases then need to be considered depending on whether $\varphi(u_i(0)) = ku_-$, $\varphi(u_i(0)) = ku_+$ or $\varphi(u_i(0)) = ku_i(0)$. Using (11) to compute $u_i(0)$ leads to

$$u_i(0) = K_{ref,i}y_{ref}(0) - F_{y,i}(0)G_{p,2,i}(0)w(0)$$

$$-F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)ku_-, \quad u_i(0) \leq u_-, \tag{12}$$

$$u_i(0) = \frac{K_{ref,i}y_{ref}(0)}{1 + F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k}$$

$$-\frac{F_{y,i}(0)G_{p,2,i}(0)w(0)}{1 + F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k}, \quad u_- < u_i(0) < u_+, \tag{13}$$

$$u_i(0) = K_{ref,i}y_{ref}(0) - F_{y,i}(0)G_{p,2,i}(0)w(0)$$

$$-F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)ku_+, \quad u_i(0) \geq u_+. \tag{14}$$

Since $w(s)$ is an un-modelled disturbance, it is reasonable to assume that the reference signal gain is selected so that the closed loop gain from $y_{ref}(0)$ to $y_i(0)$ equals 1 without the disturbance, in the (static) case (13), without saturation. Using the property of a unity static gain together with C13, (10) and (13) results in

$$
\begin{aligned}
0 &= y_{ref}(0) - y_i(0) \\
&= y_{ref}(0) - \frac{K_{ref,i}}{F_{y,i}(0)} y_{ref}(0) + \frac{1}{F_{y,i}(0)} u(0) \\
&= y_{ref}(0) - \frac{K_{ref,i}}{F_{y,i}(0)} y_{ref}(0) \\
&+ \frac{K_{ref,i}}{F_{y,i}(0)} \frac{1}{(1 + F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k)} y_{ref}(0) \\
&= \left(1 - \frac{K_{ref,i}G_{p,1,i}(0)G_{p,2,i}(0)k}{1 + F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k}\right) y_{ref}(0).
\end{aligned} \tag{15}
$$

Since C4 holds this implies the reference signal gain

$$
K_{ref,i} = F_{y,i}(0) + \frac{1}{G_{p,1,i}(0)G_{p,2,i}(0)k}. \tag{16}
$$

This is formulated as

C14) The reference signal gain is selected as $K_{ref,i} = F_{y,i}(0) + (G_{p,1,i}(0)G_{p,2,i}(0)k)^{-1}$.

Note that $K_{ref,i} \rightarrow K_{ref} = F_y(0) + (G_{p,1}(0)G_{p,2}(0)k)^{-1}$ as $i \rightarrow \infty$, which is well defined due to C4.

To conclude the discussion, $e_i(0)$ is computed in the three cases (12)-(14), using (10), (16) and C10-C14. The result corresponding to (12) becomes

$$
\begin{aligned}
e_i(0) &= y_{ref}(0) - y_i(0) \\
&= y_{ref}(0) - \frac{K_{ref,i}}{F_{y,i}(0)} y_{ref}(0) + \frac{1}{F_{y,i}(0)} u_i(0) \\
&= y_{ref}(0) - G_{p,2,i}(0)w(0) \\
&- G_{p,1,i}(0)G_{p,2,i}(0)ku_-, \; u_i(0) \leq u_-.
\end{aligned} \tag{17}
$$

A similar calculation corresponding to (13) gives

$$
\begin{aligned}
e_i(0) &= y_{ref}(0) - y_i(0) \\
&= y_{ref}(0) - \frac{K_{ref,i}}{F_{y,i}(0)} y_{ref}(0) + \frac{1}{F_{y,i}(0)} u_i(0) \\
&= \frac{F_{y,i}(0) - K_{ref,i}}{F_{y,i}(0)} y_{ref}(0)
\end{aligned}
$$

$$+\frac{1}{F_{y,i}(0)}\frac{K_{ref,i}}{(1+F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k)}y_{ref}(0)$$

$$-\frac{G_{p,2,i}(0)}{1+F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k}w(0)$$

$$=\frac{(1+F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k)(F_{y,i}(0)-K_{ref,i})}{F_{y,i}(0)(1+F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k)}$$

$$+\frac{K_{ref,i}}{F_{y,i}(0)(1+F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k)}$$

$$-\frac{G_{p,2,i}(0)}{1+F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k}w(0)$$

$$=\frac{(1+F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k)\left(-\frac{1}{G_{p,1,i}(0)G_{p,2,i}(0)k}\right)}{F_{y,i}(0)(1+F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k)}$$

$$+\frac{K_{ref,i}}{F_{y,i}(0)(1+F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k)}$$

$$-\frac{G_{p,2,i}(0)}{1+F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k}w(0)$$

