



Block Generalized Locally Toeplitz Sequences: From the Theory to the Applications

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Abstract: The theory of generalized locally Toeplitz (GLT) sequences is a powerful apparatus for computing the asymptotic spectral distribution of matrices A_n arising from virtually any kind of numerical discretization of differential equations (DEs). Indeed, when the mesh fineness parameter n tends to infinity, these matrices A_n give rise to a sequence $\{A_n\}_n$, which often turns out to be a GLT sequence or one of its “relatives”, i.e., a block GLT sequence or a reduced GLT sequence. In particular, block GLT sequences are encountered in the discretization of systems of DEs as well as in the higher-order finite element or discontinuous Galerkin approximation of scalar DEs. Despite the applicative interest, a solid theory of block GLT sequences has been developed only recently, in 2018. The purpose of the present paper is to illustrate the potential of this theory by presenting a few noteworthy examples of applications in the context of DE discretizations.

Keywords: spectral (eigenvalue) and singular value distributions; generalized locally Toeplitz sequences; discretization of systems of differential equations; higher-order finite element methods; discontinuous Galerkin methods; finite difference methods; isogeometric analysis; B-splines; curl-curl operator, time harmonic Maxwell’s equations and magnetostatic problems; linear elasticity

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1. Introduction

The theory of generalized locally Toeplitz (GLT) sequences stems from Tilli’s work on locally Toeplitz (LT) sequences [39] and from the spectral theory of Toeplitz matrices [2,10–13,30,32,38,40,41,43]. It was then carried forward in [26,27,36,37], and it has been recently extended by Barbarino in [3]. This theory is a powerful apparatus for computing the asymptotic spectral distribution of matrices arising from the numerical discretization of continuous problems, such as integral equations (IEs) and, especially, differential equations (DEs). The experience reveals that virtually any kind of numerical methods for the discretization of DEs gives rise to structured matrices A_n whose asymptotic spectral distribution, as the mesh fineness parameter n tends to infinity, can usually be computed through the theory of GLT sequences. We refer the reader to [26, Section 10.5], [27, Section 8.5], and [9,36,37] for applications of the theory of GLT sequences in the context of finite difference (FD) discretizations of DEs; to [26, Section 10.6] and [5,9,37] for the finite element (FE) case; to [7] for the finite volume (FV) case; to [26, Section 10.7], [27, Section 8.5], and [17,22,24,25,33] for the case of isogeometric analysis (IgA) discretizations, both in the collocation and Galerkin frameworks; and to [18] for a further recent

application to fractional DEs. We also refer the reader to [26, Section 10.4] and [1,34] for a look at the GLT approach for sequences of matrices coming from IE discretizations.

In the very recent work [29], starting from the original intuition by the third author [37, Section 3.3], the theory of block GLT sequences has been developed in a systematic way as an extension of the theory of GLT sequences. Such an extension is of the utmost importance in practical applications. In particular, it provides the necessary tools for computing the spectral distribution of block structured matrices arising from the discretization of systems of DEs [37, Section 3.3] and from the higher-order finite element or discontinuous Galerkin approximation of scalar/vectorial DEs [6,20,28]. The purpose of this paper is to illustrate the potential of the theory of block GLT sequences [29] and of its multivariate version — which combines the results of [29] with the “multivariate technicalities” from [27] — by presenting a few noteworthy examples of applications. Actually, the present paper can be seen as a necessary completion of the purely theoretical work [29].

The paper is organized as follows. In Section 2 we report a summary of the theory of block GLT sequences. In Section 3 we focus on the FD discretization of a model system of univariate DEs; through the theory of block GLT sequences, we compute the spectral distribution of the related discretization matrices. In Section 4 we focus on the higher-order FE approximation of the univariate diffusion equation; again, we compute the spectral distribution of the associated discretization matrices through the theory of block GLT sequences. In Section 5 we summarize the multivariate version of the theory of block GLT sequences, also known as the theory of multilevel block GLT sequences. In Section 6 we describe the general GLT approach for computing the spectral distribution of matrices arising from the discretization of systems of partial differential equations (PDEs). In Section 7 we focus on the B-spline IgA approximation of a bivariate variational problem for the curl-curl operator, which is of interest in magnetostatics; through the theory of multilevel block GLT sequences, we compute the spectral distribution of the related discretization matrices. In Section 8 we focus on the linear elasticity equations discretized through the modified FE Taylor–Hood Q1isoQ1 stable pair; again, we compute the spectral distribution of the related discretization matrices through the theory of multilevel block GLT sequences. Final considerations are collected in Section 9.

2. The Theory of Block GLT Sequences

In this section we summarize the theory of block GLT sequences, which was originally introduced in [37, Section 3.3] and has been recently revised and systematically developed in [29].

Sequences of Matrices and Block Matrix-Sequences. Throughout this paper, a sequence of matrices is any sequence of the form $\{A_n\}_n$, where A_n is a square matrix of size d_n and $d_n \rightarrow \infty$ as $n \rightarrow \infty$. Let $s \geq 1$ be a fixed positive integer independent of n ; an s -block matrix-sequence (or simply a matrix-sequence if s can be inferred from the context or we do not need/want to specify it) is a special sequence of matrices $\{A_n\}_n$ in which the size of A_n is $d_n = sn$.

