Parametric Convergence Analysis of an Adaptive Filtering Algorithm with Output Quantization

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Abstract. *This report analyses global parametric convergence properties of a recursive algorithm for identification of linear finite impulse response (FIR) dynamics with quantized output measurements. The problem is addressed by analytic calculation of the right hand side of the associated ODE for a second order model. The ODE is then analysed numerically to investigate the possibilities for proving parametric convergence based on the coupling between the switch point and the dynamic gain.*

1. Introduction

System identification of plants with quantized measurement outputs is of importance in modeling of systems with limited sensor information, establishing relationships between communication resource limitations and identification complexity, and for studying sensor networks [5]. Other applications can be found in signal processing, for instance, in echo cancellation and blind quantization [10].

The parametric convergence properties of algorithms for identification of linear dynamics with quantized output measurements have been considered using stochastic averaging theory in [2] and [3]. The local convergence result for the infinite impulse response (IIR) model case is given in [7] where the use of an approximate gradient was introduced to handle the output quantization. See also [1], [6], [8] (and the references therein) for results on convergence of linear finite impulse response (FIR) based algorithms. The work [11] is based on [6] and [7], where Ljung's result were extended to cover identification algorithms based on Wiener models. The Wiener convergence results are then reused for the FIR based algorithm in [8]. That analysis considers sufficient conditions for global output error convergence to a linear FIR model that provides an exact input-output description of the data. The identification algorithm is implemented in the software package [9]. The paper [11] contributes with an analysis of sufficient conditions for global parametric convergence of the algorithm of [11]. Counterexamples are constructed to parametric convergence using low order FIR models and a binary output quantizer. It is shown that global parametric convergence does not occur in case the input signal distribution is discrete, and when the switch point and dynamic gain are such that there is no signal energy in the neighborhood of the switch point.

This report aims to extend the results given in [11] for investigating the global parametric convergence for a second order FIR dynamical model. The averaging analysis of [3], [4] is applied and the right hand side of the associated ordinary differential equation

(ODE) of the algorithm of the low order model is calculated analytically. The stationary points are then analysed using a numerical solution of the ODE, and the global parametric convergence properties are discussed [11].

The text is organized as follows. The adaptive filtering algorithm is given in Section 2. The method of analysis and the result on global convergence are reviewed in Section 3. The main results are provided in Section 4. Numerical results are given in Section 5 and finally the report ends with conclusions in Section 6.

2. Adaptive Filtering Algorithm

The model structure is given by

$$\hat{y}(t,\theta) = \theta^\top \varphi(t) \tag{1}$$

$$\varphi(t) = (u(t-1)\dots u(t-m))^\top \tag{2}$$

$$\theta = (b_1 \dots b_m)^\top, \tag{3}$$

where θ is the unknown parameter vector and $\varphi(t)$ is the regression vector. The quantizer $f_{n,\varepsilon,\hat{y}}(\hat{y}(t,\theta))$, generates the output of the model according to

$$\hat{y}_{n,\varepsilon}(t,\theta) = f_{n,\varepsilon,\hat{y}}(\hat{y}(t,\theta)) = f_{n,\varepsilon,\hat{y}}(\theta^\top \varphi(t)). \tag{4}$$

The quantizer is defined as in [8]. A conventional monotone quantizer $\bar{f}_{n,\hat{y},i}(\hat{y}) = C_i$, with constant levels $C_i, i = 0,\dots,N$, such that $\{C_i\}_0^N$ is increasing or decreasing is used here.

The system is assumed to be described by

$$y_{n,\varepsilon}(t) = f_{n,\varepsilon,y}(y(t)) + w(t) = f_{n,\varepsilon,y}\left(\theta_0^\top \varphi(t)\right) + w(t), \tag{5}$$

where $y(t)$ denotes the output signal, $w(t)$ is a zero mean correlated disturbance independent of $u(t)$, and θ_0 is the true parameter vector.

The identification algorithm is then derived by minimization of the criterion

$$V(\theta) = \frac{1}{2}\mathbb{E}\left[\varepsilon_{n,\varepsilon}(t,\theta)\right]^2, \tag{6}$$

where $\varepsilon_{n,\varepsilon}(t,\theta) = y_{n,\varepsilon}(t) - \hat{y}_{n,\varepsilon}(t,\theta)$ is the prediction error. The negative gradient $\psi_{n,\varepsilon}(t,\theta)$ of $\varepsilon_{n,\varepsilon}(t,\theta)$ with respect to the parameter vector θ is required in the algorithm and it contains the gradient of the quantizer. However, the gradient of the quantized output would be identically zero except at the quantization steps where it would contain Dirac pulses. Hence, an algorithm based on the exact gradient would not to work in practice. To sidestep this problem, a continuously differentiable approximate gradient

$$k_{n,\hat{y}}(\hat{y}(t,\theta)) \approx df_{n,\varepsilon,\hat{y}}(\hat{y}(t,\theta))/d\hat{y}, \tag{7}$$

is used instead. Proceeding as in [7], [8], the normalized stochastic gradient algorithm is

$$\begin{aligned}
\hat{\theta}(t) &= \Big[\hat{\theta}(t-1) \\
&\quad + \frac{\mu(t)}{t}\Big(\frac{1}{r(t-1)}k_{n,\hat{y}}(\hat{y}(t))\varphi(t)\left(y_{n,\varepsilon}(t)-\hat{y}_{n,\varepsilon}(t)\right)\Big)\Big]_{D_{\mathscr{M}}} \\
r(t) &= \Big[r(t-1) \\
&\quad + \frac{\mu(t)}{t}\Big(k^2_{n,\hat{y}}(\hat{y}(t))\varphi^\top(t)\varphi(t)-r(t-1)\Big)\Big]_{D_{\mathscr{M}}} \\
\varphi(t+1) &= (u(t)\ldots u(t-m+1))^\top \\
\hat{y}(t+1) &= \hat{\theta}^\top(t)\varphi(t+1) \\
\hat{y}_{n,\varepsilon}(t+1) &= f_{n,\varepsilon,\hat{y}}(\hat{y}(t+1)),
\end{aligned} \tag{8}$$

where $\hat{\theta}(t)$ is the parameter estimate, $\mu(t)/t$ is the gain sequence, and $D_{\mathscr{M}}$ indicates the projection into the model set in order to ensure that the estimates remain in the compact model set

$$D_{\mathscr{M}} = \{(\theta,r)|\ |b_i| \leq C < \infty, i=1,\ldots,m, 0<\delta_1 \leq r \leq C < \infty\}. \tag{9}$$

It is proved in [6] that projection according to (9) is allowed in the general algorithm treated by [3] and that the projection disappears in the averaged updating direction.

3. Method of Analysis

The stochastic averaging analysis of [3], [4] is used to study the convergence properties of (8). There, the updating equations of the algorithm are related to an associated deterministic ODE using a formal change of time scale. It is shown that the asymptotic paths of the recursive algorithm will closely follow the trajectories of the associated ODE. The right hand side of the associated ODE consists of the average updating direction of the algorithm that is computed for a *fixed* value of the parameter vector. It is proved in [3] that the local (global) stability of the associated ODE implies local (global) convergence of the algorithm.

Global output error convergence is proved in [8]. That analysis requires the following assumptions

A1) $m \geq m_0$, where $m_0 = \dim(\theta_0)$ and $m = \dim(\theta)$.

A2) $\theta_0 = (b_1^0 \ldots b_{m_0}^0 0 \ldots 0)^\top \in D_{\mathscr{M}} \backslash \partial D_{\mathscr{M}}$ where θ_0 has been filled out with zeros to make $m = m_0$.

