An Interpolation–Extrapolation Algorithm for Computing the Eigenvalues of Preconditioned Banded Symmetric Toeplitz Matrices

Sven-Erik Ekström

Uppsala University, Department of Information Technology, Division of Scientific Computing,
ITC, Lägerhyddsv. 2, Hus 2, P.O. Box 337, SE-751 05 Uppsala, Sweden. Email: sven-erik.ekstrom@it.uu.se.

Carlo Garoni

University of Italian Switzerland (USI), Institute of Computational Science,
Via Giuseppe Buffi 13, 6900 Lugano, Switzerland. Email: carlo.garoni@usi.ch.

University of Insubria, Department of Science and High Technology,
Via Valleggio 11, 22100 Como, Italy. Email: carlo.garoni@uninsubria.it.

August 22, 2017

Abstract

In the past few years, Bogoya, Böttcher, Grudsky, and Maximenko obtained for the eigenvalues of a Toeplitz matrix $T_n(f)$, under suitable assumptions on the generating function f, the precise asymptotic expansion as the matrix size n goes to infinity. On the basis of several numerical experiments, it was conjectured by Serra-Capizzano that a completely analogous expansion also holds for the eigenvalues of the preconditioned Toeplitz matrix $T_n(u)^{-1}T_n(v)$, provided $f = v/u$ is monotone and further conditions on u and v are satisfied. Based on this expansion, we here propose and analyze an interpolation–extrapolation algorithm for computing the eigenvalues of $T_n(u)^{-1}T_n(v)$. We illustrate the performance of the algorithm through numerical experiments and we also present its generalization to the case where $f = v/u$ is non-monotone.

Keywords: preconditioned Toeplitz matrices, eigenvalues, asymptotic eigenvalue expansion, polynomial interpolation, extrapolation

2010 MSC: 15B05, 65F15, 65D05, 65B05

1 Introduction

A matrix of the form

$$[a_{i-j}]_{i,j=1}^{n} = \begin{bmatrix} a_0 & a_{-1} & \cdots & \cdots & a_{-(n-1)} \\ a_1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_1 & a_0 \end{bmatrix},$$

whose entries are constant along each diagonal, is called a Toeplitz matrix. Given a function $g : [-\pi, \pi] \to \mathbb{C}$ belonging to $L^1([-\pi, \pi])$, the nth Toeplitz matrix associated with g is defined as

$$T_n(g) = \left[\hat{g}_{i-j}\right]_{i,j=1}^{n},$$

where the numbers $\hat{g}_k$ are the Fourier coefficients of g,

$$\hat{g}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\mathrm{e}^{-\mathrm{i}k\theta}\mathrm{d}\theta, \qquad k \in \mathbb{Z}.$$

We refer to $\{T_n(g)\}_n$ as the Toeplitz sequence generated by g, which in turn is called the generating function of $\{T_n(g)\}_n$. It is not difficult to see that, whenever g is real, $T_n(g)$ is Hermitian for all n. Moreover, if g is non-negative and not almost everywhere equal to zero in $[-\pi,\pi]$, then $T_n(g)$ is Hermitian positive definite for all n; see [6, 10]. In the case where g is a real cosine trigonometric polynomial (RCTP), that is, a function of the form

$$g(\theta)=\hat{g}_0+2\sum_{k=1}^{m}\hat{g}_k\cos(k\theta),\qquad \hat{g}_0,\hat{g}_1,\ldots,\hat{g}_m\in\mathbb{R},\qquad m\in\mathbb{N},$$

the nth Toeplitz matrix generated by g is the real symmetric banded matrix given by

$$T_n(g)=\begin{bmatrix} \hat{g}_0 & \hat{g}_1 & \cdots & \hat{g}_m & & & & & & & \\ \hat{g}_1 & \ddots & \ddots & & \ddots & & & & & & \\ \vdots & \ddots & \ddots & \ddots & & \ddots & & & & & \\ \hat{g}_m & & \ddots & \ddots & \ddots & & \ddots & & & & \\ & \ddots & & \ddots & \ddots & \ddots & & \ddots & & & \\ & & \boxed{\hat{g}_m \;\cdots\; \hat{g}_1 \;\; \hat{g}_0 \;\; \hat{g}_1 \;\cdots\; \hat{g}_m} & & & & & & & & \\ & & & \ddots & & \ddots & \ddots & \ddots & & \ddots & \\ & & & & \ddots & & \ddots & \ddots & \ddots & & \hat{g}_m \\ & & & & & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & & & & & \ddots & & \ddots & \ddots & \hat{g}_1 \\ & & & & & & & \hat{g}_m & \cdots & \hat{g}_1 & \hat{g}_0 \end{bmatrix}.$$

Based on several numerical experiments, in [1] Serra-Capizzano formulated the following conjecture.

Conjecture 1. Let u,v be RCTPs, with $u>0$ on $(0,\pi)$, and suppose that $f=v/u$ is monotone increasing over $(0,\pi)$. Set $X_n=T_n(u)^{-1}T_n(v)$ for all n. Then, for every integer $\alpha\geq 0$, every n and every $j=1,\ldots,n$, the following asymptotic expansion holds:

$$\lambda_j(X_n)=f(\theta_{j,n})+\sum_{k=1}^{\alpha}c_k(\theta_{j,n})h^k+E_{j,n,\alpha}, \tag{1}$$

where:
- the eigenvalues of X_n are arranged in non-decreasing order, $\lambda_1(X_n)\leq\ldots\leq\lambda_n(X_n)$;[1]
- $\{c_k\}_{k=1,2,\ldots}$ is a sequence of functions from $(0,\pi)$ to $\mathbb{R}$ which depends only on u,v;
- $h=\frac{1}{n+1}$ and $\theta_{j,n}=\frac{j\pi}{n+1}=j\pi h$;
- $E_{j,n,\alpha}=O(h^{\alpha+1})$ is the remainder (the error), which satisfies the inequality $|E_{j,n,\alpha}|\leq C_\alpha h^{\alpha+1}$ for some constant C_α depending only on α,u,v.

In the case where $u=1$ identically, Conjecture 1 was originally formulated and supported through numerical experiments in [9]. In the case where $u=1$ identically and v satisfies some additional assumptions, Conjecture 1 was formally proved by Bogoya, Böttcher, Grudsky, and Maximenko in a sequel of recent papers [2, 4, 5].

Assuming Conjecture 1, in Section 2 of this paper we describe and analyze a new algorithm for computing the eigenvalues of $X_n=T_n(u)^{-1}T_n(v)$; and in Section 3 we illustrate its performance through numerical experiments. The algorithm combines the extrapolation procedure proposed in [1, 9] — which allows the computation of *some* of the eigenvalues of X_n — with an appropriate interpolation process, thus allowing the simultaneous computation of *all* the eigenvalues of X_n. In Section 4 we provide a generalization of the algorithm to the case where $f=v/u$ is non-monotone. In Section 5 we draw conclusions and suggest possible future lines of research.

2 The Algorithm

Throughout this paper, with each positive integer $n\in\mathbb{N}=\{1,2,3,\ldots\}$ we associate the stepsize $h=\frac{1}{n+1}$ and the grid points $\theta_{j,n}=j\pi h$, $j=1,\ldots,n$. For notational convenience, we will always denote a positive integer and the associated stepsize in the same way. For example, if the positive integer is n, the associated stepsize is h; if the positive integer is

[1] Note that the eigenvalues of X_n are real, because X_n is similar to the symmetric matrix $T_n(u)^{-1/2}T_n(v)T_n(u)^{-1/2}$.

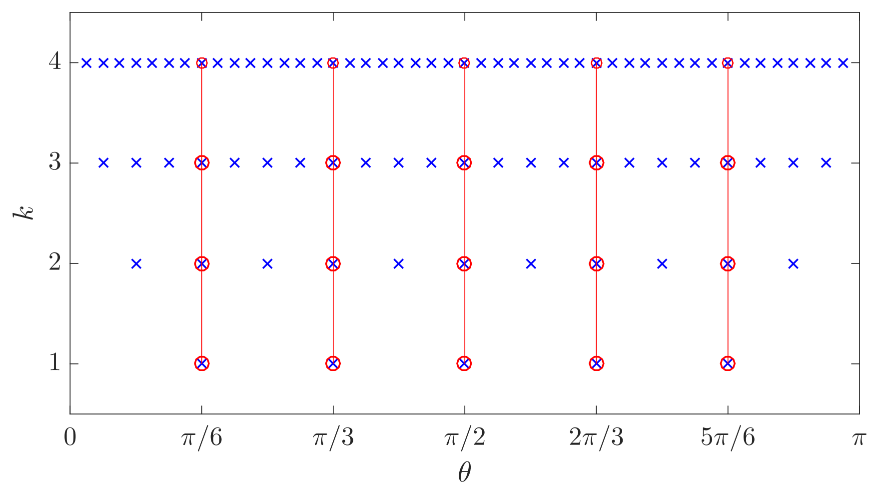

Figure 1: Representation of the grids $\{\theta_{1,n_k},\ldots,\theta_{n_k,n_k}\}$, $k=1,\ldots,\alpha$, for $n_1=5$ and $\alpha=4$.

n_1, the associated stepsize is h_1; if the positive integer is $\bar{n}$, the associated stepsize is $\bar{h}$; etc. Throughout this section, we make the following assumptions.

- u, v, f are as in Conjecture 1.
- $n, n_1, \alpha \in \mathbb{N}$ are fixed parameters and $X_n = T_n(u)^{-1}T_n(v)$.
- $n_k = 2^{k-1}(n_1+1)-1$ for $k=1,\ldots,\alpha$.
- $j_k = 2^{k-1}j_1$ for $j_1=1,\ldots,n_1$ and $k=1,\ldots,\alpha$; j_k is the index in $\{1,\ldots,n_k\}$ such that $\theta_{j_k,n_k}=\theta_{j_1,n_1}$.

A graphical representation of the grids $\{\theta_{1,n_k},\ldots,\theta_{n_k,n_k}\}$, $k=1,\ldots,\alpha$, is reported in Figure 1 for $n_1=5$ and $\alpha=4$.