$$=-\frac{G_{p,2,i}(0)}{1+F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k}w(0),$$

$$u_-<u_i(0)<u_+. \tag{18}$$

Here (16) was exploited in the last two equalities. As expected, a static sensitivity results, cf. [34]. The case corresponding to (14) parallels the case corresponding to (12) and results in

$$e_i(0)=y_{ref}(0)-y_i(0)=y_{ref}(0)-G_{p,2,i}(0)w(0)$$

$$-G_{p,1,i}(0)G_{p,2,i}(0)ku_+,\ u_i(0)\geq u_+. \tag{19}$$

The quantities $\Delta_{we,i}$ corresponding to (8) and indexed by i can then be computed as

$$\Delta_{we,i}=$$

$$\frac{y_{ref,i}(0)-G_{p,1,i}(0)G_{p,2,i}(0)ku_-}{w(0)}-G_{p,2,i}(0),\ u_i(0)\leq u_-, \tag{20}$$

$$\Delta_{we,i}=$$

$$-\frac{G_{p,2,i}(0)}{1+F_{y,i}(0)G_{p,1,i}(0)G_{p,2,i}(0)k},\ u_-<u_i(0)<u_+, \tag{21}$$

$$\Delta_{we,i}=$$

$$\frac{y_{ref,i}(0)-G_{p,1,i}(0)G_{p,2,i}(0)ku_+}{w(0)}-G_{p,2,i}(0),\ u_i(0)\geq u_+. \tag{22}$$

The cases corresponding to (20) and (22) depend on the size of the disturbance and therefore need to be excluded from the further analysis. This means that the following technical condition obtained from (13) needs to be imposed

C15)

$$u_{-} < \frac{K_{ref,i} y_{ref}(0) - F_{y,i}(0) G_{p,2,i}(0) w(0)}{1 + F_{y,i}(0) G_{p,1,i}(0) G_{p,2,i}(0) k} < u_{+}.$$

The derivation of the first result of the report can then be concluded, noting that (5), (8), (21) and the inequality of Lemma 2 imply

$$k \hat{g}_i(0) = \frac{-G_{p,2,i}(0)}{\Delta_{we,i}} - 1 \geq 1. \tag{23}$$

A rearrangement results in

Theorem 1: Consider the system given by Fig. 1 and (1) - (4). Assume that the conditions C1-15 hold, and that $T \rightarrow \infty$. Then the Popov criterion cannot imply $\mathcal{L}_2$-stability in case

$$|\Delta_{we}| \leq \lim_{i \rightarrow \infty} \frac{1}{2} G_{p,2,i}(0) = \frac{1}{2} G_{p,2}(0).$$

Note that C3 and C4 imply the positivity of the right hand side in Theorem 1. The conclusion is hence that in case the relative static regulation accuracy becomes too good (small), then the Popov criterion will not be useful for securing stability of the NCS.

Remark 2: It needs to be stressed that the Popov-criterion is not necessary for stability, therefore the trade-offs and fundamental limitations discussed in the report may not be tight. The use of e.g. IQC-methods [14], [16] might therefore strengthen the results, and this is an important topic for further research. However, there is little doubt that a constraint like that of Theorem 1 is valid, a fact that is easily realized by consideration of the parallel linear case in which the necessary and sufficient Nyquist criterion would give similar results for large classes of loop gains. One such class of loop gains could be the class with a monotonically decreasing gain as a function of the angular frequency.

V. The data flow capacity and regulation performance trade-off

The next step of the report focuses on the case of networked flow control of a flowing medium of any kind, where a set of R cascaded leaky reservoirs for this medium are to be regulated. These reservoirs form the plant node. It is assumed that the transfer function of each of the reservoirs is given by

$$G_r(s) = \frac{1}{s + \varepsilon_r}, \; r = 1, ..., R. \tag{24}$$

Each of these transfer functions hence corresponds to the block diagram of Fig. 3. As can be seen, a fraction of the reservoir contents is fed back to reduce the incoming flow to the reservoir. It is furthermore assumed that the disturbance enters between