Singular Value and Eigenvalue Distribution of a Sequence of Matrices. Let μ_k be the Lebesgue measure in \mathbb{R}^k . Throughout this paper, all the terminology from measure theory (such as “measurable set”, “measurable function”, “a.e.”, etc.) is referred to the Lebesgue measure. A matrix-valued function $f : D \subseteq \mathbb{R}^k \rightarrow \mathbb{C}^{r \times r}$ is said to be measurable (resp., continuous, Riemann-integrable, in $L^p(D)$, etc.) if its components $f_{\alpha\beta} : D \rightarrow \mathbb{C}$, $\alpha, \beta = 1, \dots, r$, are measurable (resp., continuous, Riemann-integrable, in $L^p(D)$, etc.). We denote by $C_c(\mathbb{R})$ (resp., $C_c(\mathbb{C})$) the space of continuous complex-valued functions with bounded support defined on \mathbb{R} (resp., \mathbb{C}). If $A \in \mathbb{C}^{m \times m}$, the singular values and the eigenvalues of A are denoted by $\sigma_1(A), \dots, \sigma_m(A)$ and $\lambda_1(A), \dots, \lambda_m(A)$, respectively.

Definition 1. Let $\{A_n\}_n$ be a sequence of matrices, with A_n of size d_n , and let $f : D \subset \mathbb{R}^k \rightarrow \mathbb{C}^{r \times r}$ be a measurable function defined on a set D with $0 < \mu_k(D) < \infty$.

- We say that $\{A_n\}_n$ has a (asymptotic) singular value distribution described by f , and we write $\{A_n\}_n \sim_\sigma f$, if

$$\lim_{n \rightarrow \infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\sigma_i(A_n)) = \frac{1}{\mu_k(D)} \int_D \frac{\sum_{i=1}^r F(\sigma_i(f(\mathbf{x})))}{r} d\mathbf{x}, \quad \forall F \in C_c(\mathbb{R}). \quad (1)$$

- We say that $\{A_n\}_n$ has a (asymptotic) spectral (or eigenvalue) distribution described by f , and we write $\{A_n\}_n \sim_\lambda f$, if

$$\lim_{n \rightarrow \infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\lambda_i(A_n)) = \frac{1}{\mu_k(D)} \int_D \frac{\sum_{i=1}^r F(\lambda_i(f(\mathbf{x})))}{r} d\mathbf{x}, \quad \forall F \in C_c(\mathbb{C}). \quad (2)$$

If $\{A_n\}_n$ has both a singular value and an eigenvalue distribution described by f , we write $\{A_n\}_n \sim_{\sigma,\lambda} f$.

We note that Definition 1 is well-posed because the functions $\mathbf{x} \mapsto \sum_{i=1}^r F(\sigma_i(f(\mathbf{x})))$ and $\mathbf{x} \mapsto \sum_{i=1}^r F(\lambda_i(f(\mathbf{x})))$ are measurable [29, Lemma 2.1]. Whenever we write a relation such as $\{A_n\}_n \sim_\sigma f$ or $\{A_n\}_n \sim_\lambda f$, it is understood that f is as in Definition 1; that is, f is a measurable function defined on a subset D of some \mathbb{R}^k with $0 < \mu_k(D) < \infty$ and taking values in $\mathbb{C}^{r \times r}$ for some $r \geq 1$. We refer the reader to [28, Remark 1] or [21, Section 4] for the informal meaning behind the spectral distribution (2); a completely analogous meaning can also be given for the singular value distribution (1).

The next two theorems are useful tools for computing the spectral distribution of sequences formed by Hermitian or perturbed Hermitian matrices. For the related proofs, we refer the reader to [31, Theorem 4.3] and [4, Theorem 1.1]. In what follows, the conjugate transpose of the matrix A is denoted by A^* . If $A \in \mathbb{C}^{m \times m}$ and $1 \leq p \leq \infty$, we denote by $\|A\|_p$ the Schatten p -norm of A , i.e., the p -norm of the vector $(\sigma_1(A), \dots, \sigma_m(A))$. The Schatten ∞ -norm $\|A\|_\infty$ is the largest singular value of A and coincides with the spectral norm $\|A\|$. The Schatten 1-norm $\|A\|_1$ is the sum of the singular values of A and is often referred to as the trace-norm of A . The Schatten 2-norm $\|A\|_2$ coincides with the Frobenius norm of A . For more on Schatten p -norms, see [8].

Theorem 1. Let $\{X_n\}_n$ be a sequence of matrices, with X_n Hermitian of size d_n , and let $\{P_n\}_n$ be a sequence such that $P_n \in \mathbb{C}^{d_n \times d_n}$, $P_n^* P_n = I_{\delta_n}$, $\delta_n \leq d_n$ and $\delta_n/d_n \rightarrow 1$ as $n \rightarrow \infty$. Then, $\{X_n\}_n \sim_{\sigma,\lambda} \kappa$ if and only if $\{P_n^* X_n P_n\}_n \sim_{\sigma,\lambda} \kappa$.

Theorem 2. Let $\{X_n\}_n$ and $\{Y_n\}_n$ be sequences of matrices, with X_n and Y_n of size d_n . Assume that:

- the matrices X_n are Hermitian and $\{X_n\}_n \sim_\lambda \kappa$;
- $\|Y_n\|_2 = o(\sqrt{d_n})$;

then $\{X_n + Y_n\}_n \sim_\lambda \kappa$.