2.1 Description and Formulation of the Algorithm

For each fixed $j_1=1,\ldots,n_1$ we apply α times the expansion (1) with $n=n_1,n_2,\ldots,n_\alpha$ and $j=j_1,j_2,\ldots,j_\alpha$. Since $\theta_{j_1,n_1}=\theta_{j_2,n_2}=\ldots=\theta_{j_\alpha,n_\alpha}$ (by definition of $j_2,\ldots,j_\alpha$), we obtain

$$\begin{cases} E_{j_1,n_1,0} = c_1(\theta_{j_1,n_1})h_1 + c_2(\theta_{j_1,n_1})h_1^2 + \ldots + c_\alpha(\theta_{j_1,n_1})h_1^\alpha + E_{j_1,n_1,\alpha} \\ E_{j_2,n_2,0} = c_1(\theta_{j_1,n_1})h_2 + c_2(\theta_{j_1,n_1})h_2^2 + \ldots + c_\alpha(\theta_{j_1,n_1})h_2^\alpha + E_{j_2,n_2,\alpha} \\ \quad\vdots \\ E_{j_\alpha,n_\alpha,0} = c_1(\theta_{j_1,n_1})h_\alpha + c_2(\theta_{j_1,n_1})h_\alpha^2 + \ldots + c_\alpha(\theta_{j_1,n_1})h_\alpha^\alpha + E_{j_\alpha,n_\alpha,\alpha} \end{cases} \tag{2}$$

where

$$E_{j_k,n_k,0} = \lambda_{j_k}(X_{n_k}) - f(\theta_{j_1,n_1}), \qquad k=1,\ldots,\alpha,$$

and

$$|E_{j_k,n_k,\alpha}| \le C_\alpha h_k^{\alpha+1}, \qquad k=1,\ldots,\alpha. \tag{3}$$

Let $\tilde{c}_1(\theta_{j_1,n_1}),\ldots,\tilde{c}_\alpha(\theta_{j_1,n_1})$ be the approximations of $c_1(\theta_{j_1,n_1}),\ldots,c_\alpha(\theta_{j_1,n_1})$ obtained by removing all the errors $E_{j_1,n_1,\alpha},\ldots,E_{j_\alpha,n_\alpha,\alpha}$ in (2) and by solving the resulting linear system:

$$\begin{cases} E_{j_1,n_1,0} = \tilde{c}_1(\theta_{j_1,n_1})h_1 + \tilde{c}_2(\theta_{j_1,n_1})h_1^2 + \ldots + \tilde{c}_\alpha(\theta_{j_1,n_1})h_1^\alpha \\ E_{j_2,n_2,0} = \tilde{c}_1(\theta_{j_1,n_1})h_2 + \tilde{c}_2(\theta_{j_1,n_1})h_2^2 + \ldots + \tilde{c}_\alpha(\theta_{j_1,n_1})h_2^\alpha \\ \quad\vdots \\ E_{j_\alpha,n_\alpha,0} = \tilde{c}_1(\theta_{j_1,n_1})h_\alpha + \tilde{c}_2(\theta_{j_1,n_1})h_\alpha^2 + \ldots + \tilde{c}_\alpha(\theta_{j_1,n_1})h_\alpha^\alpha \end{cases} \tag{4}$$

Note that this way of computing approximations for $c_1(\theta_{j_1,n_1}),\ldots,c_\alpha(\theta_{j_1,n_1})$ is completely analogous to the Richardson extrapolation procedure that is employed in the context of Romberg integration to accelerate the convergence of the trapezoidal rule [11, Section 3.4]. In this regard, the asymptotic expansion (1) plays here the same role as the Euler–Maclaurin summation formula [11, Section 3.3]. For more advanced studies on extrapolation methods, we refer the reader to [7]. The next theorem shows that the approximation error $|c_k(\theta_{j_1,n_1}) - \tilde{c}_k(\theta_{j_1,n_1})|$ is $O(h_1^{\alpha-k+1})$.

Theorem 1. *There exists a constant A_α depending only on α, u, v such that, for $j_1 = 1, \ldots, n_1$ and $k = 1, \ldots, \alpha$,*

$$|c_k(\theta_{j_1,n_1}) - \tilde{c}_k(\theta_{j_1,n_1})| \leq A_\alpha h_1^{\alpha-k+1}. \tag{5}$$

Proof. See Appendix A. □

Now, fix an index $j \in \{1, \ldots, n\}$. To compute an approximation of $\lambda_j(X_n)$ through the expansion (1) we would need the value $c_k(\theta_{j,n})$ for each $k = 1, \ldots, \alpha$. Of course, $c_k(\theta_{j,n})$ is not available in practice, but we can approximate it by interpolating in some way the values $\tilde{c}_k(\theta_{j_1,n_1})$, $j_1 = 1, \ldots, n_1$. For example, we may define $\tilde{c}_k(\theta)$ as the interpolation polynomial of the data $(\theta_{1,n_1}, \tilde{c}_k(\theta_{1,n_1})), \ldots, (\theta_{n_1,n_1}, \tilde{c}_k(\theta_{n_1,n_1}))$ — so that $\tilde{c}_k(\theta)$ is expected to be an approximation of $c_k(\theta)$ over the whole interval $(0, \pi)$ — and take $\tilde{c}_k(\theta_{j,n})$ as an approximation to $c_k(\theta_{j,n})$. It is known, however, that interpolation over a large number of uniform nodes is not advisable as it may give rise to spurious oscillations (Runge's phenomenon). It is therefore better to adopt another kind of approximation. An alternative could be the following: we approximate $c_k(\theta)$ by the spline function $\tilde{c}_k(\theta)$ which is linear on each interval $[\theta_{j_1,n_1}, \theta_{j_1+1,n_1}]$ and takes the value $\tilde{c}_k(\theta_{j_1,n_1})$ at θ_{j_1,n_1} for all $j_1 = 1, \ldots, n_1$. This strategy removes for sure any spurious oscillation, yet it is not accurate. In particular, it does not preserve the accuracy of approximation at the nodes θ_{j_1,n_1} established in Theorem 1, i.e., there is no guarantee that $|c_k(\theta) - \tilde{c}_k(\theta)| \leq B_\alpha h_1^{\alpha-k+1}$ for $\theta \in (0, \pi)$ or $|c_k(\theta_{j,n}) - \tilde{c}_k(\theta_{j,n})| \leq B_\alpha h_1^{\alpha-k+1}$ for $j = 1, \ldots, n$, with B_α being a constant depending only on α, u, v. As proved in Theorem 2, a local approximation strategy that preserves the accuracy (5), at least if $c_k(\theta)$ is sufficiently smooth, is the following: let $\theta^{(1)}, \ldots, \theta^{(\alpha-k+1)}$ be $\alpha-k+1$ points of the grid $\{\theta_{1,n_1}, \ldots, \theta_{n_1,n_1}\}$ which are closest to the point $\theta_{j,n}$,[2] and let $\tilde{c}_{k,j}(\theta)$ be the interpolation polynomial of the data $(\theta^{(1)}, \tilde{c}_k(\theta^{(1)})), \ldots, (\theta^{(\alpha-k+1)}, \tilde{c}_k(\theta^{(\alpha-k+1)}))$; then, we approximate $c_k(\theta_{j,n})$ by $\tilde{c}_{k,j}(\theta_{j,n})$. Note that, by selecting $\alpha - k + 1$ points from $\{\theta_{1,n_1}, \ldots, \theta_{n_1,n_1}\}$, we are implicitly assuming that $n_1 \geq \alpha - k + 1$.

Theorem 2. *Let $1 \leq k \leq \alpha$, and suppose $n_1 \geq \alpha-k+1$ and $c_k \in C^{\alpha-k+1}([0, \pi])$. For $j = 1, \ldots, n$, if $\theta^{(1)}, \ldots, \theta^{(\alpha-k+1)}$ are $\alpha - k + 1$ points of $\{\theta_{1,n_1}, \ldots, \theta_{n_1,n_1}\}$ which are closest to $\theta_{j,n}$, and if $\tilde{c}_{k,j}(\theta)$ is the interpolation polynomial of the data $(\theta^{(1)}, \tilde{c}_k(\theta^{(1)})), \ldots, (\theta^{(\alpha-k+1)}, \tilde{c}_k(\theta^{(\alpha-k+1)}))$, then*

$$|c_k(\theta_{j,n}) - \tilde{c}_{k,j}(\theta_{j,n})| \leq B_\alpha h_1^{\alpha-k+1} \tag{6}$$

for some constant B_α depending only on α, u, v.

Proof. See Appendix A. □

We are now ready to formulate our algorithm for computing all the eigenvalues of X_n.

Algorithm 1. Given $n, n_1, \alpha \in \mathbb{N}$ with $n_1 \geq \alpha$, we compute approximations of the eigenvalues of X_n as follows.

1. For $j_1 = 1, \ldots, n_1$ compute $\tilde{c}_1(\theta_{j_1,n_1}), \ldots, \tilde{c}_\alpha(\theta_{j_1,n_1})$ by solving (4).
2. For $j = 1, \ldots, n$
   - for $k = 1, \ldots, \alpha$
     - determine $\alpha - k + 1$ points $\theta^{(1)}, \ldots, \theta^{(\alpha-k+1)} \in \{\theta_{1,n_1}, \ldots, \theta_{n_1,n_1}\}$ which are closest to $\theta_{j,n}$;
     - compute $\tilde{c}_{k,j}(\theta_{j,n})$, where $\tilde{c}_{k,j}(\theta)$ is the interpolation polynomial of $(\theta^{(1)}, \tilde{c}_k(\theta^{(1)})), \ldots, (\theta^{(\alpha-k+1)}, \tilde{c}_k(\theta^{(\alpha-k+1)}))$;
   - compute $\tilde{\lambda}_j(X_n) = f(\theta_{j,n}) + \sum_{k=1}^{\alpha} \tilde{c}_{k,j}(\theta_{j,n}) h^k$.
3. Return $(\tilde{\lambda}_1(X_n), \ldots, \tilde{\lambda}_n(X_n))$ as an approximation to $(\lambda_1(X_n), \ldots, \lambda_n(X_n))$.