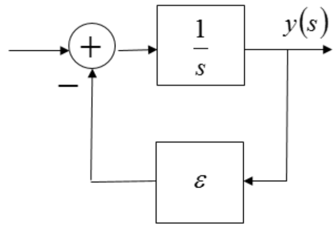

Fig. 3. Block diagram showing a feedback realization of a leaky reservoir

reservoirs r_w and r_w+1, which gives the following division between $G_{p,1}(s)$ and $G_{p,2}(s)$ of the plant of Fig. 1

$$G_{p,1} = \frac{1}{(s+\varepsilon_1)...(s+\varepsilon_{r_w})} \tag{25}$$

$$G_{p,2} = \frac{1}{(s+\varepsilon_{r_w+1})...(s+\varepsilon_R)} \tag{26}$$

To apply Theorem 1 to this system, the conditions C1-C15 need to be discussed. First C1, C2, C14 and C15 clearly need to be treated as assumptions on the NCS. C3 and C4 follow from (25) and (26), while C5 and C6 needs to be checked during the controller design. The condition C7 imposes the standard conditions on high frequency roll-off and stability of the reference signal, while (25) imposes additional requirements on more high frequency roll-off in the disturbance spectrum. C9 is normally fulfilled and can, if needed, be easily checked. The conditions C10-C13 are technical. Therefore, Theorem 1 is applicable to the flow control NCS. When the inequality of Theorem 1 is computed, this results in

Theorem 2: Consider the system given by Fig. 1, (3), (4), (25) and (26). Assume that the conditions C1, C2, C5-C15 hold, and that $T \to \infty$. Then the Popov criterion cannot imply $\mathcal{L}_2$-stability in case

$$|\Delta_{we}| \leq \frac{1}{2} \prod_{r=r_w+1}^{R} \frac{1}{\varepsilon_r}$$

This result states that in case the product of the relative static regulation accuracy and the leakage fractions of the reservoirs after the point where the disturbance enters is less than $1/2$, then $\mathcal{L}_2$-stability does not follow from the Popov criterion in case the delay is large. Since the leakage fractions $\varepsilon_r,\ r = r_w+1, ..., R$ according to Fig. 3 correspond to a reduced effective flow through the NCS, the consequence is that the flow capacity for the flowing medium is reduced by the leakage represented by $\varepsilon_r,\ r_w+1, ..., R$, with more reduction the higher the values are. It is therefore beneficial to keep $\varepsilon_r,\ r = r_w+1, ..., R$ small. Unfortunately, because of Theorem 2, this may not be possible unless the low frequency regulation accuracy is sacrificed. The stability requirement of the Popov criterion hence imposes a trade-off between the relative static regulation accuracy and the flow capacity.

VI. The wireless data rate and regulation accuracy trade-off

In the final part of the report the focus is on wireless data flow control, in particular for 5G connectivity. As stated in the introduction, an optimized data flow control solution for 5G wireless is of paramount importance for example in fading radio conditions at high carrier frequencies when techniques for multi-connectivity are used to mitigate radio shadowing. The focus is here on the downlink data flow case, from an internet controller node over a wireless interface to the plant node which interfaces to the wireless interface via a transmit data queue. The end user receives the data flow over the wireless interface, a connection that is outside the scope of the report and treated intext books like [7], [21]. In the wireless case the flow of data is strictly one-directional which is the reason why the nonlinearity of (4) is restricted to

$$\varphi(u(t)) = \begin{cases} 0, & u(t) \leq 0 \\ u(t), & 0 < u(t) < u_+ \\ u_+, & u(t) \geq u_+. \end{cases} \tag{27}$$

In order to handle the rapidly fading radio channel together with the delays associated with the data rate control and the corresponding signaling of feedback information, wireless transmit data queues are a necessity. To avoid interrupted transmission over the wireless interface the data volume in the transmit data queue needs to be sufficiently high to handle the combined effect of delay and radio fading. At the same time a too high data volume would imply an unnecessarily large queue dwell time which would add to the total latency. The conclusion is that accurate transmit queue data volume control is needed.

Since the 5G wireless systems will be connected to the internet, the basic internet data flow control schemes like the transmission control protocol (TCP) will affect the flow of data [23], [33]. TCP uses information from achnowledgement/nonacknowledgmenent (ACK/NACK) messages received at the internet data source node to control its transmit window. The consequence of that is a round trip latency that is likely to be far too high, for example in C-MTC applications were sub ms latencies may be required. The situation can be improved by application of so called active queue management (AQM) [23]. An AQM algorithm monitors the transmit data queue level and intentionally discards data packages, thereby introducing NACK messages close to the end user. This serves to reduce the maximum round trip latency to the internet source node, however this is often not enough. AQM is however important for the discussion in the present section, where a continuous time model of a transmit data queue with overlaid AQM is needed.