A3) $f_{n,\varepsilon,\hat{y}}(\hat{y})$ and $k_{n,\hat{y}}(\hat{y})$ are *a priori* known, continuous, and continuously differentiable w.r.t $\hat{y}$ such that for some $C < \infty$ and $\theta \in D_{\mathscr{M}}, |f_{n,\varepsilon,\hat{y}}(\hat{y})| \leq C(1+|\hat{y}|), |df_{n,\varepsilon,\hat{y}}(\hat{y})/d\hat{y}| + |k_{n,\hat{y}}(\hat{y})| + |dk_{n,\hat{y}}(\hat{y})/d\hat{y}| \leq C$.

A4) $\lim_{t\to\infty}\mu(t) = \mu > 0$, and $0 \leq \mu(t) < C < \infty$.

A5) $u(t)$ and $w(t)$ are realizations of bounded stationary stochastic processes. $u(t)$ is *exponentially uncorrelated* in the sense that for each $t,s,t>s$, there exits a random vector $u_s^0(t)$ ($u_s^0(s)=0$) that belongs to the $\sigma-$algebra generated by $(u(0)\ldots u(t))$ but is independent of $(u(0)\ldots u(s))$ such that $\mathbb{E}\left[|u(t)-u_s^0(t)|^4\right] \leq C\lambda^{t-s}, C<\infty, \lambda<1$. $w(t)$ is *exponentially uncorrelated* in the sense above, has zero mean and is independent of $u(t)$.

A6) $f_{n,\varepsilon,\hat{y}}(\hat{y})$ is increasing (not necessarily strictly increasing) and $k_{n,\hat{y}}(\hat{y}) \geq \delta_2 > 0$ *or* $f_{n,\varepsilon,\hat{y}}(\hat{y})$ is decreasing (not necessarily strictly decreasing) and $k_{n,\hat{y}}(\hat{y}) \leq -\delta_2 < 0$.

Then consider the following set given in [8]

$$D_{C\theta} = \{\theta | \lim_{t\to\infty} \mathbb{E}[|\hat{y}(t,\theta) - y(t)| \cdot |f_{n,\varepsilon,\hat{y}}(\hat{y}(t,\theta)) - f_{n,\varepsilon,y}(y(t))|]_{\theta=\theta_D} = 0\}. \tag{10}$$

$D_{C\theta}$ defines the possible convergence point of the algorithm. The set $D_{C\theta}$ disregards the fact that the algorithm cannot converge to locally unstable stationary points of the ODE, as stated in [3]. This fact is shown in [11] where a complete analysis is presented for a single parameter case. The result on output error convergence in [8], extended to account for local stability, is then

Lemma 1 *Assume that A1)-A6) hold for* (8). *Assume that there exists a twice differentiable, positive definite function $V(\theta,r)$ such that $dV(\theta_D(\tau),r_D(\tau))/d\tau \leq 0$ for $(\theta_D,r_D) \in D_{\mathscr{M}}\backslash\partial D_{\mathscr{M}}$, when evaluated along the solutions to*

$$\begin{aligned} \frac{d}{d\tau}\theta_D(\tau) &= \frac{\mu}{r_D(\tau)} f(\theta_D(\tau)) \\ \frac{d}{d\tau} r_D(\tau) &= \mu(g(\theta_D(\tau)) - r_D(\tau)), \end{aligned} \tag{11}$$

where $f(\theta_D)$ and $g(\theta_D)$ are defined as

$$\begin{aligned} f(\theta_D) &= \lim_{t\to\infty} \mathbb{E}\Big[k_{n,\hat{y}}(\hat{y}(t,\theta))(y_{n,\varepsilon}(t) - \hat{y}_{n,\varepsilon}(t,\theta))\varphi(t)\Big]_{|\theta=\theta_D} \\ g(\theta_D) &= \lim_{t\to\infty} \mathbb{E}[k^2_{n,\hat{y}}(\hat{y}(t,\theta))\varphi^\top(t)\varphi(t)]_{|\theta=\theta_D} > 0. \end{aligned} \tag{12}$$

Then, $\hat{\theta}(t) \to D_{C\theta} \subset \{\theta \in D_{\mathscr{M}}\backslash\partial D_{\mathscr{M}} | f_{n,\varepsilon,\hat{y}}(\hat{y}(t,\theta)) = f_{n,\varepsilon,y}(y(t)), eig_i(\partial f(\theta))/\partial\theta) \leq 0,\ i = 1,...,m,\}$ w.p.1 as $t \to \infty$, or there is a cluster point on $\partial D_{\mathscr{M}}$.

Proof 1 *The result follows from Corollary 1 of [8], using the fact that all conditions of [6] hold. The condition $eig_i(\partial f(\theta))/\partial\theta \leq 0$ follow since the stochastic gradient algorithm renders a scalar $r_D(\tau) > 0$. $eig_i(.)$ denotes the ith eigenvalue of a given matrix.*

4. Main Results

To investigate the possibilities for proving parametric convergence of (8), a second order FIR systems in combination with a binary output quantizer is defined. Second, the associated ODEs are derived. The subscript ε is dropped in the sequel.

Consider a two parameter FIR system with uniformly distributed i.i.d. input $u \sim \mathscr{U}(-u_0,u_0)$, Gaussian i.i.d. disturbance $w \sim \mathscr{N}(0,\sigma^2)$, binary quantizer with levels -1 and 1 and switch point f_0. The system and model are

$$\begin{aligned} y_n(t) &= f_{n,y}\left(b_1^0 u(t-1) + b_2^0 u(t-2)\right) + w(t) \\ \hat{y}_n(t,\theta) &= f_{n,\hat{y}}\left(b_1 u(t-1) + b_2 u(t-2)\right), \end{aligned} \tag{13}$$

with $b_1^0 = 1.0$, $b_2^0 = 1.0$ and $k_{n,\hat{y}}(\hat{y}(t,\theta)) = k_0 = 1.0$.

The associated ODEs corresponding to the test model are now derived by evaluation of the average updating direction (12). The evaluation of the average updating direction for the model, requires that all possible situations where the switch point is either inside or outside the interval $[-u_0, u_0]$ are considered.

The average updating direction for (13) becomes

$$f(\theta) = k_0 \begin{pmatrix} \lim_{t\to\infty} \mathbb{E}u(t-1) f_{n,y}(y(t)) \\ \lim_{t\to\infty} \mathbb{E}u(t-2) f_{n,y}(y(t)) \end{pmatrix} - k_0 \begin{pmatrix} \lim_{t\to\infty} \mathbb{E}u(t-1) f_{n,\hat{y}}(\hat{y}(t,\theta)) \\ \lim_{t\to\infty} \mathbb{E}u(t-2) f_{n,\hat{y}}(\hat{y}(t,\theta)) \end{pmatrix}. \tag{14}$$

Denote

$$\begin{aligned} \mathscr{T}_1 &= \lim_{t\to\infty} \mathbb{E}u(t-1) f_{n,y}(y(t)) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_1 f_{n,y}(u_1+u_2) P_{\mathscr{U}_1}(u_1) P_{\mathscr{U}_2}(u_2) du_1 du_2, \end{aligned} \tag{15}$$

$$\begin{aligned} \mathscr{T}_2 &= \lim_{t\to\infty} \mathbb{E}u(t-1) f_{n,\hat{y}}(\hat{y}(t,\theta)) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_1 f_{n,\hat{y}}(b_1u_1+b_2u_2) P_{\mathscr{U}_1}(u_1) P_{\mathscr{U}_2}(u_2) du_1 du_2, \end{aligned} \tag{16}$$

$$\begin{aligned} \mathscr{T}_3 &= \lim_{t\to\infty} \mathbb{E}u(t-2) f_{n,y}(y(t)) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_2 f_{n,y}(u_1+u_2) P_{\mathscr{U}}(u_1) P_{\mathscr{U}_2}(u_2) du_1 du_2, \end{aligned} \tag{17}$$

$$\begin{aligned} \mathscr{T}_4 &= \lim_{t\to\infty} \mathbb{E}u(t-2) f_{n,\hat{y}}(\hat{y}(t,\theta)) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_2 f_{n,\hat{y}}(b_1u_1+b_2u_2) P_{\mathscr{U}_1}(u_1) P_{\mathscr{U}_2}(u_2) du_1 du_2. \end{aligned} \tag{18}$$