Remark 1. Algorithm 1 is specifically designed for computing the eigenvalues of X_n in the case where the matrix size n is quite large. When applying this algorithm, it is implicitly assumed that n_1 and α are small (much smaller than n), so that each $n_k = 2^{k-1}(n_1 + 1) - 1$ is small as well and the computation of the eigenvalues of X_{n_k} — which is required in the first step — can be efficiently performed by any standard eigensolver (e.g., the MATLAB `eig` function).

Remark 2. If $C(n, n_1, \alpha)$ denotes the computational cost of Algorithm 1, a careful evaluation shows that

$$C(n, n_1, \alpha) \leq C\alpha^3 n + \sum_{k=1}^{\alpha} C_{\text{eig}}(n_k),$$

where C is a constant depending only on f and $C_{\text{eig}}(n_k)$ is the cost for computing the eigenvalues of X_{n_k}.

[2] These $\alpha - k + 1$ points are uniquely determined by $\theta_{j,n}$ except in the case where $\theta_{j,n}$ coincides with either a grid point θ_{j_1,n_1} or the midpoint between two consecutive grid points θ_{j_1,n_1} and θ_{j_1+1,n_1}.

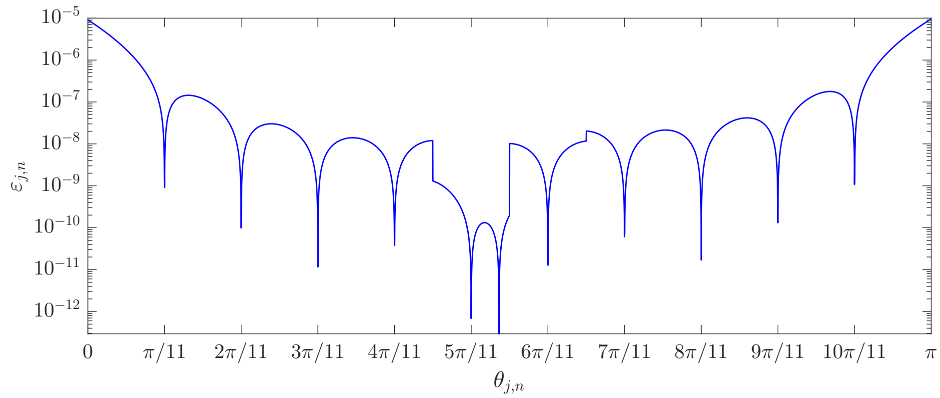

Figure 2: Example 1: errors $\varepsilon_{j,n}$ versus $\theta_{j,n}$ for $j = 1, \dots, n$ in the case where $u(\theta) = 1$, $v(\theta) = 6 - 8\cos(\theta) + 2\cos(2\theta)$, $n = 5000$, $n_1 = 10$, and $\alpha = 7$.

2.2 Error Estimate

Theorem 3. *Suppose $n \geq n_1 \geq \alpha$ and $c_k \in C^{\alpha-k+1}([0,\pi])$ for $k = 1, \dots, \alpha$. Let $(\tilde{\lambda}_1(X_n), \dots, \tilde{\lambda}_n(X_n))$ be the approximation of $(\lambda_1(X_n), \dots, \lambda_n(X_n))$ computed by Algorithm 1. Then, there exists a constant D_α depending only on α, u, v such that, for $j = 1, \dots, n$,*

$$|\lambda_j(X_n) - \tilde{\lambda}_j(X_n)| \leq D_\alpha h_1^\alpha h.$$

Proof. By (1) and Theorem 2,

$$\begin{aligned}
|\lambda_j(X_n) - \tilde{\lambda}_j(X_n)| &= \left| f(\theta_{j,n}) + \sum_{k=1}^{\alpha} c_k(\theta_{j,n}) h^k + E_{j,n,\alpha} - f(\theta_{j,n}) - \sum_{k=1}^{\alpha} \tilde{c}_{k,j}(\theta_{j,n}) h^k \right| \\
&= \left| \sum_{k=1}^{\alpha} (c_k(\theta_{j,n}) - \tilde{c}_{k,j}(\theta_{j,n})) h^k + E_{j,n,\alpha} \right| \\
&\leq B_\alpha \sum_{k=1}^{\alpha} h_1^{\alpha-k+1} h + C_\alpha h^{\alpha+1} \leq D_\alpha h_1^\alpha h,
\end{aligned}$$

where $D_\alpha = (\alpha + 1)\max(B_\alpha, C_\alpha)$. □

Remark 3. The error estimate provided in Theorem 3 suggests that the eigenvalue approximations provided by Algorithm 1 improve as n increases, i.e., as h decreases. Numerical experiments reveal that this is in fact the case (see also Example 2 below).

3 Numerical Experiments

In this section we illustrate through numerical examples the performance of Algorithm 1.

Example 1. Let

$$\begin{aligned}
u(\theta) &= 1, \\
v(\theta) &= 6 - 8\cos(\theta) + 2\cos(2\theta).
\end{aligned}$$

Note that $f(\theta) = v(\theta)/u(\theta) = v(\theta)$ is monotone increasing on $(0, \pi)$. Suppose we want to approximate the eigenvalues of $X_n = T_n(u)^{-1} T_n(v) = T_n(f)$ for $n = 5000$. Let $\tilde{\lambda}_j(X_n)$ be the approximation of $\lambda_j(X_n)$ obtained by applying Algorithm 1 with $n_1 = 10$ and $\alpha = 7$. In Figure 2 we plot the errors $\varepsilon_{j,n} = |\lambda_j(X_n) - \tilde{\lambda}_j(X_n)|$ versus $\theta_{j,n}$ for $j = 1, \dots, n$. We note that the largest errors are attained when either $\theta_{j,n} \approx 0$ or $\theta_{j,n} \approx \pi$. As highlighted also in Example 3 below, this is probably due to two concomitant factors:

- the errors $\varepsilon_{j,n}$ are supposed to be smaller for $\theta_{j,n} \in [\theta_{1,n_1}, \theta_{n_1,n_1}] = [\pi/11, 10\pi/11]$, because in this case the approximations $\tilde{c}_{k,j}(\theta_{j,n})$ computed by Algorithm 1 for the values $c_k(\theta_{j,n})$ are expected to be more accurate as the interpolation polynomial $\tilde{c}_{k,j}(\theta)$ is evaluated inside the convex hull of the interpolation nodes $\{\theta_{1,n_1}, \dots, \theta_{n_1,n_1}\}$;

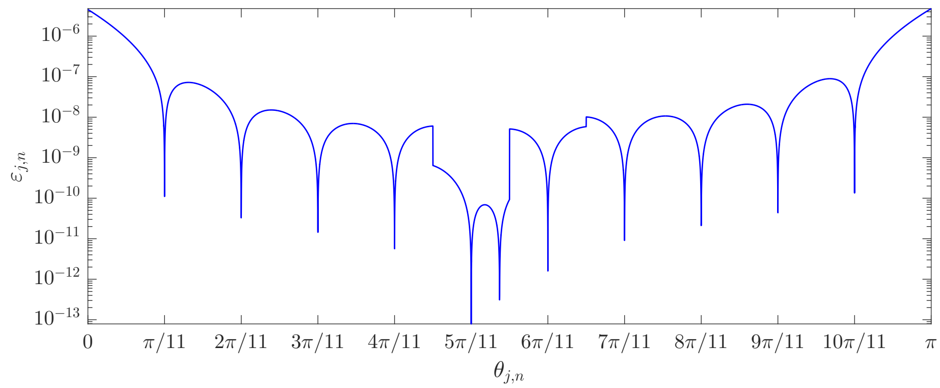

Figure 3: Example 2: errors $\varepsilon_{j,n}$ versus $\theta_{j,n}$ for $j = 1, \dots, n$ in the case where $u(\theta) = 1$, $v(\theta) = 6 - 8\cos(\theta) + 2\cos(2\theta)$, $n = 10000$, $n_1 = 10$, and $\alpha = 7$.

- $\theta = 0$ and $\theta = \pi$ are the two points on $[0, \pi]$ where f' vanishes, which means that the monotonicity of f is "weak" around these points (recall that Algorithm 1 works under the assumption that f is monotone as in Conjecture 1).

In reference to the previous discussion, we note that the maximum error for $\theta_{j,n} \in [\theta_{1,n_1}, \theta_{n_1,n_1}]$ is given by

$$\max\{\varepsilon_{j,n} : \ \theta_{j,n} \in [\theta_{1,n_1}, \theta_{n_1,n_1}]\} \approx 1.7803 \cdot 10^{-7},$$

which is about two order of magnitude less than

$$\max_{j=1,\dots,n} \varepsilon_{j,n} \approx 9.5167 \cdot 10^{-6}.$$

A careful look at Figure 2 shows that, aside from the exceptional minimum attained inside the interval $(5\pi/11, 6\pi/11)$, the local minima of $\varepsilon_{j,n}$ are attained when $\theta_{j,n}$ is approximately equal to some of the grid points θ_{j_1,n_1}, $j_1 = 1, \dots, n_1$. This is no surprise, because for $\theta_{j,n} = \theta_{j_1,n_1}$ we have $\tilde{c}_{k,j}(\theta_{j,n}) = \tilde{c}_k(\theta_{j_1,n_1})$ and $c_k(\theta_{j,n}) = c_k(\theta_{j_1,n_1})$, which means that the error of the approximation $\tilde{c}_{k,j}(\theta_{j,n}) \approx c_k(\theta_{j,n})$ reduces to the error of the approximation $\tilde{c}_k(\theta_{j_1,n_1}) \approx c_k(\theta_{j_1,n_1})$; that is, we are not introducing further error due to the interpolation process. To conclude, we make the following observation: for α, u, v as in this example, Theorem 3 yields

$$D_\alpha \geq \frac{\max_{j=1,\dots,n} \varepsilon_{j,n}}{h_1^\alpha h} \approx 9.2745 \cdot 10^5 > \alpha^\alpha = 8.23543 \cdot 10^5.$$

This suggests that, unfortunately, the best constant D_α for which the error estimate of Theorem 3 is satisfied grows very quickly with α.