To derive such a model the incoming data flow to the transmit data queue is denoted $\bar{u}(s)$. The outgoing data rate over the wireless interface is denoted $w(s)$, and it acts as a disturbance on the transmit data queue. As a model of AQM, data discarding proportional to the the data volume of the transmit data queue is selected. The proportionality factor is denoted ε. The choice of model is reasonable since a high data volume needs more discards to reduce the data volume. This results in the following differential equation for the data volume in the transmit data queue

$$\frac{dy(t)}{dt} = \bar{u}(t) - w(t) - \varepsilon y(t). \tag{28}$$

A Laplace transformation results in

$$y(s) = \frac{1}{s+\varepsilon}\left(\bar{u}(s) - w(s)\right). \tag{29}$$

This means that

$$G_{p,1}(s) = 1, \tag{30}$$

$$G_{p,2}(s) = \frac{1}{s+\varepsilon}. \tag{31}$$

It is now evident that the disturbance enters together with the incoming data flow before the AQM transmit data queue dynamics $(s+\varepsilon)^{-1}$. Theorem 2 is therefore applicable. The following trade off between the data rate and the relative static regulation accuracy therefore holds

Theorem 3: Consider the data flow NCS given by Fig. 1, (3), (27), (30) and (31) with a maximum incoming data rate u_+, a specified relative regulation accuracy Δ_{we} and a data discard rate ε. Assume that the conditions C1, C2, C5-C15 hold, and that $T \to \infty$. Then the Popov criterion cannot imply $\mathcal{L}_2$-stability in case

$$|\Delta_{we}|\varepsilon \leq \frac{1}{2},$$

which represents an end user static data rate reduction from the incoming maximum data rate u_+ to $u_+ - \varepsilon y(0)$, where $y(0)$ is the static data volume of the transmit data queue terminating the NCS.

Remark 3: The consequence of Theorem 3 is therefore a trade-off between the maximum effective data rate of the NCS as counted from the controlling node to the wireless interface of the plant node, and the relative static regulation accuracy of the NCS. In case the incoming data rate is interpreted as the Shannon capacity [7], [21] of the network interface, the result quantifies a data rate loss associated with a requirement of closed loop $\mathcal{L}_2$ stability.

Noting that the inequality of Theorem 3 means that $\varepsilon > (2\Delta_{we})^{-1}$ needs to hold in case the low frequency properties of the NCS should not prevent closed loop stability, the following corollary to Theorem 3 follows:

Corollary 1: Consider the data flow NCS given by Fig. 1 (3), (27), (30)and (29), with an incoming network interface data rate capacity u_+ and a specified relative regulation accuracy Δ_{we}. Assume that the conditions C1, C2, C5-C15 hold, and that the Popov criterion is met in the static limit where $T \to \infty$. Then the static capacity as counted to the entry point of the transmit data queue is associated by a capacity that is less than

$$u_+ - \frac{1}{2}\frac{y(0)}{\Delta_{we}}.$$

The effective rate reduction is dependent on $y(0)$ and Δ_{we}. The reduction disappears only when $y(0) = 0$ or $\Delta_{we} \to \infty$. Both these cases are however infeasible in the wireless applications. The case with $y(0) = 0$ would mean that the transmit data queue is empty on average, i.e. identically zero, which contradicts the reason why the queue is needed at all, namely to mitigate the combined effect of radio fading and flow control loop delay. The case with $\Delta_{we} \to \infty$ would mean that there are no static requirements on the static feedback control error, which is clearly also infeasible since it would mean that any average value would be allowed in case of a disturbance.

VII. Numerical validation of Theorem 3 for an LQG based flow controller

In order to illustrate the obtained result the LQG controller described in [35] is used, when applied for wireless downlink data flow control. Since the data flow control application benefits from feedforward from the wireless rate, and since the controller of [35] combines feedback and feedforward, the present numerical example also does so. The combination of feedback and feedforward is similar to the approach used in [25] and [26]. The inclusion of feedforward means that some additional assumptions on the external feedforward signal would appear to be needed. In the example below the feedforward signal is however selected to be constant, adding bias and thereby only changing the operation point. The feedforward inclusion is therefore of little concern for the validation of Theorem 3, as long as C15 holds. However, since the feedforward relates to the practical data flow control implementation of [35], the choice here is to include feedforward.

The NCS plant of Fig. 1 hence consists of a wireless transmit queue with an AQM algorithm with a data discard rate ε. The transmit data queue receives incoming data packets from the controller node with rate $\bar{u}(s)$ and it is emptied by the scheduled downlink wireless data rate $w(s)$. As described in the previous section the transmit data queue can therefore be described as the leaky integrator

$$y(s) = \frac{1}{s+\varepsilon}\left(\bar{u}(s) - w(s)\right). \tag{32}$$

The controller of the transmit data queue resides in the upstream controller node. It receives measurements of the transmit queue data volume $y(s)$ and the wireless data rate $w_{FF}(s)$ over a network interface with a delay $(1-\alpha)T$. The controller then computes a control signal $u(s)$ based on $\bar{u}(s)$, $w_{FF}(s)$ and $y(s)$ with the LQG state space techniques of [35]. It can be noted that the controller incorporates delay compensation for the nominal delays of the loop by embedding of a rational delay approxmation in the state space model underpinning the controller design [28]. The following Padé approximation is used