Block Toeplitz Matrices. Given a function $f : [-\pi, \pi] \rightarrow \mathbb{C}^{s \times s}$ in $L^1([-\pi, \pi])$, its Fourier coefficients are denoted by

$$f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} d\theta \in \mathbb{C}^{s \times s}, \quad k \in \mathbb{Z},$$

where the integrals are computed componentwise. The n th block Toeplitz matrix generated by f is defined as

$$T_n(f) = [f_{i-j}]_{i,j=1}^n \in \mathbb{C}^{sn \times sn}.$$

It is not difficult to see that all the matrices $T_n(f)$ are Hermitian when f is Hermitian a.e.

Block Diagonal Sampling Matrices. For $n \in \mathbb{N}$ and $a : [0, 1] \rightarrow \mathbb{C}^{s \times s}$, we define the block diagonal sampling matrix $D_n(a)$ as the diagonal matrix

$$D_n(a) = \text{diag}_{i=1, \dots, n} a\left(\frac{i}{n}\right) = \begin{bmatrix} a\left(\frac{1}{n}\right) & & & \\ & a\left(\frac{2}{n}\right) & & \\ & & \ddots & \\ & & & a(1) \end{bmatrix} \in \mathbb{C}^{sn \times sn}.$$

Zero-Distributed Sequences. A sequence of matrices $\{Z_n\}_n$ such that $\{Z_n\}_n \sim_\sigma 0$ is referred to as a zero-distributed sequence. Note that, for any $r \geq 1$, $\{Z_n\}_n \sim_\sigma 0$ is equivalent to $\{Z_n\}_n \sim_\sigma O_r$ (throughout this paper, O_m and I_m denote the $m \times m$ zero matrix and the $m \times m$ identity matrix, respectively). Proposition 1 provides an important characterization of zero-distributed sequences together with a useful sufficient condition for detecting such sequences. Throughout this paper we use the natural convention $1/\infty = 0$.

Proposition 1. *Let $\{Z_n\}_n$ be a sequence of matrices, with Z_n of size d_n .*

- $\{Z_n\}_n$ is zero-distributed if and only if $Z_n = R_n + N_n$ with $\text{rank}(R_n)/d_n \rightarrow 0$ and $\|N_n\| \rightarrow 0$ as $n \rightarrow \infty$.
- $\{Z_n\}_n$ is zero-distributed if there exists a $p \in [1, \infty]$ such that $\|Z_n\|_p / (d_n)^{1/p} \rightarrow 0$ as $n \rightarrow \infty$.

Approximating Classes of Sequences. The notion of approximating classes of sequences (a.c.s.) is the fundamental concept on which the theory of block GLT sequences is based.

Definition 2. *Let $\{A_n\}_n$ be a sequence of matrices, with A_n of size d_n , and let $\{\{B_{n,m}\}_m\}_n$ be a sequence of sequences of matrices, with $B_{n,m}$ of size d_n . We say that $\{\{B_{n,m}\}_m\}_n$ is an approximating class of sequences (a.c.s.) for $\{A_n\}_n$ if the following condition is met: for every m there exists n_m such that, for $n \geq n_m$,*

$$A_n = B_{n,m} + R_{n,m} + N_{n,m}, \quad \text{rank}(R_{n,m}) \leq c(m)d_n, \quad \|N_{n,m}\| \leq \omega(m),$$

where $n_m, c(m), \omega(m)$ depend only on m and $\lim_{m \rightarrow \infty} c(m) = \lim_{m \rightarrow \infty} \omega(m) = 0$.

Roughly speaking, $\{\{B_{n,m}\}_m\}_n$ is an a.c.s. for $\{A_n\}_n$ if, for large m , the sequence $\{B_{n,m}\}_n$ approximates $\{A_n\}_n$ in the sense that A_n is eventually equal to $B_{n,m}$ plus a small-rank matrix (with respect to the matrix size d_n) plus a small-norm matrix. It turns out that, for each fixed sequence of positive integers d_n such that $d_n \rightarrow \infty$, the notion of a.c.s. is a notion of convergence in the space

$$\mathcal{E} = \{\{A_n\}_n : A_n \in \mathbb{C}^{d_n \times d_n}\}.$$

More precisely, there exists a pseudometric $d_{\text{a.c.s.}}$ in \mathcal{E} such that $\{\{B_{n,m}\}_m\}_n$ is an a.c.s. for $\{A_n\}_n$ if and only if $d_{\text{a.c.s.}}(\{B_{n,m}\}_n, \{A_n\}_n) \rightarrow 0$ as $m \rightarrow \infty$. We will therefore use the convergence notation $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ to indicate that $\{\{B_{n,m}\}_m\}_n$ is an a.c.s. for $\{A_n\}_n$. A useful criterion to identify an a.c.s. is provided in the next proposition [26, Corollary 5.3].

Proposition 2. *Let $\{A_n\}_n$ be a sequence of matrices, with A_n of size d_n , let $\{\{B_{n,m}\}_m\}_n$ be a sequence of sequences of matrices, with $B_{n,m}$ of size d_n , and let $p \in [1, \infty]$. Suppose that for every m there exists n_m such that, for $n \geq n_m$,*

$$\|A_n - B_{n,m}\|_p \leq \varepsilon(m, n)(d_n)^{1/p},$$

where $\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \varepsilon(m, n) = 0$. Then $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$.