Example 2. Let u, v, f be as in Example 1. Suppose we want to approximate the eigenvalues of $X_n = T_n(u)^{-1} T_n(v) = T_n(f)$ for $n = 10000$. Let $\tilde{\lambda}_j(X_n)$ be the approximation of $\lambda_j(X_n)$ obtained by applying Algorithm 1 with $n_1 = 10$ and $\alpha = 7$ as in Example 1. In Figure 3 we plot the errors $\varepsilon_{j,n} = |\lambda_j(X_n) - \tilde{\lambda}_j(X_n)|$ versus $\theta_{j,n}$ for $j = 1, \dots, n$. We note that the errors in Figure 3 are smaller than in Figure 2. This shows that the eigenvalue approximations provided by Algorithm 1 improve as n increases (see also Remark 3).

Example 3. Let

$$\begin{aligned} u(\theta) &= 1, \\ v(\theta) &= -\frac{1}{4} - \frac{1}{2}\cos(\theta) + \frac{1}{4}\cos(2\theta) - \frac{1}{12}\cos(3\theta). \end{aligned}$$

Note that $f(\theta) = v(\theta)/u(\theta) = v(\theta)$ is monotone increasing on $(0, \pi)$. Suppose we want to approximate the eigenvalues of $X_n = T_n(u)^{-1} T_n(v) = T_n(f)$ for $n = 10000$. Let $\tilde{\lambda}_j^{(m)}(X_n)$ be the approximation of $\lambda_j(X_n)$ obtained by applying Algorithm 1 with $n_1 = 10 \cdot 2^{m-1}$ and $\alpha = 5$. In Figure 4 we plot the errors $\varepsilon_{j,n}^{(m)} = |\lambda_j(X_n) - \tilde{\lambda}_j^{(m)}(X_n)|$ versus $\theta_{j,n}$ for $j = 1, \dots, n$ and $m = 1, 2, 3, 4$. We see from the figure that, as m increases, the error decreases rather quickly everywhere except in a neighborhood of the point $\theta = \pi/3$ where f' vanishes. Actually, the three points of $[0, \pi]$ where f' vanishes are 0, $\pi/3$, π, and these are precisely the points around which the error is higher than elsewhere. We remark that, as in Examples 1 and 2, the error $\varepsilon_{j,n}^{(m)}$ attains its local minima when $\theta_{j,n}$ is approximately equal to some of the nodes $\theta_{1,n_1^{(m)}}, \dots, \theta_{n_1^{(m)},n_1^{(m)}}$.

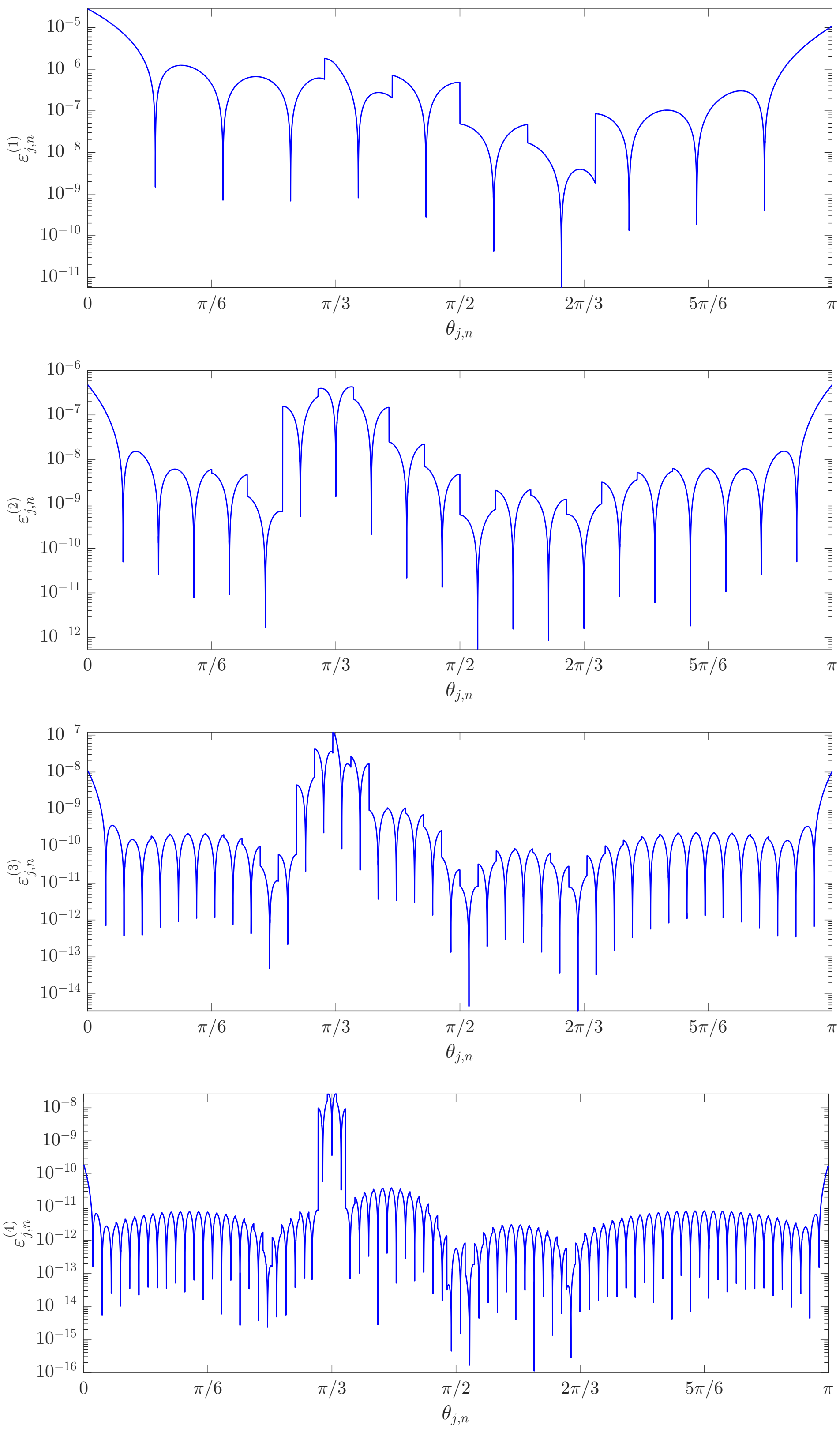

Figure 4: Example 3: errors $\varepsilon_{j,n}^{(m)}$ versus $\theta_{j,n}$ for $j = 1, \ldots, n$, in the case where $u(\theta) = 1$, $v(\theta) = -\frac{1}{4} - \frac{1}{2}\cos(\theta) + \frac{1}{4}\cos(2\theta) - \frac{1}{12}\cos(3\theta)$, $n = 10000$, $n_1 = 10 \cdot 2^{m-1}$, and $\alpha = 5$.

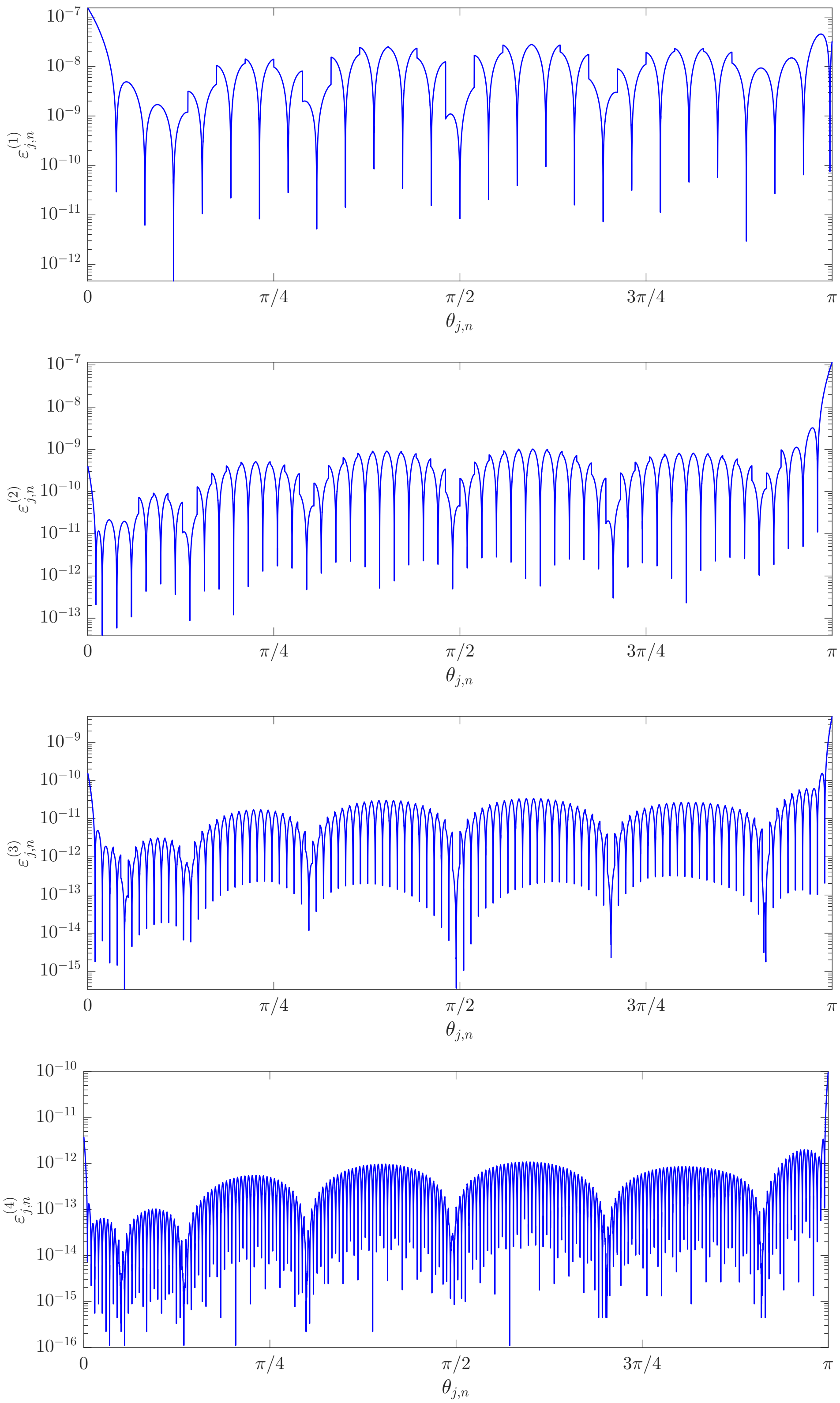

Figure 5: Example 4: errors $\varepsilon^{(m)}_{j,n}$ versus $\theta_{j,n}$ for $j = 1, \dots, n$, in the case where $u(\theta) = 1$, $v(\theta) = \frac{301}{400} - \cos(\theta) + \frac{1}{5}\cos(2\theta) + \frac{1}{10}\cos(3\theta) - \frac{1}{20}\cos(4\theta) + \frac{1}{400}\cos(6\theta)$, $n = 10000$, $n_1 = 25 \cdot 2^{m-1}$, and $\alpha = 5$.