$$G_T(s) =$$

$$\frac{(T^d)^4 s^4 - 20(T^d)^3 s^3 + 180(T^d)^2 s^2 - 840 T^d s + 1680}{(T^d)^4 s^4 + 20(T^d)^3 s^3 + 180(T^d)^2 s^2 + 840 T^d s + 1680}. \tag{33}$$

where T^d is the designing delay, i.e the delay applied in the design. Furthermore, the $\mathcal{L}_2$ stability region is pre-computed to allow selection of controller parameters that provide robust stability. The computed control signal defines the downlink data rate $u(s)$ with which data is sent from the buffer in the controller node to the transmit data queue in the plant node. Due to the downlink network interface there is a delay αT until the commanded data rate takes effect as $\bar{u}(s)$ at the input of the transmit data queue. Another important part of the downlink data transfer is the saturation

$$\varphi(u(t)) = \begin{cases} 0, & u(t) \leq 0 \\ u(t), & 0 < u(t) < u_+ \\ u_+, & u(t) \geq u_+, \end{cases} \tag{34}$$

that results since the flow of data is one-directional. The system quantities appearing in Fig. 1 are therefore

$$G_{p,1}(s) = 1, \tag{35}$$

$$G_{p,2}(s) = \frac{1}{s + \varepsilon}. \tag{36}$$

The quantity $F_y(s)$ is directly computed from the state space controller of [35] using equation (29) of that paper. The principles for tuning of the state space controller is described at length in [35], which the reader is referred to for further details.

The parameters used here are the same as defined in section 7.2. and 7.3 of [35]. This means that $\varepsilon = 0.1\ s^{-1}$ and $T^d = 0.06\ s$ were used together with the control signal penalty $Q_2 = 0.01\ s^2 bit^{-2}$. The state penalty matrix $\mathbf{Q}_1$ was one for the output state ($y(t)$) and zero for all other elements. The covariance matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ were tuned to balance feedback and feedforward. The size of the variation of the wireless rate is reflected by $R_1(2,2)$. Since the mean value of this quantity was close to $1e6\ bits(s^{-1})$, $R_1(2,2) = 1e10\ (bits)^2(s^{-2})$ was selected. Since the feedforward signal of [35] is integrated with the dynamic queue model there is no more stochastic external signal and $R_1(i,j) = 0, i, j \neq 2, 2$. The measurement error for the transmit queue data volume, $R_2(1,1)$ is roughly 0.1%. Since the data volume was close to $1e6\ bits$, $R_2(1,1) = 1e6\ (bits)^2$ was selected. A similar order of magnitude was observed for the expected measurement error of the wireless rate. This quantity was used for fine tuning, resulting in $R_2(2,2) = 16e6\ (bits)^2(s^{-2})$. The off-diagonal elements of $\mathbf{R}_2$ were all 0. The reader is referred to [35] for a detailed discussion on how these parameters affect the LQG controller design method. The static reference signal gain was adjusted to 1. What matters most in this report is however the quantity $F_y(s)$ and the associated relative regulation accuracy that results. For this reason $F_y(s)$ and the sensitivity function $S(s)$ that result from the above tuning are depicted in Fig. 4 and Fig. 5, respectively. It can be seen from Fig. 5 that the static sensitivity is $-28\ dB$, which corresponds to a linear value of

$$S(0) = 0.0398. \tag{37}$$

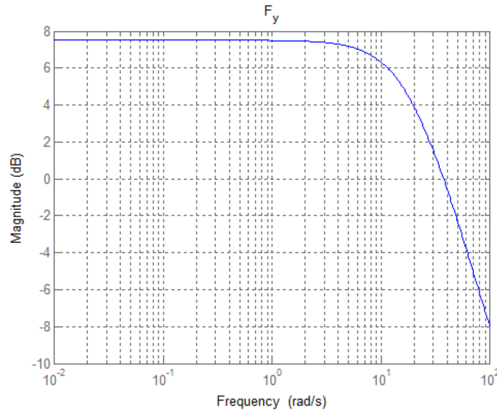

Fig. 4. Bode plot of the feedback controller transfer function.

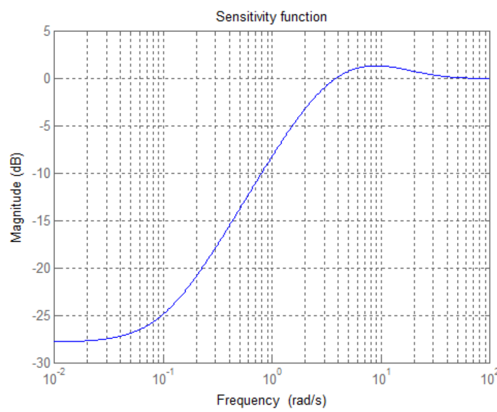

Fig. 5. Bode plot of the sensitivity function.