$\mathscr{T}_1$ can be written as

$$\mathscr{T}_1 = \int_{-\infty}^{\infty} u_1 P_{\mathscr{U}_1}(u_1) \underbrace{\left(\int_{-\infty}^{\infty} P_{\mathscr{U}_2}(u_2) f_{n,y}(u_1+u_2) du_2\right)}_{I_2} du_1. \tag{19}$$

Evaluating $f_{n,y}(u_1+u_2)$ for the cases of

$$f_{n,y}(u_1+u_2) = \begin{cases} 1 & u_1+u_2 > f_0 \\ -1 & u_1+u_2 < f_0 \end{cases}, \tag{20}$$

yields

$$I_2 = \int_{-\infty}^{f_0-u_1} (-1) P_{\mathscr{U}_2}(u) du_2 + \int_{f_0-u_1}^{\infty} (1) P_{\mathscr{U}_2}(u_2) du_2. \tag{21}$$

Therefore

$$I_2 = \begin{cases} 1 & f_0 - u_1 < -u_0 \\ \frac{u_1-f_0}{u_0} & |f_0-u_1| \le u_0 \\ -1 & f_0-u_1 > u_0 \end{cases}. \tag{22}$$

Substituting (22) into (19) gives

$$\begin{aligned}\mathscr{T}_1 &= \int_{-\infty}^{f_0-u_0} (-1)u_1 P_{\mathscr{U}_1} du_1 \\ &+ \int_{f_0-u_0}^{f_0+u_0} (\frac{u_1-f_0}{u_0}) u_1 P_{\mathscr{U}_1(u_1)} du_1 + \int_{f_0+u_0}^{\infty} (1) u_1 P_{\mathscr{U}_1}(u_1) du_1.\end{aligned} \tag{23}$$

Next, $\mathscr{T}_1$ is calculated for all possible cases where the switch points are either inside or outside the support interval of the input signal

i) $f_0 < 0$

$$\mathscr{T}_1 = \frac{1}{6u_0^2}\left(k_2^3 + u_0^3\right) - \frac{f_0}{4u_0^2}\left(k_2^2 - u_0^2\right) + \frac{1}{4u_0}\left(u_0^2 - k_2^2\right), \tag{24}$$

ii) $f_0 > 0$

$$\mathscr{T}_1 = \frac{1}{6u_0^2}\left(u_0^3 - k_1^3\right) - \frac{f_0}{4u_0^2}\left(u_0^2 - k_1^2\right) + \frac{1}{4u_0}\left(u_0^2 - k_1^2\right), \tag{25}$$

where

$$k_1 = f_0 - u_0, \quad k_2 = f_0 + u_0. \tag{26}$$

Similarly, $\mathscr{T}_3$ is obtained. Summarizing the results gives

$$\begin{aligned}&\mathscr{T}_1 = \mathscr{T}_3 \\ &= \begin{cases} \frac{1}{6u_0^2}\left(k_2^3 + u_0^3\right) - \frac{f_0}{4u_0^2}\left(k_2^2 - u_0^2\right) + \frac{1}{4u_0}\left(u_0^2 - k_2^2\right) & f_0 < 0 \\ \frac{1}{6u_0^2}\left(u_0^3 - k_1^3\right) - \frac{f_0}{4u_0^2}\left(u_0^2 - k_1^2\right) + \frac{1}{4u_0}\left(u_0^2 - k_1^2\right) & f_0 > 0 \end{cases}.\end{aligned} \tag{27}$$

Next, $\mathscr{T}_1$ and $\mathscr{T}_4$ are calculated for the cases below

$$\begin{cases} b_1 = b_2 = 0 \\ b_1 > 0, b_2 = 0 \\ b_1 < 0, b_2 = 0 \\ b_1 = 0, b_2 > 0 \\ b_1 = 0, b_2 < 0 \\ b_1 = b_2 = b, b > 0 \\ b_1 = b_2 = b, b < 0 \\ b_1 > 0, b_2 > 0, b_1 \neq b_2 \\ b_1 < 0, b_2 < 0, b_1 \neq b_2 \\ b_1 > 0, b_2 < 0 \\ b_1 < 0, b_2 > 0 \end{cases}. \tag{28}$$

i) $b_1 = b_2 = 0$:

$$\mathscr{T}_2 = \mathscr{T}_4 = 0. \tag{29}$$

ii) $b_1 \neq 0, b_2 = 0$:

$$\mathscr{T}_2 = \mathscr{T}_4 = \int_{-\infty}^{\infty} u_1 f_{n,\hat{y}}(b_1 u_1) P_{\mathscr{U}_1}(u_1) du_1 \tag{30}$$

$$= \begin{cases} 0 & |\frac{f_0}{b_1}| > u_0 \\ \frac{1}{2u_0}\left(u_0^2 - (\frac{f_0}{b_1})^2\right) & |\frac{f_0}{b_1}| \leq u_0, b_1 > 0 \\ -\frac{1}{2u_0}\left(u_0^2 - (\frac{f_0}{b_1})^2\right) & |\frac{f_0}{b_1}| \leq u_0, b_1 < 0 \end{cases}, \tag{31}$$

iii) $b_1 = 0, b_2 \neq 0$:

$$\mathscr{T}_2 = \mathscr{T}_4 = \int_{-\infty}^{\infty} u_2 f_{n,\hat{y}}(b_2 u_2) P_{\mathscr{U}_2}(u_2) du_2 \tag{32}$$

$$= \begin{cases} 0 & |\frac{f_0}{b_2}| > u_0 \\ \frac{1}{2u_0}\left(u_0^2 - (\frac{f_0}{b_2})^2\right) & |\frac{f_0}{b_2}| \leq u_0, b_2 > 0 \\ -\frac{1}{2u_0}\left(u_0^2 - (\frac{f_0}{b_2})^2\right) & |\frac{f_0}{b_2}| \leq u_0, b_2 < 0 \end{cases}. \tag{33}$$

iv) $b_1 = b_2 = b > 0$:

$$\mathscr{T}_2 = \mathbb{E} f_{n,\hat{y}}(b(u_1 + u_2)) u_1 \tag{34}$$

$$= \int_{-\infty}^{\infty} u_1 P_{\mathscr{U}_1}(u_1) \underbrace{\left(\int_{-\infty}^{\infty} P_{\mathscr{U}_2}(u_2) f_{n,\hat{y}}(b(u_1+u_2)) du_2\right)}_{I_2} du_1 \tag{35}$$

Calculating I_2 for the cases of

$$f_{n,\hat{y}}(b(u_1+u_2)) = \begin{cases} 1 & b(u_1+u_2) > f_0 \\ -1 & b(u_1+u_2) < f_0 \end{cases} \tag{36}$$

yields

$$I_2 = \begin{cases} 1 & u_1 > (\frac{f_0}{b} + u_0) \\ \frac{1}{u_0}(u_1 - \frac{f_0}{b}) & (\frac{f_0}{b} - u_0) \leq u_1 \leq (\frac{f_0}{b} + u_0) \\ -1 & u_1 < (\frac{f_0}{b} - u_0) \end{cases} \tag{37}$$