Example 4. Let

$$\begin{aligned} u(\theta) &= 1, \\ v(\theta) &= \frac{301}{400} - \cos(\theta) + \frac{1}{5}\cos(2\theta) + \frac{1}{10}\cos(3\theta) - \frac{1}{20}\cos(4\theta) + \frac{1}{400}\cos(6\theta). \end{aligned}$$

Note that $f(\theta) = v(\theta)/u(\theta) = v(\theta)$ is monotone increasing on $(0, \pi)$ and $f'(\theta) = 0$ only for $\theta = 0, \pi$.[3] Suppose we want to approximate the eigenvalues of $X_n = T_n(u)^{-1}T_n(v) = T_n(f)$ for $n = 10000$. Let $\tilde{\lambda}_j^{(m)}(X_n)$ be the approximation of $\lambda_j(X_n)$ obtained by applying Algorithm 1 with $n_1 = 25 \cdot 2^{m-1}$ and $\alpha = 5$. In Figure 5 we plot the errors $\varepsilon_{j,n}^{(m)} = |\lambda_j(T_n(f)) - \tilde{\lambda}_j^{(m)}(T_n(f))|$ versus $\theta_{j,n}$ for $j = 1, \ldots, n$ and $m = 1, 2, 3, 4$. Considerations analogous to those of Example 3 apply also in this case.

Example 5. Let

$$\begin{aligned} u(\theta) &= 3 + 2\cos(\theta), \\ v(\theta) &= 2 - \cos(\theta) - \cos(2\theta). \end{aligned}$$

Note that $f(\theta) = v(\theta)/u(\theta) = 1 - \cos(\theta)$ is monotone increasing on $(0, \pi)$ and $f'(\theta) = 0$ only for $\theta = 0, \pi$. Suppose we want to approximate the eigenvalues of $X_n = T_n(u)^{-1}T_n(v)$ for $n = 5000$. Let $\tilde{\lambda}_j(X_n)$ be the approximation of $\lambda_j(X_n)$ obtained by applying Algorithm 1 with $n_1 = 50 \cdot 2^{m-1}$ and $\alpha = 4$. The graph of the errors $\varepsilon_{j,n} = |\lambda_j(X_n) - \tilde{\lambda}_j(X_n)|$ versus $\theta_{j,n}$ is shown in Figure 6 for $j = 1, \ldots, n$ and $m = 1, 2, 3, 4$.

Example 6. The last example is suggested by the cubic B-spline isogeometric analysis discretization of second-order eigenvalue problems [10, Section 10.7.3]. Let

$$\begin{aligned} u(\theta) &= 1208 + 1191\cos(\theta) + 120\cos(2\theta) + \cos(3\theta), \\ v(\theta) &= 40 - 15\cos(\theta) - 24\cos(2\theta) - \cos(3\theta). \end{aligned}$$

It can be shown that $u(\theta) > 0$ on $(0, \pi)$,

$$f(\theta) = \frac{v(\theta)}{u(\theta)} = \frac{40 - 15\cos(\theta) - 24\cos(2\theta) - \cos(3\theta)}{1208 + 1191\cos(\theta) + 120\cos(2\theta) + \cos(3\theta)}$$

is monotone increasing on $(0, \pi)$, and $f'(\theta) = 0$ only for $\theta = 0, \pi$. Suppose we want to approximate the eigenvalues of $X_n = T_n(u)^{-1}T_n(v)$ for $n = 5000$. Let $\tilde{\lambda}_j(X_n)$ be the approximation of $\lambda_j(X_n)$ obtained by applying Algorithm 1 with $n_1 = 50 \cdot 2^{m-1}$ and $\alpha = 4$ (as in Example 5). The graph of the errors $\varepsilon_{j,n} = |\lambda_j(X_n) - \tilde{\lambda}_j(X_n)|$ versus $\theta_{j,n}$ is shown in Figure 7 for $j = 1, \ldots, n$ and $m = 1, 2, 3, 4$.

4 Generalization to the Non-Monotone Case

With reference to Conjecture 1, suppose that the function $f = v/u$ is monotone decreasing on $(0, \pi)$. Then, $-f = -v/u$ is monotone increasing on $(0, \pi)$ and, moreover, $T_n(u)^{-1}T_n(v) = -T_n(u)^{-1}T_n(-v)$. This immediately implies that Algorithm 1 allows one to compute the eigenvalues of $T_n(u)^{-1}T_n(v)$ even in the case where $f = v/u$ is monotone decreasing on $(0, \pi)$: it suffices to apply the algorithm with $X_n = T_n(u)^{-1}T_n(-v)$. Some limitations on the applicability of Algorithm 1 arise when f is non-monotone on $(0, \pi)$. This is precisely the case we are going to investigate in this section. We begin by formulating the following conjecture.

Conjecture 2. Let u, v be RCTPs, with $u > 0$ on $(0, \pi)$, and suppose that $f = v/u$ restricted to the interval $I \subseteq (0, \pi)$ is monotone and $f^{-1}(f(I)) = I$. Set $X_n = T_n(u)^{-1}T_n(v)$ for all n. Then, for every integer $\alpha \geq 0$, every n and every $j = 1, \ldots, n$ such that $\theta_{j,n} \in I$, the following asymptotic expansion holds:

$$\lambda_{\rho_n(j)}(X_n) = f(\theta_{j,n}) + \sum_{k=1}^{\alpha} c_k(\theta_{j,n})h^k + E_{j,n,\alpha}, \tag{7}$$

where:

[3] Note that we always have $g'(0) = g'(\pi) = 0$ whenever $g(\theta)$ is an RCTP.

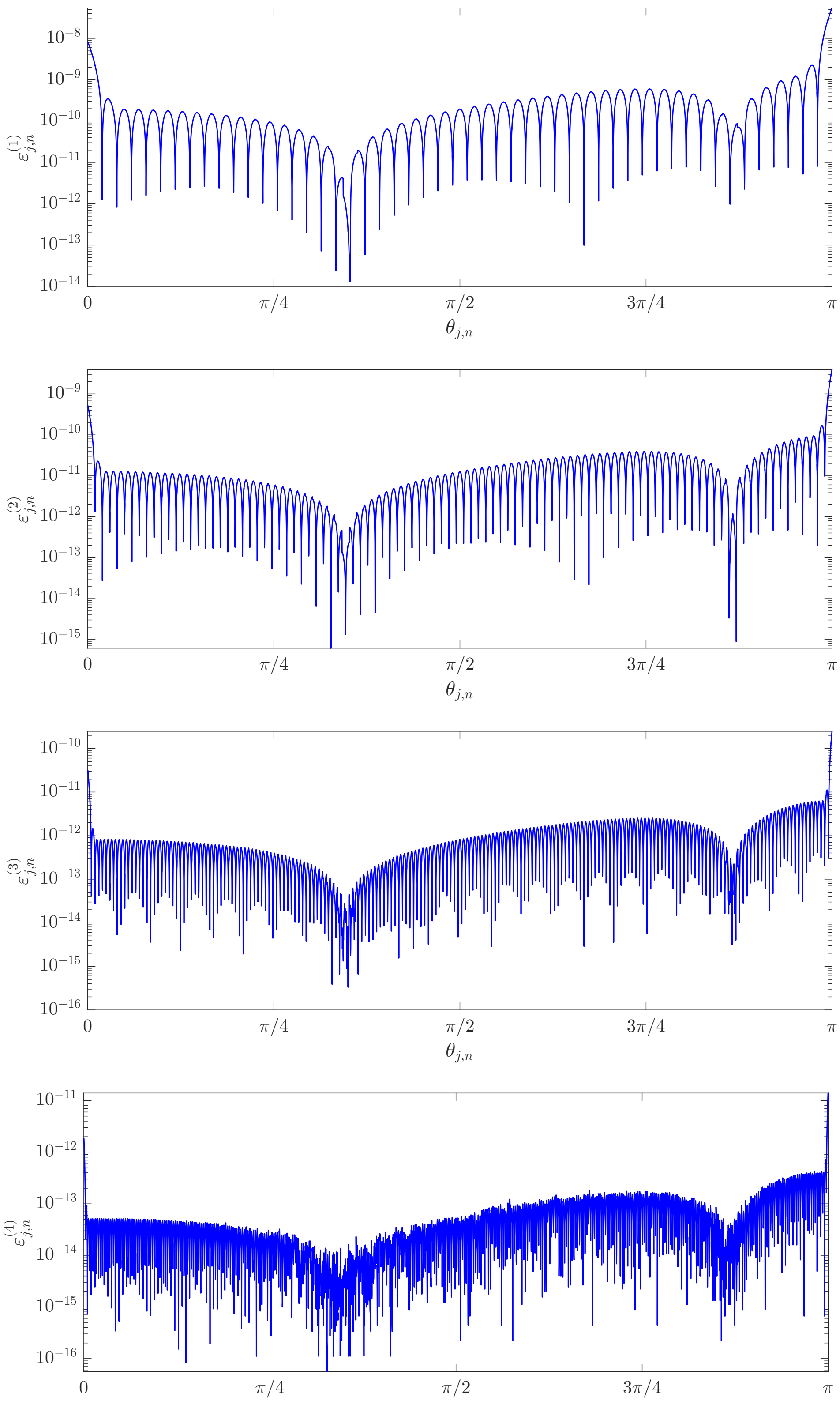

Figure 6: Example 5: errors $\varepsilon_{j,n}$ versus $\theta_{j,n}$ for $j = 1, \ldots, n$, in the case where $u(\theta) = 3 + 2\cos(\theta)$, $v(\theta) = 2 - \cos(\theta) - \cos(2\theta)$, $n = 5000$, $n_1 = 50 \cdot 2^{m-1}$, and $\alpha = 4$.