The reference signal was selected to give a queue packet dwell time of $0.125\ ms$, as in [35]. The wireless rate signal was however selected to be constant to allow stability properties to be studied with simulation. The value $1e7\ (bits)s^{-1}$ was selected, to obtain a rate significantly different than what was assumed for the tuning of $\mathbf{R}_1$, to avoid treatment of a case described exactly of the selected tuning parameters. Furthermore, the system was initialized with an empty transmit data queue. Theorem 3 can then be validated numerically.

To do so, the conditions C1, C2, C5 - C15 first need to be verified for the application at hand. C1 follows from (34) with $\tilde{k} = k = 1$. The assumption C2 may be questionable for practical wireless networked internet flow control systems, however it is somewhat justified by the field trial results of [33] where a similar algorithm as the one used in the present report operated on real internet traffic with excellent results. The condition C5 is true since the model underpinning the controller design does not have a direct term, see [35]. The condition C6 needs to be checked during the controller design and it is true in the present case. The signal conditions C7 and C8 are not strictly valid for constant signals, they could however be replaced with

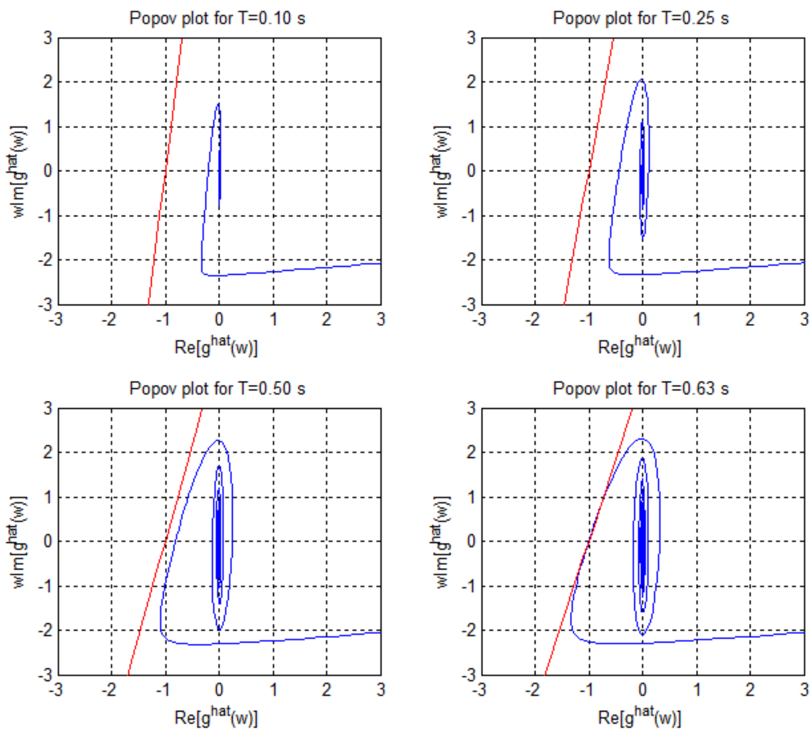

Fig. 6. Popov plots for the wireless NCS. The delay $T = 0.63$ s represents the stability limit predicted by the Popov criterion.

a very slowly exponentially decaying signal for simulation purposes, therefore they are not considerd to be a limitation here. C9 follows by inspection of Fig. 6 which shows Popov plots for the studied NCS for varying values of the loop delay T. The conditions C10 - C13 are technical conditions only used in the proof of the main results. Their validity was discussed in the initial part of section IV. The condition C14 was secured in the controller design, while C15 means that the signals need to be positive since u_+ is selected to a very large value in the present evaluation. Hence the conditions of Theorem 3 hold.

Then the relative static regulation accuracy needs to be determined. Referring to C11-C13, (21), (34), (36) and (37) it follows that

$$|\Delta_{we}| = |\lim_{i\to\infty} \Delta_{we,i}|$$

$$= \left| \frac{1}{1 + F_y(0)G_{p,1}(0)G_{p,2}(0)} G_{p,2}(0) \right|$$

$$= S(0)G_{p,2}(0) = \frac{0.0398}{\varepsilon} = \frac{0.0398}{0.10} = 0.398. \quad (38)$$

Applying Theorem 3 then gives

$$|\Delta_{we}|\varepsilon = 0.0398 < \frac{1}{2}. \quad (39)$$

Theorem 3 therefore states that $\mathcal{L}_2$-stability cannot be proved by the Popov criterion when $T \to \infty$, a fact that is also obvious

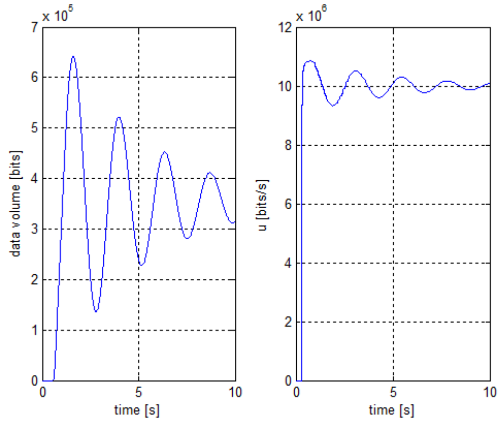

Fig. 7. Transmit queue data volume (left) and control signal (right), for $T = 0.5\ s$.