If $X \in \mathbb{C}^{m_1 \times m_2}$ and $Y \in \mathbb{C}^{\ell_1 \times \ell_2}$ are any two matrices, the tensor (Kronecker) product of X and Y is the $m_1 \ell_1 \times m_2 \ell_2$ matrix defined as follows:

$$X \otimes Y = [x_{ij}Y]_{\substack{i=1,\dots,m_1 \\ j=1,\dots,m_2}} = \begin{bmatrix} x_{11}Y & \cdots & x_{1m_2}Y \\ \vdots & & \vdots \\ x_{m_11}Y & \cdots & x_{m_1m_2}Y \end{bmatrix}.$$

We recall that the tensor product operation \otimes is associative and bilinear. Moreover,

$$\|X \otimes Y\| = \|X\| \|Y\|, \quad (3)$$

$$\text{rank}(X \otimes Y) = \text{rank}(X)\text{rank}(Y), \quad (4)$$

$$(X \otimes Y)^T = X^T \otimes Y^T. \quad (5)$$

Finally, if X_1, X_2 can be multiplied and Y_1, Y_2 can be multiplied, then

$$(X_1 \otimes Y_1)(X_2 \otimes Y_2) = (X_1 X_2) \otimes (Y_1 Y_2). \quad (6)$$

Lemma 1. For $i, j = 1, \dots, s$, let $\{A_{n,ij}\}_n$ be a sequence of matrices and suppose that $\{B_{n,ij}^{(m)}\}_n \xrightarrow{\text{a.c.s.}} \{A_{n,ij}\}_n$. Then

$$[B_{n,ij}^{(m)}]_{i,j=1}^s \xrightarrow{\text{a.c.s.}} [A_{n,ij}]_{i,j=1}^s.$$

Proof. Let E_{ij} be the $s \times s$ matrix having 1 in position (i, j) and 0 elsewhere. Note that

$$[A_{n,ij}]_{i,j=1}^s = \sum_{i,j=1}^s E_{ij} \otimes A_{n,ij}, \quad [B_{n,ij}^{(m)}]_{i,j=1}^s = \sum_{i,j=1}^s E_{ij} \otimes B_{n,ij}^{(m)}. \quad (7)$$

Since $\{B_{n,ij}^{(m)}\}_n \xrightarrow{\text{a.c.s.}} \{A_{n,ij}\}_n$, it is clear from (3), (4) and the definition of a.c.s. that

$$E_{ij} \otimes B_{n,ij}^{(m)} \xrightarrow{\text{a.c.s.}} E_{ij} \otimes A_{n,ij}, \quad i, j = 1, \dots, s. \quad (8)$$

Now, if $\{B_{n,m}^{[k]}\}_n \xrightarrow{\text{a.c.s.}} \{A_n^{[k]}\}_n$ for $k = 1, \dots, K$ then $\{\sum_{k=1}^K B_{n,m}^{[k]}\}_n \xrightarrow{\text{a.c.s.}} \{\sum_{k=1}^K A_n^{[k]}\}_n$ (this an obvious consequence of the definition of a.c.s.). Thus, the thesis follows from (7) and (8). \square

Block GLT Sequences. Let $s \geq 1$ be a fixed positive integer. An s -block GLT sequence (or simply a GLT sequence if s can be inferred from the context or we do not need/want to specify it) is a special s -block matrix-sequence $\{A_n\}_n$ equipped with a measurable function $\kappa : [0, 1] \times [-\pi, \pi] \rightarrow \mathbb{C}^{s \times s}$, the so-called symbol. We use the notation $\{A_n\}_n \sim_{\text{GLT}} \kappa$ to indicate that $\{A_n\}_n$ is a GLT sequence with symbol κ . The symbol of a GLT sequence is unique in the sense that if $\{A_n\}_n \sim_{\text{GLT}} \kappa$ and $\{A_n\}_n \sim_{\text{GLT}} \zeta$ then $\kappa = \zeta$ a.e. in $[0, 1] \times [-\pi, \pi]$. The main properties of s -block GLT sequences proved in [29] are listed below. If A is a matrix, we denote by A^\dagger the Moore–Penrose pseudoinverse of A (recall that $A^\dagger = A^{-1}$ whenever A is invertible). If $f_m, f : D \subseteq \mathbb{R}^k \rightarrow \mathbb{C}^{r \times r}$ are measurable matrix-valued functions, we say that f_m converges to f in measure (resp., a.e., in $L^p(D)$, etc.) if $(f_m)_{\alpha\beta}$ converges to $f_{\alpha\beta}$ in measure (resp., a.e., in $L^p(D)$, etc.) for all $\alpha, \beta = 1, \dots, r$.

GLT1. If $\{A_n\}_n \sim_{\text{GLT}} \kappa$ then $\{A_n\}_n \sim_{\sigma} \kappa$. If moreover each A_n is Hermitian then $\{A_n\}_n \sim_{\lambda} \kappa$.

GLT2. We have:

- $\{T_n(f)\}_n \sim_{\text{GLT}} \kappa(x, \theta) = f(\theta)$ if $f : [-\pi, \pi] \rightarrow \mathbb{C}^{s \times s}$ is in $L^1([-\pi, \pi])$;
- $\{D_n(a)\}_n \sim_{\text{GLT}} \kappa(x, \theta) = a(x)$ if $a : [0, 1] \rightarrow \mathbb{C}^{s \times s}$ is Riemann-integrable;
- $\{Z_n\}_n \sim_{\text{GLT}} \kappa(x, \theta) = O_s$ if and only if $\{Z_n\}_n \sim_{\sigma} 0$.