Substituting (37) into (34) gives

$$\mathscr{T}_2 = \int_{-\infty}^{l_1} (-1) u_1 P_{\mathscr{U}_1}(u_1) du_1 \tag{38}$$

$$+ \int_{l_1}^{l_2} \frac{1}{u_0}(u_1^2 - \frac{f_0}{b} u_1) P_{\mathscr{U}_1}(u_1) du_1 + \int_{l_2}^{\infty} (1) u_1 P_{\mathscr{U}_1}(u_1) du_1 \tag{39}$$

where $l_1 = \frac{f_0}{b_1} - u_0, l_2 = \frac{f_0}{b_1} + u_0$. Next, $\mathscr{T}_2$ is calculated for all possible cases where the switch point is either inside or outside the interval where the input is non-zero. The same result is obtained for $\mathscr{T}_4$

$$\mathscr{T}_2 = \mathscr{T}_4 = \begin{cases} 0 & l_2 < -u_0 \\ t_1 & |l_2| \leq u_0, l_1 < -u_0 \\ t_2 & |l_1| \leq u_0, l_2 > u_0 \\ 0 & l_1 > u_0 \end{cases}, \tag{40}$$

where

$$t_1 = \frac{1}{6u_0^2}\left(l_2^3 + u_0^3\right) - \frac{f_0}{4u_0^2 b}\left(l_2^2 - u_0^2\right) + \frac{1}{4u_0}\left(u_0^2 - l_2^2\right), \tag{41}$$

$$t_2 = \frac{1}{6u_0^2}\left(u_0^3 - l_1^3\right) - \frac{f_0}{4u_0^2 b}\left(u_0^2 - l_1^2\right) + \frac{1}{4u_0}\left(u_0^2 - l_1^2\right). \tag{42}$$

v) $b_1 = b_2 = b < 0$:

$$\mathscr{T}_2 = \mathscr{T}_4 = \begin{cases} 0 & l_2 < -u_0 \\ -t_1 & |l_2| \leq u_0, l_1 < -u_0 \\ -t_2 & |l_1| \leq u_0, l_2 > u_0 \\ 0 & l_1 > u_0 \end{cases}. \tag{43}$$

vi) $b_1 > 0, b_2 > 0, b_1 \neq b_2$:

$$\mathscr{T}_2 = \begin{cases} 0 & n_2 < -u_0 \\ t_3 & |n_2| \leq u_0, n_1 < -u_0 \\ t_4 & |n_2| \leq u_0, |n_1| \leq u_0 \\ t_5 & |n_1| \leq u_0, n_2 > u_0 \\ 0 & n_1 > u_0 \\ \frac{b_1 u_0}{3b_2} & n_1 < -u_0, n_2 > u_0 \end{cases}, \tag{44}$$

$$\mathscr{T}_4 = \begin{cases} 0 & m_2 < -u_0 \\ t_6 & |m_2| \leq u_0, m_1 < -u_0 \\ t_7 & |m_2| \leq u_0, |m_1| \leq u_0 \\ t_8 & |m_1| \leq u_0, m_2 > u_0 \\ 0 & m_1 > u_0 \\ \frac{b_2 u_0}{3b_1} & m_1 < -u_0, m_2 > u_0 \end{cases}, \tag{45}$$

where

$$t_3 = \frac{1}{4u_0}\left(u_0^2 - n_2^2\right) - \frac{f_0}{4b_2 u_0^2}\left(n_2^2 - u_0^2\right) + \frac{b_1}{6b_2 u_0^2}\left(n_2^3 + u_0^3\right), \tag{46}$$

$$t_4 = \frac{u_0}{2} - \frac{1}{4u_0}\left(n_2^2 + n_1^2\right) - \frac{f_0}{4b_2 u_0^2}\left(n_2^2 - n_1^2\right) + \frac{b_1}{6b_2 u_0^2}\left(n_2^3 - n_1^3\right), \tag{47}$$

$$t_5 = \frac{1}{4u_0}\left(u_0^2 - n_1^2\right) - \frac{f_0}{4b_2 u_0^2}\left(u_0^2 - n_1^2\right) + \frac{b_1}{6b_2 u_0^2}\left(u_0^3 - n_1^3\right), \tag{48}$$

$$t_6 = \frac{1}{4u_0}\left(u_0^2 - m_2^2\right) - \frac{f_0}{4b_1 u_0^2}\left(m_2^2 - u_0^2\right) + \frac{b_2}{6b_1 u_0^2}\left(m_2^3 + u_0^3\right), \tag{49}$$

$$t_7 = \frac{u_0}{2} - \frac{1}{4u_0}\left(m_2^2 + m_1^2\right) - \frac{f_0}{4b_1u_0^2}\left(m_2^2 - m_1^2\right) + \frac{b_2}{6b_1u_0^2}\left(m_2^3 - m_1^3\right), \tag{50}$$

$$t_8 = \frac{1}{4u_0}\left(u_0^2 - m_1^2\right) - \frac{f_0}{4b_1u_0^2}\left(u_0^2 - m_1^2\right) + \frac{b_2}{6b_1u_0^2}\left(u_0^3 - m_1^3\right), \tag{51}$$

with

$$\begin{aligned} n_1 &= \frac{(f_0 - b_2u_0)}{b_1}, \quad n_2 = \frac{(f_0 + b_2u_0)}{b_1}, \\ m_1 &= \frac{(f_0 - b_1u_0)}{b_2}, \quad m_2 = \frac{(f_0 + b_1u_0)}{b_2}. \end{aligned} \tag{52}$$

vii) $b_1 < 0, b_2 < 0, b_1 \neq b_2$:

$$\mathscr{T}_2 = \begin{cases} 0 & n_2 < -u_0 \\ -t_3 & |n_2| \leq u_0, n_1 < -u_0 \\ -t_4 & |n_2| \leq u_0, |n_1| \leq u_0 \\ -t_5 & |n_1| \leq u_0, n_2 > u_0 \\ 0 & n_1 > u_0 \\ -\frac{b_1u_0}{3b_2} & n_1 < -u_0, n_2 > u_0 \end{cases}, \tag{53}$$

$$\mathscr{T}_4 = \begin{cases} 0 & m_2 < -u_0 \\ -t_6 & |m_2| \leq u_0, m_1 < -u_0 \\ -t_7 & |m_2| \leq u_0, |m_1| \leq u_0 \\ -t_8 & |m_1| \leq u_0, m_2 > u_0 \\ 0 & m_1 > u_0 \\ -\frac{b_2u_0}{3b_1} & m_1 < -u_0, m_2 > u_0 \end{cases}. \tag{54}$$

viii) $b_1 > 0, b_2 < 0$:

$$\mathscr{T}_2 = \begin{cases} 0 & n_1 < -u_0 \\ t_9 & |n_1| \leq u_0, n_2 < -u_0 \\ t_4 & |n_1| \leq u_0, |n_2| \leq u_0 \\ t_{10} & |n_2| \leq u_0, n_1 > u_0 \\ 0 & n_2 > u_0 \\ -\frac{b_1u_0}{3b_2} & n_2 < -u_0, n_1 > u_0 \end{cases}, \tag{55}$$

$$\mathscr{T}_4 = \begin{cases} 0 & m_1 < -u_0 \\ t_{11} & |m_1| \leq u_0, m_2 < -u_0 \\ t_7 & |m_1| \leq u_0, |m_2| \leq u_0 \\ t_{12} & |m_2| \leq u_0, m_1 > u_0 \\ 0 & m_2 > u_0 \\ -\frac{b_2u_0}{3b_1} & m_2 < -u_0, m_1 > u_0 \end{cases}, \tag{56}$$