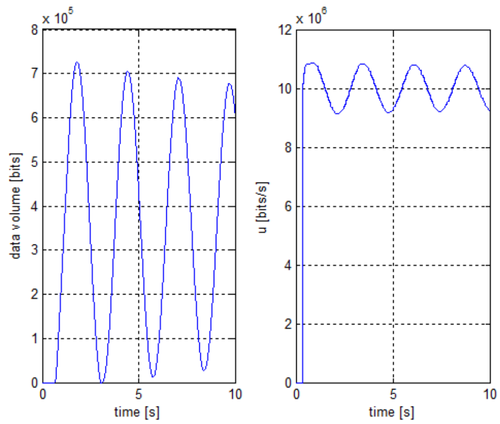

Fig. 8. Transmit queue data volume (left) and control signal (right), for $T = 0.6\ s$.

from Fig. 6. In order to evaluate the conclusion, simulations of the closed loop NCS were carried out with signals selected as described above. The simulations were carried out for $T = 0.5\ s$, $T = 0.6\ s$, $T = 0.7\ s$ and $T = 1.0\ s$. The results are depicted in Fig. 7, Fig. 8, Fig. 9 and Fig. 10, respectively. It can be observed that the system is stable for $T = 0.5\ s$ and $T = 0.6\ s$, but that a limit cycle occurs for $T = 0.7\ s$ and $T = 1.0\ s$. This is perfectly consistent with Theorem 3 and also exactly as predicted by Fig. 6.

VIII. Discussion

The results of the report that relate the relative static regulation accuracy to the low frequency gain of the system and the flow capacity in networked flow control were obtained from the Popov stability criterion. The reason why the Popov criterion was used is the saturation that was assumed to be present in the feedback loop. The analysis is however not believed to be limited to the case with saturation, also linear NCSs should be subject to similar tradeoffs, at least if additional constraints are imposed on the loop gain. It is for example intuitively clear that the Nyquist criterion [31] should lead to the same result

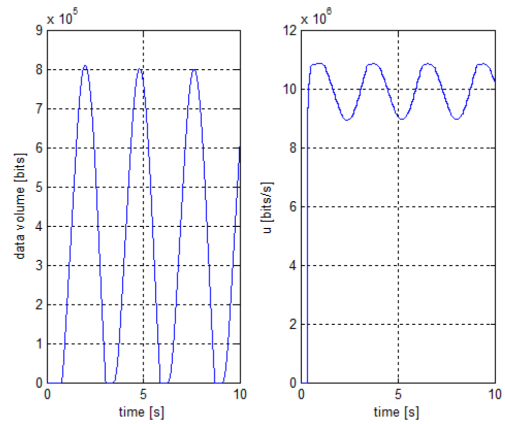

Fig. 9. Transmit queue data volume (left) and control signal (right), for $T = 0.7\ s$.

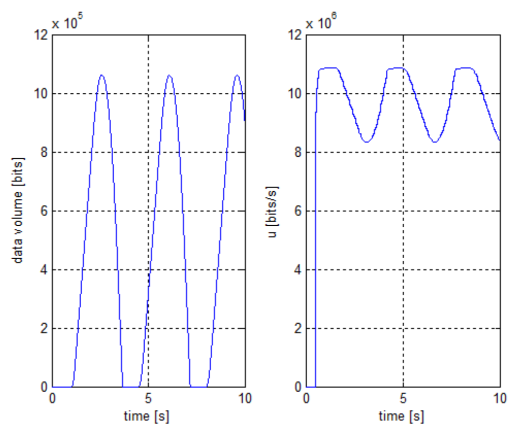

Fig. 10. Transmit queue data volume (left) and control signal (right), for $T = 1.0\ s$.

as in the present report in case the loop gain is a monotonically decreasing function of the angular frequency. Another reason why it is believed that it is the delay that is the key dynamic feature is the fact that there is no direct sign of the sector gain k of the nonlinearity in any of Theorem 1, Theorem 2 and Theorem 3.