GLT3. If $\{A_n\}_n \sim_{\text{GLT}} \kappa$ and $\{B_n\}_n \sim_{\text{GLT}} \zeta$ then:

- $\{A_n^*\}_n \sim_{\text{GLT}} \kappa^*$;

- $\{\alpha A_n + \beta B_n\}_n \sim_{\text{GLT}} \alpha \kappa + \beta \zeta$ for all $\alpha, \beta \in \mathbb{C}$;
- $\{A_n B_n\}_n \sim_{\text{GLT}} \kappa \zeta$;
- $\{A_n^\dagger\}_n \sim_{\text{GLT}} \kappa^{-1}$ provided that κ is invertible a.e.

GLT 4. $\{A_n\}_n \sim_{\text{GLT}} \kappa$ if and only if there exist s -block GLT sequences $\{B_{n,m}\}_n \sim_{\text{GLT}} \kappa_m$ such that $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ and $\kappa_m \rightarrow \kappa$ in measure.

3. FD Discretization of a System of DEs

Consider the following system of DEs:

$$\begin{cases} -a_{11}(x)u_1''(x) + a_{12}(x)u_2'(x) = f_1(x), & x \in (0,1), \\ a_{21}(x)u_1'(x) + a_{22}(x)u_2(x) = f_2(x), & x \in (0,1), \\ u_1(0) = 0, \quad u_1(1) = 0, \\ u_2(0) = 0, \quad u_2(1) = 0. \end{cases} \quad (9)$$

In this section we consider the classical central FD discretization of (9). Through the theory of block GLT sequences we show that the corresponding sequence of (normalized) FD discretization matrices enjoys a spectral distribution described by a 2×2 matrix-valued function, where 2 is the number of equations that compose the system (9). In what follows, we use the following notation:

$$\text{tridiag}_{j=1,\dots,n} \left[\beta_j \mid \alpha_j \mid \gamma_j \right] = \begin{bmatrix} \alpha_1 & \gamma_1 & & & \\ \beta_2 & \alpha_2 & \gamma_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{n-1} & \alpha_{n-1} & \gamma_{n-1} \\ & & & \beta_n & \alpha_n \end{bmatrix}.$$

The parameters $\alpha_j, \beta_j, \gamma_j$ may be either scalars or $s \times s$ blocks for some $s > 1$, in which case the previous matrix is a block tridiagonal matrix.

3.1. FD Discretization

Let $n \geq 1$, set $h = \frac{1}{n+1}$ and $x_j = jh$ for $j = 0, \dots, n+1$. Using the classical central FD schemes $(-1, 2, -1)$ and $\frac{1}{2}(-1, 0, 1)$ for the discretization of, respectively, the (negative) second derivative and the first derivative, for each $j = 1, \dots, n$ we obtain the following approximations:

$$\begin{aligned} [-a_{11}(x)u_1''(x) + a_{12}(x)u_2'(x)]|_{x=x_j} &\approx a_{11}(x_j) \frac{-u_1(x_{j+1}) + 2u_1(x_j) - u_1(x_{j-1}))}{h^2} \\ &\quad + a_{12}(x_j) \frac{u_2(x_{j+1}) - u_2(x_{j-1}))}{2h}, \\ [a_{21}(x)u_1'(x) + a_{22}(x)u_2(x)]|_{x=x_j} &\approx a_{21}(x_j) \frac{u_1(x_{j+1}) - u_1(x_{j-1}))}{2h} + a_{22}(x_j)u_2(x_j). \end{aligned}$$

This means that the nodal values of the solutions u_1, u_2 of (9) satisfy approximately the equations

$$\begin{aligned} a_{11}(x_j) [-u_1(x_{j+1}) + 2u_1(x_j) - u_1(x_{j-1}))] + \frac{h}{2} a_{12}(x_j) [u_2(x_{j+1}) - u_2(x_{j-1})] &= h^2 f_1(x_j), \\ \frac{1}{2} a_{21}(x_j) [u_1(x_{j+1}) - u_1(x_{j-1})] + h a_{22}(x_j) u_2(x_j) &= h f_2(x_j), \end{aligned}$$

for $j = 1, \dots, n$. We then approximate the solution u_1 (resp., u_2) by the piecewise linear function that takes the value $u_{1,j}$ (resp., $u_{2,j}$) at x_j for all $j = 0, \dots, n+1$, where $u_{1,0} = u_{1,n+1} = u_{2,0} = u_{2,n+1} = 0$ and the vectors $\mathbf{u}_1 = (u_{1,1}, \dots, u_{1,n})^T$ and $\mathbf{u}_2 = (u_{2,1}, \dots, u_{2,n})^T$ solve the linear system

$$\begin{aligned} a_{11}(x_j) [-u_{1,j+1} + 2u_{1,j} - u_{1,j-1}] + \frac{h}{2}a_{12}(x_j) [u_{2,j+1} - u_{2,j-1}] &= h^2 f_1(x_j), & j = 1, \dots, n, \\ \frac{1}{2}a_{21}(x_j) [u_{1,j+1} - u_{1,j-1}] + ha_{22}(x_j)u_{2,j} &= hf_2(x_j), & j = 1, \dots, n. \end{aligned} \quad (10)$$