where

$$t_9 = \frac{1}{4u_0}\left(u_0^2 - n_1^2\right) + \frac{f_0}{4b_2u_0^2}\left(n_1^2 - u_0^2\right) - \frac{b_1}{6b_2u_0^2}\left(n_1^3 + u_0^3\right), \tag{57}$$

$$t_{10} = \frac{1}{4u_0}\left(u_0^2 - n_2^2\right) + \frac{f_0}{4b_2u_0^2}\left(u_0^2 - n_2^2\right) - \frac{b_1}{6b_2u_0^2}\left(u_0^3 - n_2^3\right), \tag{58}$$

$$t_{11} = \frac{1}{4u_0}\left(u_0^2 - m_1^2\right) + \frac{f_0}{4b_1u_0^2}\left(m_1^2 - u_0^2\right) - \frac{b_2}{6b_1u_0^2}\left(m_1^3 + u_0^3\right), \tag{59}$$

$$t_{12} = \frac{1}{4u_0}\left(u_0^2 - m_2^2\right) + \frac{f_0}{4b_1u_0^2}\left(u_0^2 - m_2^2\right) - \frac{b_2}{6b_1u_0^2}\left(u_0^3 - m_2^3\right). \tag{60}$$

ix) $b_1 < 0, b_2 > 0$:

$$\mathscr{T}_2 = \begin{cases} 0 & n_1 < -u_0 \\ -t_9 & |n_1| \le u_0, n_2 < -u_0 \\ -t_4 & |n_1| \le u_0, |n_2| \le u_0 \\ -t_{10} & |n_2| \le u_0, n_1 > u_0 \\ 0 & n_2 > u_0 \\ \frac{b_1u_0}{3b_2} & n_2 < -u_0, n_1 > u_0 \end{cases}, \tag{61}$$

$$\mathscr{T}_4 = \begin{cases} 0 & m_1 < -u_0 \\ -t_{11} & |m_1| \le u_0, m_2 < -u_0 \\ -t_7 & |m_1| \le u_0, |m_2| \le u_0 \\ -t_{12} & |m_2| \le u_0, m_1 > u_0 \\ 0 & m_2 > u_0 \\ \frac{b_2u_0}{3b_1} & m_2 < -u_0, m_1 > u_0 \end{cases}. \tag{62}$$

The final expressions for$\mathscr{T}_2, \mathscr{T}_4$ are given in Fig. 1 and Fig. 2, respectively,

The expression for $g(\theta)$ is readily obtained as

$$g(\theta) = k_0^2 \lim_{t\to\infty} \mathbb{E}\left[u(t-1)^2 + u(t-2)^2\right] = \frac{2(k_0u_0)^2}{3}. \tag{63}$$

For the low order model with one parameter, it is shown in [11] that provided that $0 < |f_0| \le u_0$, it follows that the algorithm (8) is globally convergent to the true parameter vector b_1^0. The following theorem given in [11], states this result for the one parameter case.

Theorem 1 *[11] Assume that A1) and A6) hold for the algorithm* (8). *Then, if* $0 < |f_0| \le u_0$ *holds,* $\hat{b}_1(t) \to b_1^0$ *w.p.1 as* $t \to \infty$, *or there is a cluster point on (9).*

The condition $0 < |f_0| \le u_0$ requires that the switch point of the quantizer is not zero. The problem is likely to be caused by the symmetry between the quantizer and the input signal distribution.

5. Numerical Solution

In this section, the result of Theorem 1 for the model (13) is illustrated numerically.

The associated ODE (11) was solved numerically for different initial conditions for the model (13). The output quantizer used a switch point of $f_0 = 0.5$. The noise had a standard deviation of 1.0. The solutions are shown in Fig. 3 as function of time. The 3-dimensional phase diagrams for the model (13) are plotted in Fig. 4 a). It can be seen that the point $(b_1^* \; b_2^* \; r_D^*)^\top = (1.0 \; 1.0 \; 0.6667)^\top$ appears to be a globally asymptotically stable stationary point of the ODE. Correspondingly initialized simulations of the algorithm, with $\mu(t) = 8/(50+t)$, running from $t = 20000$ to 300000 are depicted in Fig. 4 b).

Comparing with Fig. 4 a) it is observed that the asymptotic paths of the algorithm stay close to the trajectories of the ODE as predicted by the averaging theory [3].

5.1. Counterexample - Switch point approaching zero

In this set of simulations, the case where the switch point approaches zero is treated. Solutions of the ODE are obtained numerically for switch point values that are selected increasingly close to zero. The noise had a standard deviation of 0.5.

The switch points of the output quantizer were selected as $f_0 = 0.7, 0.5, 0.3, 0.1$. Fig. 5 and Fig. 6 show the trajectories of the associated ODEs and the asymptotic paths of the algorithm, respectively. The algorithm was run for 50000000 steps, with $\mu = 30/(50+t)$. As can be seen, the trajectories behave in a more singular manner when the switch point approaches zero, approaching a one dimensional asymptotic low gain convergence. This is a clear indication that the parameters cannot be estimated when $f_0 = 0$ as also is shown in Theorem 1 [11] for the one parameter model.

5.2. Counterexample - Reduced signal energy

Finally, we consider the case where the signal energy in the neighborhood of the switch point is reduced. The switch point was $f_0 = 0.5$ and the noise had the standard deviation 1.0.

The dynamic gains for the model (13) were selected as $b_1^0 = 0.7, 0.4, 0.3, 0.1$ and $b_2^0 = 0.8, 0.3, 0.2, 0.1$. The trajectories of the associated ODEs and the asymptotic paths of the algorithm are illustrated in Fig. 7 and Fig. 8, respectively. The algorithm was run for 1000000 steps with $\mu = 8/(50+t)$. Is is observed that by decreasing the dynamic gain of the system the trajectories fail to converge to the stationary point due to lack of enough input signal energy in the neighborhood of the step of the quantizer. This occurs when $b_1^0 + b_2^0 < 0.5$. The same result can be seen from Fig. 8 for the asymptotic paths of the algorithm.

6. Discussion and Conclusions

6.1. Discussion

As it is shown by the counterexample 5.2, if there is no energy in the neighborhood of the switch point, convergence does not follow. The condition on the signal energy in the switch point seems to be of crucial importance. Moreover, it was illustrated by the counterexample 5.1 that the symmetry situation between the binary quantizer and the distribution of $y(t)$ can also lead to failure of parametric convergence. Thus the binary quantizer requires $\mathbb{E}y(t) \neq f_0$. This means that any sufficient conditions for parametric convergence at least need to include S1 and S2 given by:

S1) The probability distribution $p_y(Y)$ of $y(t)$ fulfills $p_y(Y) > \delta > 0$ in a neighborhood of at least one switch point of the quantizer.

S2) In the binary quantizer case, $\mathbb{E}y(t) \neq f_0$, for symmetric input distributions, where f_0 denotes the switch point.

6.2. Conclusions

The report investigated sufficient conditions for global parametric convergence in identification of linear systems with qunatized output measurements. The new conditions were studied and found by defining a second order system with two parameters and a binary quantizer function. The associated ODEs were computed analytically and were analysed by simulation. Two counterexamples to parametric convergence were used to formulate restricting sufficient conditions that need to be fulfilled to derive results on global parametric convergence. These conditions appears to be fundamental for the related identification algorithms.