A consequence of the report is obviously that caution is needed when integrating control is applied for systems with delay (and saturation). Integrating control is of course not excluded, however an integrator together with a large delay may require a careful stability analysis. Another consequence of the report is the perhaps surprising tradeoff between the static relative regulation accuracy and the flow capacity. This result builds on the assumption that the flow controlling NCS aims at controlling a volume in a terminal reservoir. In such situations the report proves that the regulation cannot be perfect in case an unmodelled disturbance is present. The relative static regulation accuracy must then be traded against the capacity of the flow that is controlled by the NCS.

The most important result of the report may be Theorem 3 that proves that there is a trade-off between the achievable static regulation accuracy of the wireless transmit data queue, and the achievable data rate as counted to the transmit data queue. The data rate reduction needed to comply with $\mathcal{L}_2$-stability of the flow control feedback loop is assumed to be obtained by data packet discarding, typically by application of AQM. In case the incoming maximum data rate is interpreted as a Shannon capacity [7], [21], Theorem 3 indicates a need for an additional reduction in case also closed loop $\mathcal{L}_2$-stability is required. The effect of the stability requirement therefore seems to provide another link to information theory [7]. At this point it is interesting to note that the data rate theorems of e.g. [3] and [17] are also focused on feedback stability, more precisely stabilization using a minimum amount of information of the quantized NCS signals. Delay is another effect that reduces the available state information used for feedback control.

There are two main limitations of the report. The first one follows since the applied controller is a linear one, therefore it is not yet clear if more advanced nonlinear control strategies could relax the obtained tradeoffs. There are many possible nonlinear methods that could be applied, e.g. adaptive control [2] or MPC [11], [36]. It is also noted that the plant should fit into the framework of positive systems, since one-directional flow rates and data volumes are inherently positive quantities. Therefore a closer look on fundamental limitations for positive system could be fruitful to enhance the understanding of the present problem, see e.g. [9]. Secondly, the Popov criterion is not a necessary condition for $\mathcal{L}_2$-stability. Therefore there may be regions where the system is stable, despite the fact that the Popov inequality is violated. The results of the report still hold though, because of their reference to the Popov criterion. Given the results of the present report, it would of course be very interesting to investigate what could be proved with IQC theory [16], also under alternative conditions like time varying delays [14]. It is not expected that the major conclusion of the report would be invalidated by such an analysis, the numerical example, the intuitive geometric frequency domain stability interpretation, and the discussion around the *necessary and sufficient* Nyquist criterion above in this section, all indicate that the trade-offs discussed in this publication are of a fundamental kind.

IX. Conclusions

The report analysed steady state properties of a nonlinear NCS, with a saturation in the loop when subject to very long delay. Three main conclusions were obtained, all related to the relative static regulation accuracy in the presence of an un-modelled disturbance. The first result shows that there is a trade-off between the relative static regulation accuracy and $\mathcal{L}_2$-stability as expressed in terms of the Popov criterion. More exactly, in case the relative static regulation accuracy is set to a too low value, i.e. too good, then stability does not follow from the Popov criterion. This is in line with previous results that state the same fact in case the low frequency gain is set too high, or the low frequency sensitivity is designed to be to small.

Based on this result general flow control of any medium was considered. It was proved that the product of the relative static regulation accuracy of the level of a terminating reservoir, and the leakage rates of the reservoirs in a chain to the terminating reservoir, needs to be less than $1/2$, or stability cannot be obtained with the Popov criterion. Small leakage and accurate static regulation therefore seems to be inconsistent with fulfilment of the Popov criterion.

The case of wireless internet data flow control was then considered. In that application the data volume of a single wireless transmit data queue is rate controlled by means of an incoming data rate, disturbed by the wireless outgoing data rate. For this NCS, it was proved that if the product of the data rate and the data discard rate associated with the queue is less than $1/2$, then the data flow control loop stability cannot be guaranteed by the Popov criterion. This result may be the most important one of the report since the stability requirement seems to require a data rate reduction counted to the transmit data queue, as compared to the Shannon capacity of the network interface that carries the incoming data flow. Finally, a numerical example illustrated and validated this result. It was shown that a limit cycle indeed results for long delay in a case where this is predicted to be a possibility by the results of the report.

Since the Popov criterion is not necessary for stability, an interesting topic for further research would be to apply more general tools, like IQC, to strengthen the results of this report. In addition to this it would be interesting to see if similar tradeoffs as in the present report exist when nonlinear control strategies like adaptive control, MPC, or the theory of positive systems are applied. One question is then if it is possible to obtain a generalization to non-static cases, e.g. enabling modeling and analysis of more general network interface properties. It would also be interesting to further investigate the connections to information theory that were indicated in the report. Reasons for this include the facts that long codes and long delay occur when studying channel capacity and that delay tends to limit the information available for feedback control.

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