This linear system can be rewritten in matrix form as follows:

$$A_n \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} h^2 \mathbf{f}_1 \\ h \mathbf{f}_2 \end{bmatrix}, \quad (11)$$

where $\mathbf{f}_1 = [f_1(x_j)]_{j=1}^n$, $\mathbf{f}_2 = [f_2(x_j)]_{j=1}^n$,

$$A_n = \begin{bmatrix} K_n(a_{11}) & hH_n(a_{12}) \\ H_n(a_{21}) & hM_n(a_{22}) \end{bmatrix} = \begin{bmatrix} K_n(a_{11}) & H_n(a_{12}) \\ H_n(a_{21}) & M_n(a_{22}) \end{bmatrix} \begin{bmatrix} I_n & O_n \\ O_n & hI_n \end{bmatrix}, \quad (12)$$

and

$$\begin{aligned} K_n(a_{11}) &= \text{tridiag}_{j=1, \dots, n} \left[-a_{11}(x_j) \mid 2a_{11}(x_j) \mid -a_{11}(x_j) \right] = \left(\text{diag}_{j=1, \dots, n} a_{11}(x_j) \right) T_n(2 - 2 \cos \theta), \\ H_n(a_{12}) &= \text{tridiag}_{j=1, \dots, n} \left[-\frac{1}{2}a_{12}(x_j) \mid 0 \mid \frac{1}{2}a_{12}(x_j) \right] = \left(\text{diag}_{j=1, \dots, n} a_{12}(x_j) \right) T_n(-i \sin \theta), \\ H_n(a_{21}) &= \text{tridiag}_{j=1, \dots, n} \left[-\frac{1}{2}a_{21}(x_j) \mid 0 \mid \frac{1}{2}a_{21}(x_j) \right] = \left(\text{diag}_{j=1, \dots, n} a_{21}(x_j) \right) T_n(-i \sin \theta), \\ M_n(a_{22}) &= \text{diag}_{j=1, \dots, n} a_{22}(x_j). \end{aligned}$$

In view of (12), the linear system (11) is equivalent to

$$B_n \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} h^2 \mathbf{f}_1 \\ h \mathbf{f}_2 \end{bmatrix}, \quad (13)$$

where $\mathbf{v}_1 = \mathbf{u}_1$, $\mathbf{v}_2 = h\mathbf{u}_2$, and

$$B_n = \begin{bmatrix} K_n(a_{11}) & H_n(a_{12}) \\ H_n(a_{21}) & M_n(a_{22}) \end{bmatrix}. \quad (14)$$

Let $v_{1,1}, \dots, v_{1,n}$ and $v_{2,1}, \dots, v_{2,n}$ be the components of \mathbf{v}_1 and \mathbf{v}_2 , respectively. When writing the linear system (10) in the form (13), we are implicitly assuming the following.

- The unknowns are sorted as follows:

$$\begin{bmatrix} [v_{1,j}]_{j=1, \dots, n} \\ [v_{2,j}]_{j=1, \dots, n} \end{bmatrix} = \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ \vdots \\ v_{1,n} \\ v_{2,1} \\ v_{2,2} \\ \vdots \\ v_{2,n} \end{bmatrix}. \quad (15)$$

- The equations are sorted as follows, in accordance with the ordering (15) for the unknowns:

$$\left[\begin{array}{c} \left[a_{11}(x_j) [-v_{1,j+1} + 2v_{1,j} - v_{1,j-1}] + \frac{1}{2}a_{12}(x_j) [v_{2,j+1} - v_{2,j-1}] = hf_1(x_j) \right]_{j=1,\dots,n} \\ \left[\frac{1}{2}a_{21}(x_j) [u_{1,j+1} - u_{1,j-1}] + a_{22}(x_j)v_{2,j} = hf_2(x_j) \right]_{j=1,\dots,n} \end{array} \right]. \quad (16)$$

Suppose we decide to change the ordering for both the unknowns and the equations. More precisely, suppose we opt for the following orderings.

- The unknowns are sorted as follows:

$$\left[\begin{array}{c} v_{1,j} \\ v_{2,j} \end{array} \right]_{j=1,\dots,n} = \left[\begin{array}{c} v_{1,1} \\ v_{2,1} \\ v_{1,2} \\ v_{2,2} \\ \vdots \\ v_{1,n} \\ v_{2,n} \end{array} \right]. \quad (17)$$

- The equations are sorted as follows, in accordance with the ordering (17) for the unknowns:

$$\left[\begin{array}{c} a_{11}(x_j) [-v_{1,j+1} + 2v_{1,j} - v_{1,j-1}] + \frac{1}{2}a_{12}(x_j) [v_{2,j+1} - v_{2,j-1}] = hf_1(x_j) \\ \frac{1}{2}a_{21}(x_j) [v_{1,j+1} - v_{1,j-1}] + a_{22}(x_j)v_{2,j} = hf_2(x_j) \end{array} \right]_{j=1,\dots,n}. \quad (18)$$

The matrix C_n associated with the linear system (10) assuming the new orderings (17) and (18) is the 2×2 block tridiagonal matrix given by

$$C_n = \text{tridiag}_{j=1,\dots,n} \left[\begin{array}{cc|cc|cc} -a_{11}(x_j) & -\frac{1}{2}a_{12}(x_j) & 2a_{11}(x_j) & 0 & -a_{11}(x_j) & \frac{1}{2}a_{12}(x_j) \\ -\frac{1}{2}a_{21}(x_j) & 0 & 0 & a_{22}(x_j) & \frac{1}{2}a_{21}(x_j) & 0 \end{array} \right]. \quad (19)$$