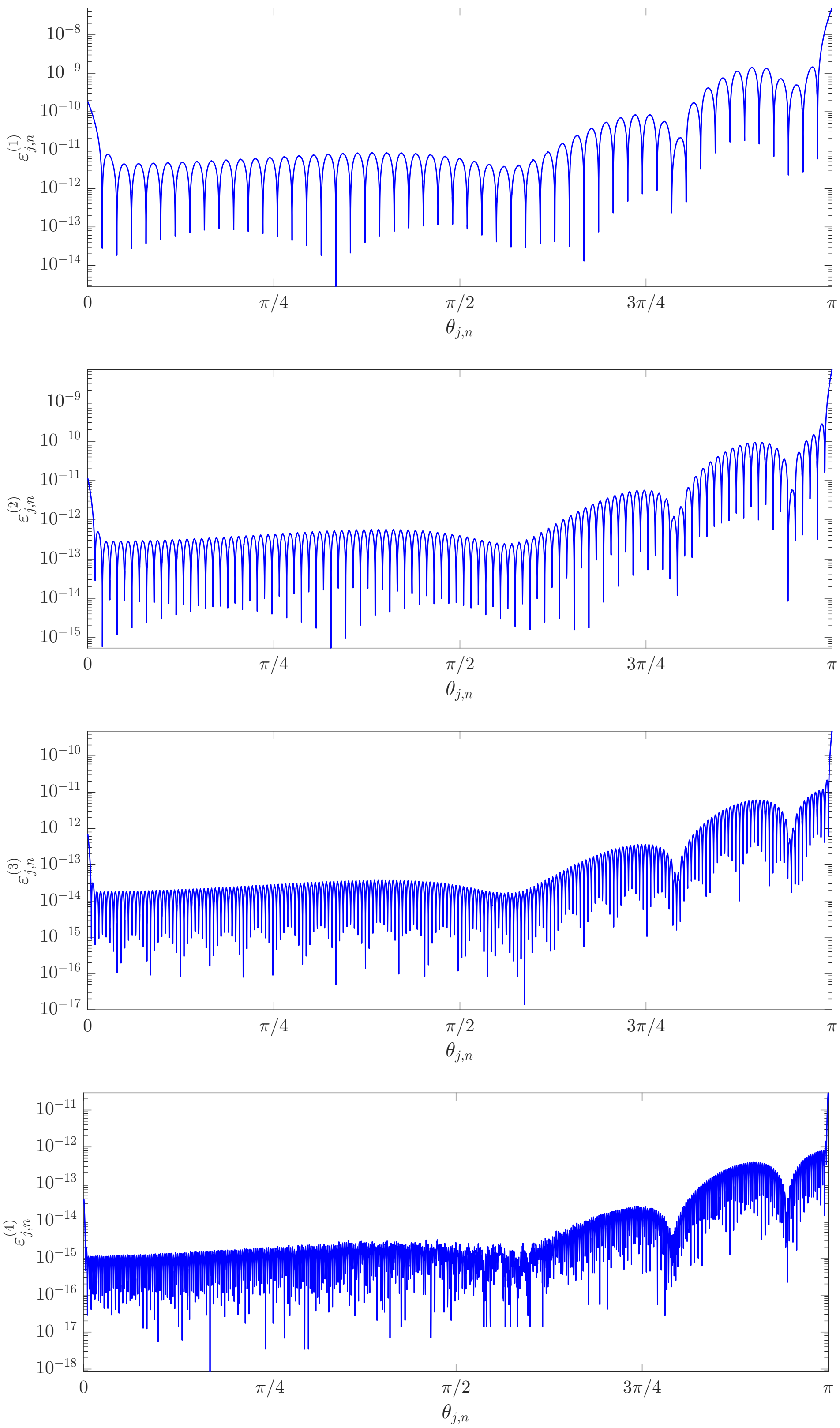

Figure 7: Example 6: errors $\varepsilon_{j,n}$ versus $\theta_{j,n}$ for $j = 1, \ldots, n$, in the case where $u(\theta) = 1208 + 1191\cos(\theta) + 120\cos(2\theta) + \cos(3\theta)$, $v(\theta) = 40 - 15\cos(\theta) - 24\cos(2\theta) - \cos(3\theta)$, $n = 5000$, $n_1 = 50 \cdot 2^{m-1}$, and $\alpha = 4$.

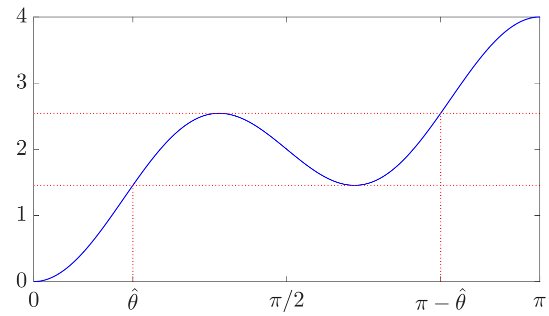

Figure 8: Example 7: graph of $f(\theta) = v(\theta)/u(\theta) = 2 - \cos(\theta) - \cos(3\theta)$ over $(0, \pi)$.

- the eigenvalues of X_n are arranged in non-decreasing order, $\lambda_1(X_n) \le \ldots \le \lambda_n(X_n)$;
- $\rho_n = \sigma_n^{-1}$, where σ_n is a permutation of $\{1, \ldots, n\}$ such that $f(\theta_{\sigma_n(1),n}) \le \ldots \le f(\theta_{\sigma_n(n),n})$;
- $\{c_k\}_{k=1,2,\ldots}$ is a sequence of functions from I to $\mathbb{R}$ which depends only on u, v;
- $h = \frac{1}{n+1}$ and $\theta_{j,n} = \frac{j\pi}{n+1} = j\pi h$;
- $E_{j,n,\alpha} = O(h^{\alpha+1})$ is the error, which satisfies the inequality $|E_{j,n,\alpha}| \le C_\alpha h^{\alpha+1}$ for some constant C_α depending only on α, u, v.

Conjecture 2 is clearly an extension of Conjecture 1. Indeed, in the case where f is monotone increasing on $(0, \pi)$, if we take $I = (0, \pi)$ and we note that both σ_n and ρ_n reduce to the identity on $\{1, \ldots, n\}$, we see that Conjecture 2 reduces to Conjecture 1. Conjecture 2 is based on the numerical experiments carried out in [1, 9]. In the case where $u = 1$ identically, it was already formulated in [9]. In the case where $u = 1$ identically and $\alpha = 0$, it can be formally proved by adapting the argument used by Bogoya, Böttcher, Grudsky, and Maximenko in the proof of [3, Theorem 1.6].

In the situation described in Conjecture 2, we propose the following natural modification of Algorithm 1 for computing the eigenvalues of X_n corresponding to the the interval I (that is, the eigenvalues $\lambda_{\rho_n(j)}(X_n)$ corresponding to points $\theta_{j,n} \in I$). In what follows, for any integer n_1 we denote by $n_1(I)$ the cardinality of $\{\theta_{1,n_1}, \ldots, \theta_{n_1,n_1}\} \cap I$.

Algorithm 2. With the notation introduced in Conjecture 2, given three integers $n, n_1, \alpha \in \mathbb{N}$ with $n_1(I) \ge \alpha$, we compute approximations of the eigenvalues $\{\lambda_{\rho_n(j)}(X_n) :\ \theta_{j,n} \in I\}$ as follows.

1. For $j_1 = 1, \ldots, n_1$ such that $\theta_{j_1,n_1} \in I$ compute $\tilde{c}_1(\theta_{j_1,n_1}), \ldots, \tilde{c}_\alpha(\theta_{j_1,n_1})$ by solving the linear system

$$\begin{cases} E_{j_1,n_1,0} = \tilde{c}_1(\theta_{j_1,n_1})h_1 + \tilde{c}_2(\theta_{j_1,n_1})h_1^2 + \ldots + \tilde{c}_\alpha(\theta_{j_1,n_1})h_1^\alpha \\ E_{j_2,n_2,0} = \tilde{c}_1(\theta_{j_1,n_1})h_2 + \tilde{c}_2(\theta_{j_1,n_1})h_2^2 + \ldots + \tilde{c}_\alpha(\theta_{j_1,n_1})h_2^\alpha \\ \quad\vdots \\ E_{j_\alpha,n_\alpha,0} = \tilde{c}_1(\theta_{j_1,n_1})h_\alpha + \tilde{c}_2(\theta_{j_1,n_1})h_\alpha^2 + \ldots + \tilde{c}_\alpha(\theta_{j_1,n_1})h_\alpha^\alpha \end{cases} \tag{8}$$

   where $n_k = 2^{k-1}(n_1 + 1) - 1$, $j_k = 2^{k-1}j_1$, and

$$E_{j_k,n_k,0} = \lambda_{\rho_{n_k}(j_k)}(X_{n_k}) - f(\theta_{j_1,n_1}), \qquad k = 1, \ldots, \alpha.$$

2. For $j = 1, \ldots, n$ such that $\theta_{j,n} \in I$
   - for $k = 1, \ldots, \alpha$
     - determine $\alpha - k + 1$ points $\theta^{(1)}, \ldots, \theta^{(\alpha-k+1)} \in \{\theta_{1,n_1}, \ldots, \theta_{n_1,n_1}\} \cap I$ which are closest to $\theta_{j,n}$;
     - compute $\tilde{c}_{k,j}(\theta_{j,n})$, where $\tilde{c}_{k,j}(\theta)$ is the interpolation polynomial of $(\theta^{(1)}, \tilde{c}_k(\theta^{(1)})), \ldots, (\theta^{(\alpha-k+1)}, \tilde{c}_k(\theta^{(\alpha-k+1)}))$;
   - compute $\tilde{\lambda}_{\rho_n(j)}(X_n) = f(\theta_{j,n}) + \sum_{k=1}^{\alpha} \tilde{c}_{k,j}(\theta_{j,n})h^k$.
3. Return $\{\tilde{\lambda}_{\rho_n(j)}(X_n) :\ \theta_{j,n} \in I\}$ as an approximation to $\{\lambda_{\rho_n(j)}(X_n) :\ \theta_{j,n} \in I\}$.