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$$
\mathscr{T}_2 = \begin{cases}
0 & b_1 = b_2 = 0 \\
0 & b_1 \neq 0, b_2 = 0, |f_0/b_1| > 0 \\
\frac{1}{2u_0}(u_0^2 - (\frac{f_0}{b_1})^2) & b_1 > 0, b_2 = 0, |f_0/b_1| \leq 0 \\
-\frac{1}{2u_0}(u_0^2 - (\frac{f_0}{b_1})^2) & b_1 < 0, b_2 = 0, |f_0/b_1| \leq 0 \\
0 & b_1 = 0, b_2 \neq 0, |f_0/b_2| > 0 \\
\frac{1}{2u_0}(u_0^2 - (\frac{f_0}{b_2})^2) & b_1 = 0, b_2 > 0, |f_0/b_2| \leq 0 \\
-\frac{1}{2u_0}(u_0^2 - (\frac{f_0}{b_2})^2) & b_1 = 0, b_2 < 0, |f_0/b_2| \leq 0 \\
0 & b_1 = b_2 = b \neq 0, l_2 < -u_0 \\
t_1 & b_1 = b_2 = b > 0, |l_2| \leq u_0, l_1 < -u_0 \\
t_2 & b_1 = b_2 = b > 0, |l_1| \leq u_0, l_2 > u_0 \\
0 & b_1 = b_2 = b > 0, l_1 > u_0 \\
-t_1 & b_1 = b_2 = b < 0, |l_2| \leq u_0, l_1 < -u_0 \\
-t_2 & b_1 = b_2 = b < 0, |l_1| \leq u_0, l_2 > u_0 \\
0 & b_1 = b_2 = b < 0, l_1 > u_0 \\
0 & b_1 > 0, b_2 > 0, b_1 \neq b_2, n_2 < -u_0 \\
t_3 & b_1 > 0, b_2 > 0, b_1 \neq b_2, |n_2| \leq u_0, n_1 < -u_0 \\
t_4 & b_1 > 0, b_2 > 0, b_1 \neq b_2, |n_1| \leq u_0, |n_2| \leq u_0 \\
t_5 & b_1 > 0, b_2 > 0, b_1 \neq b_2, |n_1| \leq u_0, n_2 > u_0 \\
0 & b_1 > 0, b_2 > 0, b_1 \neq b_2, n_1 > u_0 \\
\frac{b_1 u_0}{3b_2} & b_1 > 0, b_2 > 0, b_1 \neq b_2, n_1 < -u_0, n_2 > u_0 \\
0 & b_1 < 0, b_2 < 0, b_1 \neq b_2, n_2 < -u_0 \\
-t_3 & b_1 < 0, b_2 < 0, b_1 \neq b_2, |n_2| \leq u_0, n_1 < -u_0 \\
-t_4 & b_1 < 0, b_2 < 0, b_1 \neq b_2, |n_1| \leq u_0, |n_2| \leq u_0 \\
-t_5 & b_1 < 0, b_2 < 0, b_1 \neq b_2, |n_1| \leq u_0, n_2 > u_0 \\
0 & b_1 < 0, b_2 < 0, b_1 \neq b_2, n_1 > u_0 \\
-\frac{b_1 u_0}{3b_2} & b_1 < 0, b_2 < 0, b_1 \neq b_2, n_1 < -u_0, n_2 > u_0 \\
0 & b_1 > 0, b_2 < 0, n_1 < -u_0 \\
t_9 & b_1 > 0, b_2 < 0, |n_1| \leq u_0, n_2 < -u_0 \\
t_4 & b_1 > 0, b_2 < 0, |n_1| \leq u_0, |n_2| \leq u_0 \\
t_{10} & b_1 > 0, b_2 < 0, |n_2| \leq u_0, n_1 > u_0 \\
0 & b_1 > 0, b_2 < 0, n_2 > u_0 \\
-\frac{b_1 u_0}{3b_2} & b_1 > 0, b_2 < 0, n_2 < -u_0, n_1 > u_0 \\
0 & b_1 < 0, b_2 > 0, n_1 < -u_0 \\
-t_9 & b_1 < 0, b_2 > 0, |n_1| \leq u_0, n_2 < -u_0 \\
-t_4 & b_1 < 0, b_2 > 0, |n_1| \leq u_0, |n_2| \leq u_0 \\
-t_{10} & b_1 < 0, b_2 > 0, |n_2| \leq u_0, n_1 > u_0 \\
0 & b_1 < 0, b_2 > 0, n_2 > u_0 \\
\frac{b_1 u_0}{3b_2} & b_1 < 0, b_2 > 0, n_2 < -u_0, n_1 > u_0
\end{cases},
$$

Fig. 1: The evaluated expression for $\mathscr{T}_2$ in (14).

$$
\mathscr{T}_4 = \begin{cases}
0 & b_1 = b_2 = 0 \\
0 & b_1 \neq 0, b_2 = 0, |f_0/b_1| > 0 \\
\frac{1}{2u_0}(u_0^2 - (\frac{f_0}{b_1})^2) & b_1 > 0, b_2 = 0, |f_0/b_1| \leq 0 \\
-\frac{1}{2u_0}(u_0^2 - (\frac{f_0}{b_1})^2) & b_1 < 0, b_2 = 0, |f_0/b_1| \leq 0 \\
0 & b_1 = 0, b_2 \neq 0, |f_0/b_2| > 0 \\
\frac{1}{2u_0}(u_0^2 - (\frac{f_0}{b_2})^2) & b_1 = 0, b_2 > 0, |f_0/b_2| \leq 0 \\
-\frac{1}{2u_0}(u_0^2 - (\frac{f_0}{b_2})^2) & b_1 = 0, b_2 < 0, |f_0/b_2| \leq 0 \\
0 & b_1 = b_2 = b \neq 0, l_2 < -u_0 \\
t_1 & b_1 = b_2 = b > 0, |l_2| \leq u_0, l_1 < -u_0 \\
t_2 & b_1 = b_2 = b > 0, |l_1| \leq u_0, l_2 > u_0 \\
0 & b_1 = b_2 = b > 0, l_1 > u_0 \\
-t_1 & b_1 = b_2 = b < 0, |l_2| \leq u_0, l_1 < -u_0 \\
-t_2 & b_1 = b_2 = b < 0, |l_1| \leq u_0, l_2 > u_0 \\
0 & b_1 = b_2 = b < 0, l_1 > u_0 \\
0 & b_1 > 0, b_2 > 0, b_1 \neq b_2, n_2 < -u_0 \\
t_6 & b_1 > 0, b_2 > 0, b_1 \neq b_2, |n_2| \leq u_0, n_1 < -u_0 \\
t_7 & b_1 > 0, b_2 > 0, b_1 \neq b_2, |n_1| \leq u_0, |n_2| \leq u_0 \\
t_8 & b_1 > 0, b_2 > 0, b_1 \neq b_2, |n_1| \leq u_0, n_2 > u_0 \\
0 & b_1 > 0, b_2 > 0, b_1 \neq b_2, n_1 > u_0 \\
\frac{b_2 u_0}{3b_1} & b_1 > 0, b_2 > 0, b_1 \neq b_2, n_1 < -u_0, n_2 > u_0 \\
0 & b_1 < 0, b_2 < 0, b_1 \neq b_2, n_2 < -u_0 \\
-t_6 & b_1 < 0, b_2 < 0, b_1 \neq b_2, |n_2| \leq u_0, n_1 < -u_0 \\
-t_7 & b_1 < 0, b_2 < 0, b_1 \neq b_2, |n_1| \leq u_0, |n_2| \leq u_0 \\
-t_8 & b_1 < 0, b_2 < 0, b_1 \neq b_2, |n_1| \leq u_0, n_2 > u_0 \\
0 & b_1 < 0, b_2 < 0, b_1 \neq b_2, n_1 > u_0 \\
-\frac{b_2 u_0}{3b_1} & b_1 < 0, b_2 < 0, b_1 \neq b_2, n_1 < -u_0, n_2 > u_0 \\
0 & b_1 > 0, b_2 < 0, m_1 < -u_0 \\
t_{11} & b_1 > 0, b_2 < 0, |m_1| \leq u_0, m_2 < -u_0 \\
t_7 & b_1 > 0, b_2 < 0, |m_1| \leq u_0, |m_2| \leq u_0 \\
t_{12} & b_1 > 0, b_2 < 0, |m_2| \leq u_0, m_1 > u_0 \\
0 & b_1 > 0, b_2 < 0, m_2 > u_0 \\
-\frac{b_2 u_0}{3b_1} & b_1 > 0, b_2 < 0, m_2 < -u_0, m_1 > u_0 \\
0 & b_1 < 0, b_2 > 0, m_1 < -u_0 \\
-t_{11} & b_1 < 0, b_2 > 0, |m_1| \leq u_0, m_2 < -u_0 \\
-t_7 & b_1 < 0, b_2 > 0, |m_1| \leq u_0, |m_2| \leq u_0 \\
-t_{12} & b_1 < 0, b_2 > 0, |m_2| \leq u_0, m_1 > u_0 \\
0 & b_1 < 0, b_2 > 0, m_2 > u_0 \\
\frac{b_2 u_0}{3b_1} & b_1 < 0, b_2 > 0, m_2 < -u_0, m_1 > u_0
\end{cases},
$$