The matrix C_n is similar to B_n . Indeed, by permuting both rows and columns of B_n according to the permutation $1, n+1, 2, n+2, \dots, n, 2n$ we obtain C_n . More precisely, let $\mathbf{e}_1, \dots, \mathbf{e}_n$ and $\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_{2n}$ be the vectors of the canonical basis of \mathbb{R}^n and \mathbb{R}^{2n} , respectively, and let Π_n be the permutation matrix associated with the permutation $1, n+1, 2, n+2, \dots, n, 2n$, that is,

$$\Pi_n = \left[\begin{array}{c} \tilde{\mathbf{e}}_1^T \\ \tilde{\mathbf{e}}_{n+1}^T \\ \tilde{\mathbf{e}}_2^T \\ \tilde{\mathbf{e}}_{n+2}^T \\ \vdots \\ \tilde{\mathbf{e}}_n^T \\ \tilde{\mathbf{e}}_{2n}^T \end{array} \right] = \left[\begin{array}{c} I_2 \otimes \mathbf{e}_1^T \\ I_2 \otimes \mathbf{e}_2^T \\ \vdots \\ I_2 \otimes \mathbf{e}_n^T \end{array} \right]. \quad (20)$$

Then $C_n = \Pi_n B_n \Pi_n^T$.

3.2. GLT Analysis of the FD Discretization Matrices

The main result of this section (Theorem 3) shows that $\{C_n\}_n$ is a block GLT sequence whose spectral distribution is described by a 2×2 matrix-valued symbol, which is obtained by replacing the

matrix-sequences $\{K_n(a_{11})\}_n, \{H_n(a_{12})\}_n, \{H_n(a_{21})\}_n, \{M_n(a_{22})\}_n$ appearing in the expression (14) of B_n with the corresponding symbols $a_{11}(x)(2 - 2 \cos \theta), -ia_{12}(x) \sin \theta, -ia_{21}(x) \sin \theta, a_{22}(x)$. In this regard, we note that, assuming for instance $a_{11}, a_{12}, a_{21}, a_{22} \in C([0, 1])$, we have

$$\{K_n(a_{11})\}_n \sim_{\text{GLT}} a_{11}(x)(2 - 2 \cos \theta), \quad (21)$$

$$\{H_n(a_{12})\}_n \sim_{\text{GLT}} -ia_{12}(x) \sin \theta, \quad (22)$$

$$\{H_n(a_{21})\}_n \sim_{\text{GLT}} -ia_{21}(x) \sin \theta, \quad (23)$$

$$\{M_n(a_{22})\}_n \sim_{\text{GLT}} a_{22}(x). \quad (24)$$

To prove (21), it suffices to observe that

$$\begin{aligned} \|K_n(a_{11}) - D_n(a_{11})T_n(2 - 2 \cos \theta)\| &\leq \left\| \text{diag}_{j=1, \dots, n} a_{11}(x_j) - D_n(a_{11}) \right\| \|T_n(2 - 2 \cos \theta)\| \\ &= \max_{j=1, \dots, n} \left| a_{11}(x_j) - a_{11}\left(\frac{j}{n}\right) \right| \|T_n(2 - 2 \cos \theta)\| \leq 4\omega_{a_{11}}(h), \end{aligned}$$

where $\omega_{a_{11}}(\cdot)$ is the modulus of continuity of a_{11} . Since $\omega_{a_{11}}(h) \rightarrow 0$ as $n \rightarrow \infty$, it follows from Proposition 1 that $\{K_n(a_{11}) - D_n(a_{11})T_n(2 - 2 \cos \theta)\}_n \sim_{\sigma} 0$, and so **GLT 2** and **GLT 3** immediately yield (21). The relations (22)–(24) are proved in the same way.

Theorem 3. Suppose that $a_{11}, a_{12}, a_{21}, a_{22} \in C([0, 1])$. Then

$$\{C_n\}_n \sim_{\text{GLT}} \kappa(x, \theta) = \begin{bmatrix} a_{11}(x)(2 - 2 \cos \theta) & -ia_{12}(x) \sin \theta \\ -ia_{21}(x) \sin \theta & a_{22}(x) \end{bmatrix} \quad (25)$$

and

$$\{C_n\}_n \sim_{\sigma} \kappa(x, \theta). \quad (26)$$

If moreover $a_{21} = -a_{12}$ then we also have

$$\{C_n\}_n \sim_{\lambda} \kappa(x, \theta). \quad (27)$$

Proof. From (19) we have

$$\begin{aligned} C_n &= \text{tridiag}_{j=1, \dots, n} \left[\begin{array}{cc|cc|cc} -a_{11}(x_j) & -\frac{1}{2}a_{12}(x_j) & 2a_{11}(x_j) & 0 & -a_{11}(x_j) & \frac{1}{2}a_{12}(x_j) \\ -\frac{1}{2}a_{21}(x_j) & 0 & 0 & a_{22}(x_j) & \frac{1}{2}a_{21}(x_j) & 0 \end{array} \right] \\ &= \text{tridiag}_{j=1, \dots, n} \left[\begin{array}{cc|cc|cc} -a_{11}(x_j) & 0 & 2a_{11}(x_j) & 0 & -a_{11}(x_j) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \\ &\quad + \text{tridiag}_{j=1, \dots, n} \left[\begin{array}{cc|cc|cc} 0 & -\frac{1}{2}a_{12}(x_j) & 0 & 0 & 0 & \frac{1}{2}a_{12}(x_j) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \\ &\quad + \text{tridiag}_{j=1, \dots, n} \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}a_{21}(x_j) & 0 & 0 & 0 & \frac{1}{2}a_{21}(x_j) & 0 \end{array} \right] \\ &\quad + \text{tridiag}_{j=1, \dots, n} \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{22}(x_j) & 0 & 0 \end{array} \right] \\ &= \text{diag}_{j=1, \dots, n} a_{11}(x_j) I_2 \cdot \text{tridiag}_{j=1, \dots, n} \left[\begin{array}{cc|cc|cc} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