Example 7. Let

$$u(\theta) = 1,$$
$$v(\theta) = 2 - \cos(\theta) - \cos(3\theta).$$

The graph of $f(\theta) = v(\theta)/u(\theta) = v(\theta)$ is depicted in Figure 8. The hypotheses of Conjecture 2 are satisfied with either $I = (0, \hat{\theta})$ or $I = (\pi - \hat{\theta}, \pi)$, where $\hat{\theta} = 0.61547970867038...$ To fix the ideas, let $I = (0, \hat{\theta})$. Note that any permutation

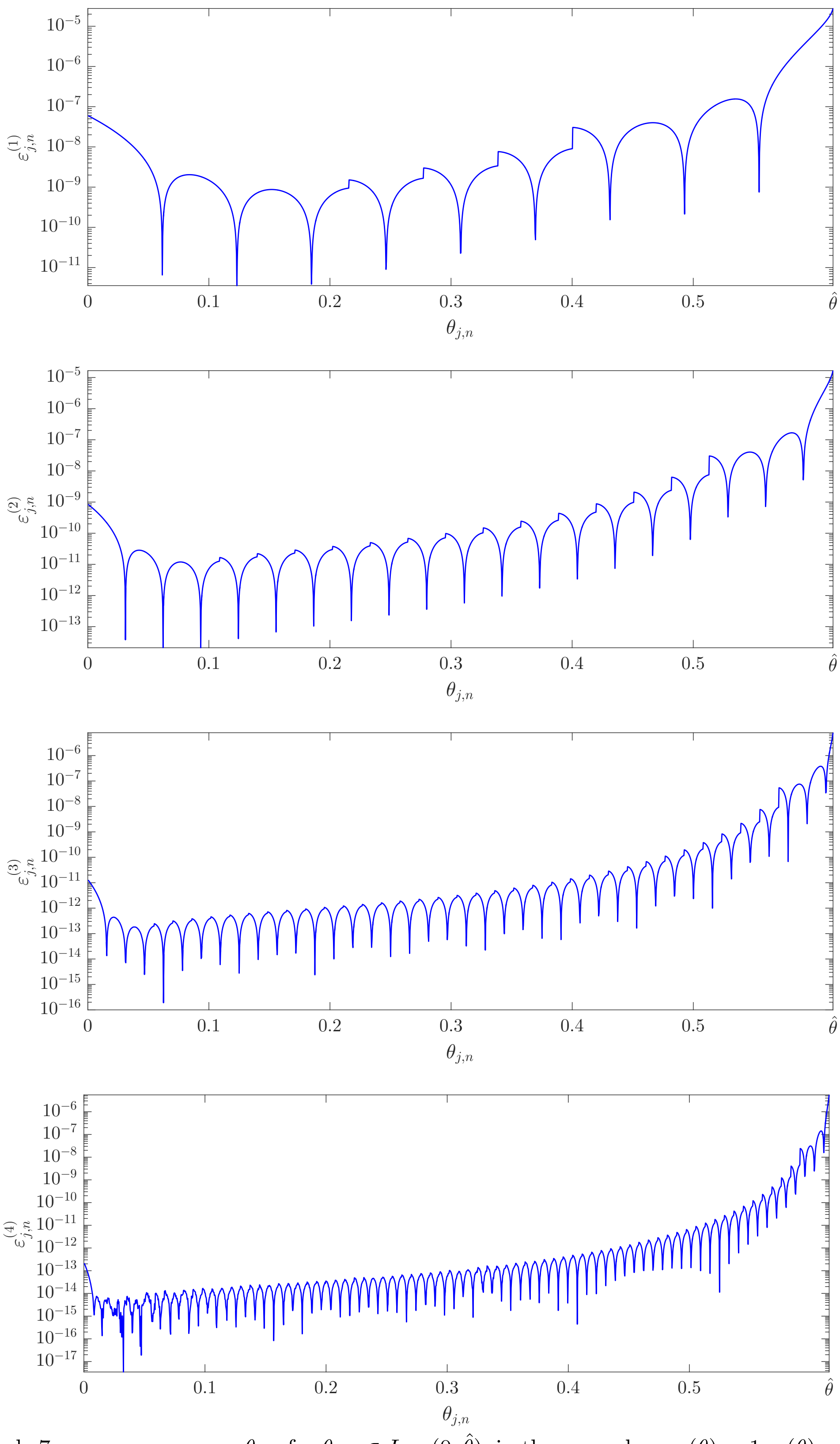

Figure 9: Example 7: errors $\varepsilon_{j,n}$ versus $\theta_{j,n}$ for $\theta_{j,n} \in I = (0, \hat{\theta})$, in the case where $u(\theta) = 1$, $v(\theta) = 2 - \cos(\theta) - \cos(3\theta)$, $n = 10000$, $n_1 = 50 \cdot 2^{m-1}$, and $\alpha = 5$.

σ_n which sorts the samples $f(\theta_{1,n}),\ldots,f(\theta_{n,n})$ in non-decreasing order is such that $\sigma_n(j)=j$ whenever $\theta_{j,n}\in I$. As a consequence, $\rho_n(j)=j$ whenever $\theta_{j,n}\in I$. Set $X_n=T_n(u)^{-1}T_n(v)=T_n(f)$ and let $\{\tilde\lambda_j(X_n):\ \theta_{j,n}\in I\}$ be the approximation of $\{\lambda_j(X_n):\ \theta_{j,n}\in I\}$ obtained for $n=10000$ by applying Algorithm 2 with $n_1=50\cdot 2^{m-1}$ and $\alpha=5$. The graph of the errors $\varepsilon_{j,n}=|\lambda_j(X_n)-\tilde\lambda_j(X_n)|$ versus $\theta_{j,n}$ is shown in Figure 9 for $\theta_{j,n}\in I$ and $m=1,2,3,4$. We note that the error $\varepsilon_{j,n}$ tends to increase as $\theta_{j,n}$ moves toward $\hat\theta$, that is, as $\theta_{j,n}$ approaches to exit the interval I over which f satisfies the assumptions of Conjecture 2. Moreover, in a neighborhood of $\hat\theta$ the error decreases very slowly. This phenomenon is related to the fact that the expansion (7) does not hold in $[\hat\theta,\pi-\hat\theta]$ and, in fact, the errors $E_{j,n,0}=\lambda_{\rho_n(j)}(X_n)-f(\theta_{j,n})$ have a wild behavior inside this interval; see [9, Figure 7].

5 Conclusions and Perspectives

We have proposed and analyzed an interpolation–extrapolation algorithm for computing the eigenvalues of preconditioned banded symmetric Toeplitz matrices of the form $T_n(u)^{-1}T_n(v)$, where u,v are RCTPs, $u>0$ on $(0,\pi)$, and $f=v/u$ is monotone on $(0,\pi)$. We have illustrated the performance of the algorithm through numerical experiments, and we have presented its generalization to the case where $f=v/u$ is non-monotone. We conclude by suggesting two possible future lines of research.

- Algorithm 1, as well as its generalized version for the non-monotone case (Algorithm 2), is based on a local interpolation strategy, as described in Section 2.1. An interesting topic for future research could be the following: try another kind of approximation (for example, an higher-order spline approximation) to see whether this reduces the errors and accelerates the convergence of both these algorithms.
- Understand whether an asymptotic eigenvalue expansion analogous to (7) holds without the hypothesis that f restricted to some interval $I\subseteq(0,\pi)$ is monotone and satisfies $f^{-1}(f(I))=I$. Such a result would eliminate any limitation in the applicability of Algorithm 2 (provided that the latter is properly modified according to the new expansion).

A Appendix

This appendix collects the proofs of Theorems 1 and 2.

Proof of Theorem 1. We follow the argument in [1, Section 2]. Equations (2) and (4) can be rewritten as

$$A(h_1,\ldots,h_1)\mathbf{c}(j_1)=\mathbf{E}_0(j_1)-\mathbf{E}_\alpha(j_1) \tag{9}$$

$$A(h_1,\ldots,h_1)\tilde{\mathbf{c}}(j_1)=\mathbf{E}_0(j_1), \tag{10}$$

where

$$\mathbf{c}(j_1)=\begin{bmatrix}c_1(\theta_{j_1,n_1})\\ \vdots\\ c_\alpha(\theta_{j_1,n_1})\end{bmatrix},\qquad \tilde{\mathbf{c}}(j_1)=\begin{bmatrix}\tilde c_1(\theta_{j_1,n_1})\\ \vdots\\ \tilde c_\alpha(\theta_{j_1,n_1})\end{bmatrix},\qquad \mathbf{E}_0(j_1)=\begin{bmatrix}E_{j_1,n_1,0}\\ \vdots\\ E_{j_\alpha,n_\alpha,0}\end{bmatrix},\qquad \mathbf{E}_\alpha(j_1)=\begin{bmatrix}E_{j_1,n_1,\alpha}\\ \vdots\\ E_{j_\alpha,n_\alpha,\alpha}\end{bmatrix}, \tag{11}$$

and

$$A(h_1,\ldots,h_\alpha)=\mathrm{diag}(h_1,\ldots,h_\alpha)\,V(h_1,\ldots,h_\alpha), \tag{12}$$

with $V(h_1,\ldots,h_\alpha)$ being the Vandermonde matrix associated with the nodes $h_1,\ldots,h_\alpha$,

$$V(h_1,\ldots,h_\alpha)=\begin{bmatrix}1 & h_1 & h_1^2 & \cdots & h_1^{\alpha-1}\\ 1 & h_2 & h_2^2 & \cdots & h_2^{\alpha-1}\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & h_\alpha & h_\alpha^2 & \cdots & h_\alpha^{\alpha-1}\end{bmatrix}.$$