Fig. 2: The evaluated expression for $\mathscr{T}_4$ in (14).

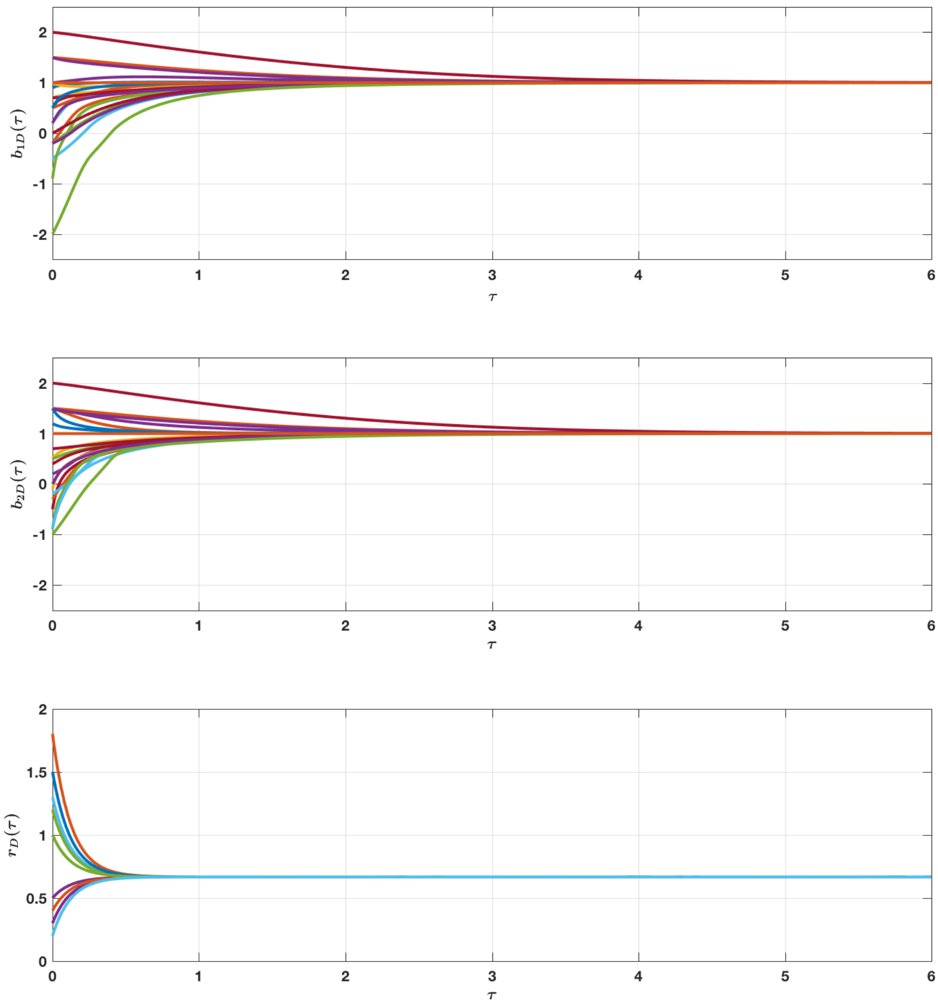

Fig. 3: The solution of (11) as a function of τ for different initial conditions.

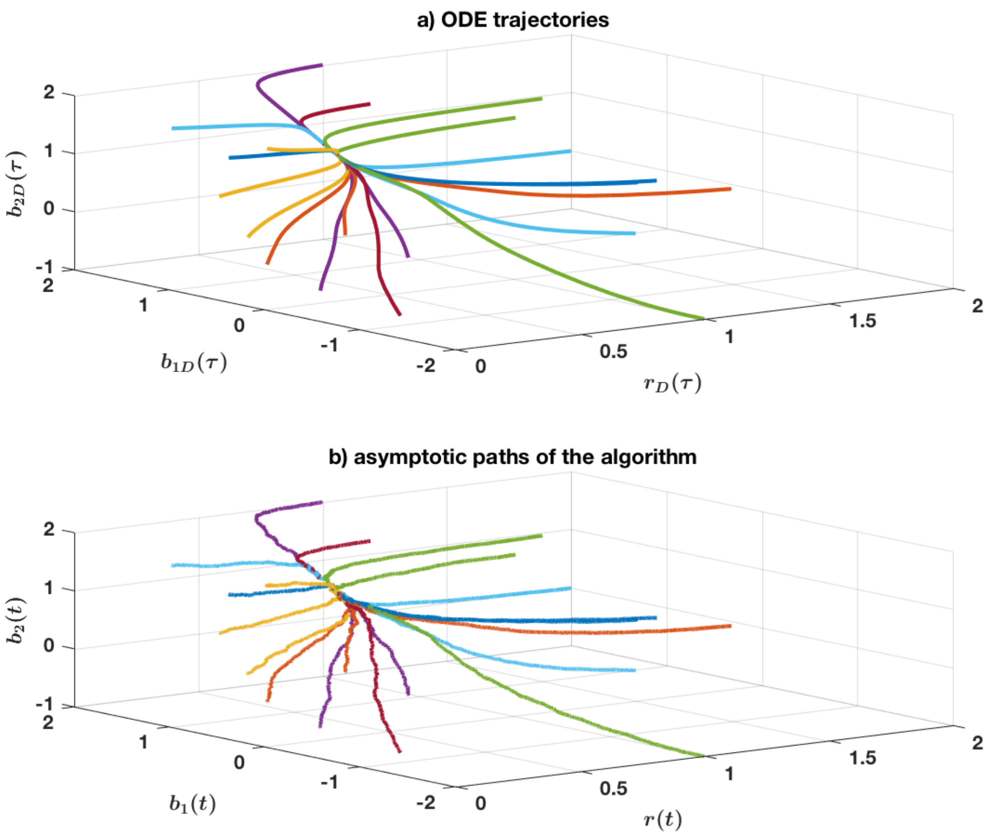

Fig. 4: **a)** Trajectories of the ODE (11) and **b)** asymptotic paths of the algorithm for the model (13).