By (9), (10), and (12), we have

$$\tilde{\mathbf{c}}(j_1)-\mathbf{c}(j_1)=A(h_1,\ldots,h_\alpha)^{-1}\mathbf{E}_\alpha(j_1)=V(h_1,\ldots,h_\alpha)^{-1}\mathbf{F}_\alpha(j_1),$$

where

$$\mathbf{F}_\alpha(j_1) = \mathrm{diag}(h_1, \ldots, h_\alpha)^{-1}\mathbf{E}_\alpha(j_1) = \begin{bmatrix} E_{j_1,n_1,\alpha}/h_1 \\ \vdots \\ E_{j_\alpha,n_\alpha,\alpha}/h_\alpha \end{bmatrix}.$$

Note that, by (3),

$$|(\mathbf{F}_\alpha(j_1))_k| = |E_{j_k,n_k,\alpha}/h_k| \le C_\alpha h_k^\alpha, \qquad k = 1, \ldots, \alpha. \tag{13}$$

The inverse of $V(h_1, \ldots, h_\alpha)$ is explicitly given by

$$(V(h_1, \ldots, h_\alpha)^{-1})_{ij} = \begin{cases} (-1)^{\alpha-i} \dfrac{\displaystyle\sum_{\substack{1\le k_1<\ldots<k_{\alpha-i}\le\alpha \\ k_1,\ldots,k_{\alpha-i}\ne j}} h_{k_1}\cdots h_{k_{\alpha-i}}}{\displaystyle\prod_{\substack{1\le k\le\alpha \\ k\ne j}} (h_j - h_k)}, & 1 \le i < \alpha, \\ \dfrac{1}{\displaystyle\prod_{\substack{1\le k\le\alpha \\ k\ne j}} (h_j - h_k)}, & i = \alpha. \end{cases} \tag{14}$$

Taking into account (13) and the equation $h_k = 2^{1-k}h_1$ for $k = 1, \ldots, \alpha$, we obtain the following.

- For $i = \alpha$,

$$\begin{aligned} |\tilde{c}_\alpha(\theta_{j_1,n_1}) - c_\alpha(\theta_{j_1,n_1})| &= |(\tilde{\mathbf{c}}(j_1) - \mathbf{c}(j_1))_\alpha| = \left|\sum_{j=1}^{\alpha}(V(h_1, \ldots, h_\alpha)^{-1})_{\alpha j}(\mathbf{F}_\alpha(j_1))_j\right| \\ &\le \sum_{j=1}^{\alpha} \frac{|(\mathbf{F}_\alpha(j_1))_j|}{\displaystyle\prod_{\substack{1\le k\le\alpha \\ k\ne j}} |h_j - h_k|} \le \sum_{j=1}^{\alpha} \frac{C_\alpha h_j^\alpha}{h_j^{\alpha-1}\displaystyle\prod_{\substack{1\le k\le\alpha \\ k\ne j}} |1 - h_k/h_j|} \\ &= C_\alpha h_1 \sum_{j=1}^{\alpha} \frac{2^{1-j}}{\displaystyle\prod_{\substack{1\le k\le\alpha \\ k\ne j}} |1 - 2^{j-k}|} = A(\alpha)h_1, \end{aligned}$$

  with $A(\alpha)$ depending only on α, u, v just like C_α.

- For $1 \le i < \alpha$,

$$\begin{aligned} |\tilde{c}_i(\theta_{j_1,n_1}) - c_i(\theta_{j_1,n_1})| &= |(\tilde{\mathbf{c}}(j_1) - \mathbf{c}(j_1))_i| = \left|\sum_{j=1}^{\alpha}(V(h_1, \ldots, h_\alpha)^{-1})_{ij}(\mathbf{F}_\alpha(j_1))_j\right| \\ &\le \sum_{j=1}^{\alpha} \frac{|(\mathbf{F}_\alpha(j_1))_j| \displaystyle\sum_{\substack{1\le k_1<\ldots<k_{\alpha-i}\le\alpha \\ k_1,\ldots,k_{\alpha-i}\ne j}} h_{k_1}\cdots h_{k_{\alpha-i}}}{\displaystyle\prod_{\substack{1\le k\le\alpha \\ k\ne j}} |h_j - h_k|} \le \sum_{j=1}^{\alpha} \frac{C_\alpha h_j^\alpha \displaystyle\sum_{\substack{1\le k_1<\ldots<k_{\alpha-i}\le\alpha \\ k_1,\ldots,k_{\alpha-i}\ne j}} h_{k_1}\cdots h_{k_{\alpha-i}}}{h_j^{\alpha-1}\displaystyle\prod_{\substack{1\le k\le\alpha \\ k\ne j}} |1 - h_k/h_j|} \\ &= C_\alpha h_1^{\alpha-i+1} \sum_{j=1}^{\alpha} \frac{2^{1-j} \displaystyle\sum_{\substack{1\le k_1<\ldots<k_{\alpha-i}\le\alpha \\ k_1,\ldots,k_{\alpha-i}\ne j}} 2^{1-k_1}\cdots 2^{1-k_{\alpha-i}}}{\displaystyle\prod_{\substack{1\le k\le\alpha \\ k\ne j}} |1 - 2^{j-k}|} = A(\alpha, i)h_1^{\alpha-i+1}, \end{aligned}$$

  with $A(\alpha, i)$ depending only on α, i, u, v.

In conclusion, Theorem 1 is proved with $A_\alpha = \max_{i=1,\ldots,\alpha} A(\alpha, i)$, where $A(\alpha, \alpha) = A(\alpha)$. □

Proof of Theorem 2. Let $L_1, \ldots, L_{\alpha-k+1}$ be the Lagrange polynomials associated with the nodes $\theta^{(1)}, \ldots, \theta^{(\alpha-k+1)}$,

$$L_r(\theta) = \prod_{\substack{s=1\\ s\neq r}}^{\alpha-k+1} \frac{\theta - \theta^{(s)}}{\theta^{(r)} - \theta^{(s)}}, \qquad r = 1, \ldots, \alpha - k + 1.$$

The interpolation polynomial of the data $(\theta^{(1)}, \tilde{c}_k(\theta^{(1)})), \ldots, (\theta^{(\alpha-k+1)}, \tilde{c}_k(\theta^{(\alpha-k+1)}))$ is

$$\tilde{c}_{k,j}(\theta) = \sum_{r=1}^{\alpha-k+1} \tilde{c}_k(\theta^{(r)}) L_r(\theta)$$

and the interpolation polynomial of the data $(\theta^{(1)}, c_k(\theta^{(1)})), \ldots, (\theta^{(\alpha-k+1)}, c_k(\theta^{(\alpha-k+1)}))$ is

$$p(\theta) = \sum_{r=1}^{\alpha-k+1} c_k(\theta^{(r)}) L_r(\theta).$$

Considering that $\theta^{(1)}, \ldots, \theta^{(\alpha-k+1)}$ are $\alpha - k + 1$ points from $\{\theta_{1,n_1}, \ldots, \theta_{n_1,n_1}\}$ which are closest to $\theta_{j,n}$, the length of the smallest interval I containing the nodes $\theta^{(1)}, \ldots, \theta^{(\alpha-k+1)}$ and the point $\theta_{j,n}$ is bounded by $(\alpha - k + 1)\pi h_1$. Hence, by Theorem 1, for all $\theta \in I$ we have

$$\begin{aligned}
|\tilde{c}_{k,j}(\theta) - p(\theta)| &\le \sum_{r=1}^{\alpha-k+1} |\tilde{c}_{k,j}(\theta^{(r)}) - c_k(\theta^{(r)})| \prod_{\substack{s=1\\ s\neq r}}^{\alpha-k+1} \frac{|\theta - \theta^{(s)}|}{|\theta^{(r)} - \theta^{(s)}|} \\
&\le \sum_{r=1}^{\alpha-k+1} A_\alpha h_1^{\alpha-k+1} \prod_{\substack{s=1\\ s\neq r}}^{\alpha-k+1} \frac{(\alpha - k + 1)\pi h_1}{\pi h_1} \\
&= A_\alpha h_1^{\alpha-k+1} (\alpha - k + 1)^{\alpha-k+1}.
\end{aligned} \tag{15}$$

Since $c_k \in C^{\alpha-k+1}([0, \pi])$ by assumption, from interpolation theory we know that for every $\theta \in I$ there exists $\xi(\theta) \in I$ such that

$$c_k(\theta) - p(\theta) = \frac{c_k^{(\alpha-k+1)}(\xi(\theta))}{(\alpha - k + 1)!} \prod_{r=1}^{\alpha-k+1} (\theta - \theta^{(r)});$$

see, e.g., [8, Theorem 3.1.1]. Thus, for all $\theta \in I$ we have

$$\begin{aligned}
|c_k(\theta) - p(\theta)| &\le \frac{|c_k^{(\alpha-k+1)}(\xi(\theta))|}{(\alpha - k + 1)!} \prod_{r=1}^{\alpha-k+1} |\theta - \theta^{(r)}| \\
&\le \frac{\|c_k^{(\alpha-k+1)}\|_\infty}{(\alpha - k + 1)!} \prod_{r=1}^{\alpha-k+1} (\alpha - k + 1)\pi h_1 \\
&= \frac{(\alpha - k + 1)^{\alpha-k+1} \pi^{\alpha-k+1} \|c_k^{(\alpha-k+1)}\|_\infty}{(\alpha - k + 1)!} h_1^{\alpha-k+1}.
\end{aligned} \tag{16}$$

From (15) and (16) we obtain

$$|c_k(\theta) - \tilde{c}_{k,j}(\theta)| \le B(k, \alpha) h_1^{\alpha-k+1} \le B_\alpha h_1^{\alpha-k+1}, \qquad \theta \in I, \tag{17}$$

where

$$B(k, \alpha) = \frac{(\alpha - k + 1)^{\alpha-k+1} \pi^{\alpha-k+1} \|c_k^{(\alpha-k+1)}\|_\infty}{(\alpha - k + 1)!} + A_\alpha (\alpha - k + 1)^{\alpha-k+1}$$

and $B_\alpha = \max_{i=1,\ldots,\alpha} B(i, \alpha)$. Since $\theta_{j,n} \in I$, it is clear that (6) follows from (17). $\square$

Acknowledgements

The research of Sven-Erik Ekström is cofinanced by the Graduate School in Mathematics and Computing (FMB) and Uppsala University. Carlo Garoni is a Marie-Curie fellow of the Italian INdAM (Istituto Nazionale di Alta Matematica) under grant agreement PCOFUND-GA-2012-600198.

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