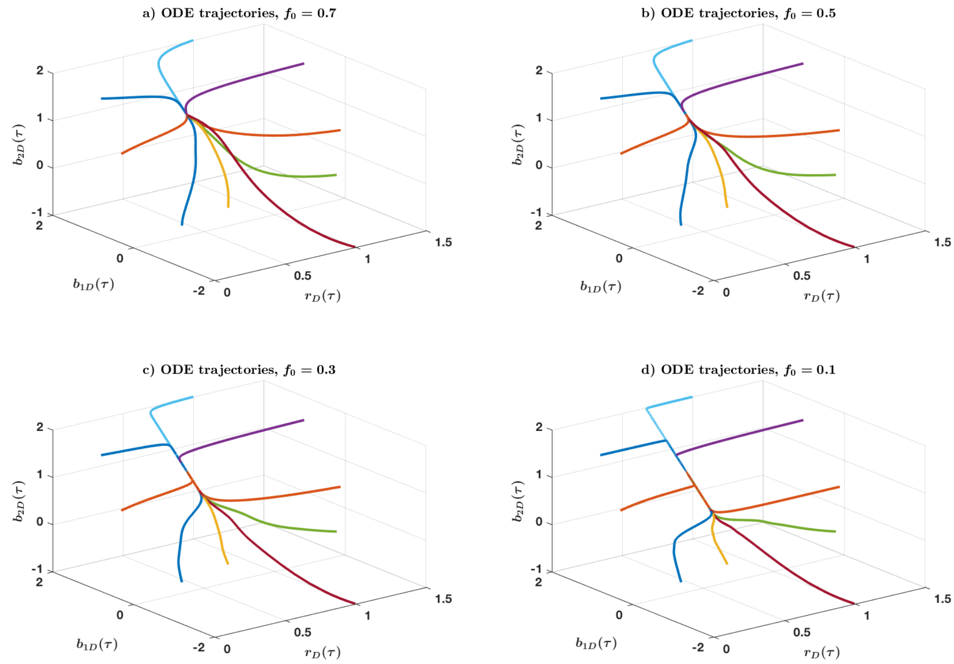

Fig. 5: ODE Trajectories (11) when the switch approaches zero.

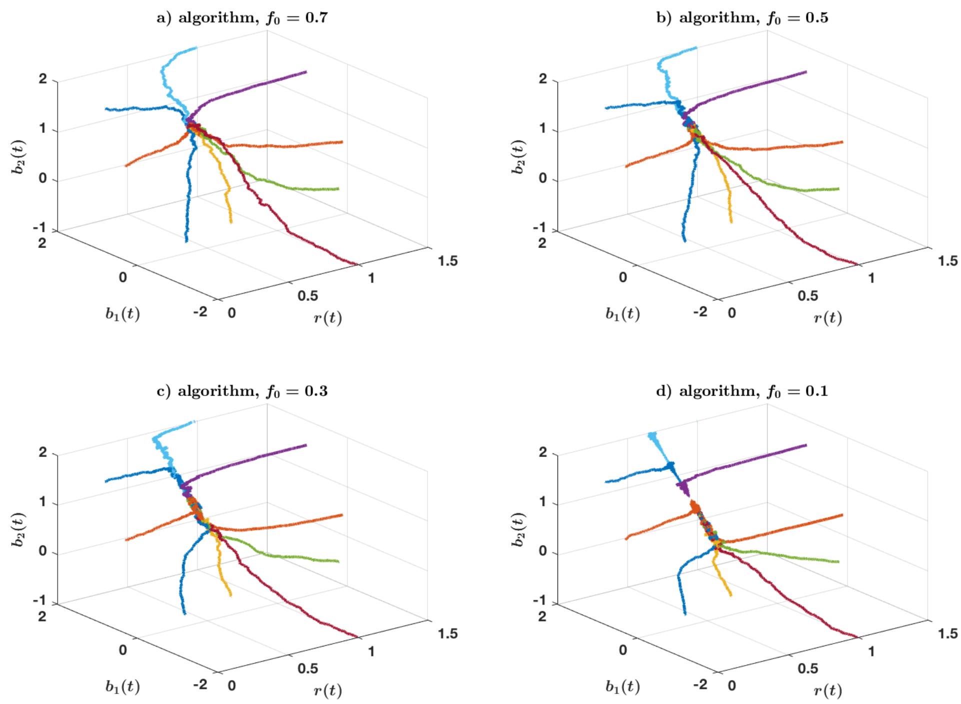

Fig. 6: Asymptotic paths of the algorithm (8) when the switch approaches zero.

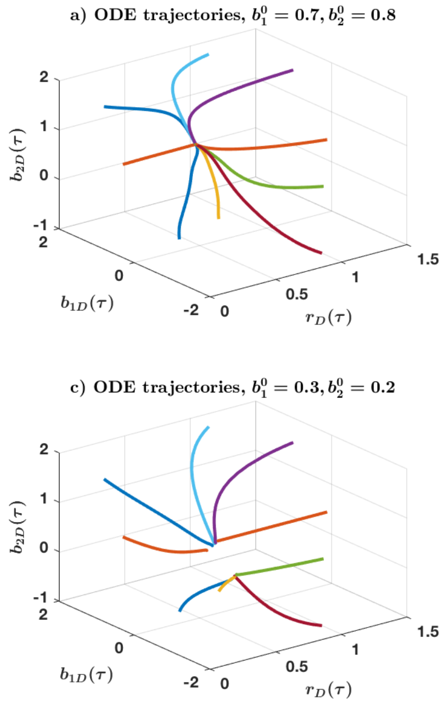

Fig. 7: ODE Trajectories (11) when the signal energy is reduced.

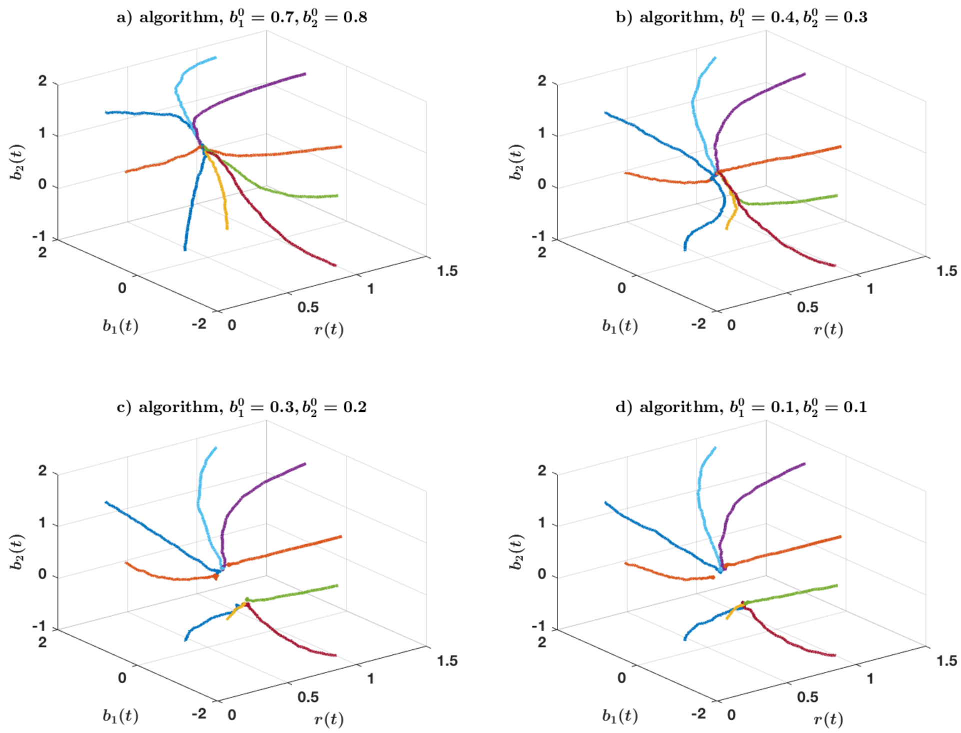

Fig. 8: Asymptotic paths of the algorithm when the signal energy is reduced.