$$\begin{aligned}
& + \operatorname{diag}_{j=1,\dots,n} a_{12}(x_j) I_2 \cdot \operatorname{tridiag}_{j=1,\dots,n} \left[\begin{array}{cc|cc} 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \middle| \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right] \\
& + \operatorname{diag}_{j=1,\dots,n} a_{21}(x_j) I_2 \cdot \operatorname{tridiag}_{j=1,\dots,n} \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{array} \middle| \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right] \\
& + \operatorname{diag}_{j=1,\dots,n} a_{22}(x_j) I_2 \cdot \operatorname{tridiag}_{j=1,\dots,n} \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \middle| \begin{array}{c} 0 \\ 0 \end{array} \right] \\
= & \operatorname{diag}_{j=1,\dots,n} a_{11}(x_j) I_2 \cdot T_n((2 - 2 \cos \theta) E_{11}) \\
& + \operatorname{diag}_{j=1,\dots,n} a_{12}(x_j) I_2 \cdot T_n((-i \sin \theta) E_{12}) \\
& + \operatorname{diag}_{j=1,\dots,n} a_{21}(x_j) I_2 \cdot T_n((-i \sin \theta) E_{21}) \\
& + \operatorname{diag}_{j=1,\dots,n} a_{22}(x_j) I_2 \cdot T_n(E_{22}), \tag{28}
\end{aligned}$$

where E_{pq} is the 2×2 matrix having 1 in position (p, q) and 0 elsewhere. It is clear that, for every $p, q = 1, 2$,

$$\left\| \operatorname{diag}_{j=1,\dots,n} a_{pq}(x_j) I_2 - D_n(a_{pq} I_2) \right\| \leq \omega_{a_{pq}}(h) \rightarrow 0$$

as $n \rightarrow \infty$; hence, by Proposition 1, GLT 2 and GLT 3,

$$\left\{ \operatorname{diag}_{j=1,\dots,n} a_{pq}(x_j) I_2 \right\}_n \sim_{\text{GLT}} a(x) I_2, \quad p, q = 1, 2.$$

As a consequence, the decomposition (28), GLT 2 and GLT 3 imply (25), which in turn implies (26) by GLT 1. It only remains to prove (27) in the case where $a_{21} = -a_{12}$. In this case, we have

$$C_n = \operatorname{tridiag}_{j=1,\dots,n} \left[\begin{array}{cc|cc} -a_{11}(x_j) & -\frac{1}{2}a_{12}(x_j) & 2a_{11}(x_j) & 0 \\ \frac{1}{2}a_{12}(x_j) & 0 & 0 & a_{22}(x_j) \end{array} \middle| \begin{array}{c} -a_{11}(x_j) \\ -\frac{1}{2}a_{12}(x_j) \end{array} \right].$$

Consider the symmetric approximation of C_n given by

$$\tilde{C}_n = \operatorname{tridiag}_{j=1,\dots,n} \left[\begin{array}{cc|cc} -a_{11}(x_{j-1}) & -\frac{1}{2}a_{12}(x_{j-1}) & 2a_{11}(x_j) & 0 \\ \frac{1}{2}a_{12}(x_{j-1}) & 0 & 0 & a_{22}(x_j) \end{array} \middle| \begin{array}{c} -a_{11}(x_j) \\ -\frac{1}{2}a_{12}(x_j) \end{array} \right].$$

It is not difficult to see that $\|C_n - \tilde{C}_n\| \rightarrow 0$ as $n \rightarrow \infty$ by invoking the inequality

$$\|X\| \leq \sqrt{\left(\max_{i=1,\dots,n} \sum_{j=1}^n |x_{ij}| \right) \left(\max_{j=1,\dots,n} \sum_{i=1}^n |x_{ij}| \right)}, \quad X \in \mathbb{C}^{n \times n}; \tag{29}$$

see, e.g., [26, Section 2.4.1]. Therefore:

- in view of the decomposition $\tilde{C}_n = C_n + (\tilde{C}_n - C_n)$, we have $\{\tilde{C}_n\}_n \sim_{\text{GLT}} \kappa(x, \theta)$ by (25), Proposition 1, GLT 2 and GLT 3, so in particular $\{\tilde{C}_n\}_n \sim_{\lambda} \kappa(x, \theta)$ by GLT 1 as \tilde{C}_n is symmetric;
- $\|C_n - \tilde{C}_n\|_2 \leq \sqrt{n} \|C_n - \tilde{C}_n\| = o(\sqrt{2n})$ as $n \rightarrow \infty$.

Thus, (27) follows from Theorem 2. \square

4. Higher-Order FE Discretization of the Diffusion Equation

Consider the diffusion equation

$$\begin{cases} -(a(x)u'(x))' = f(x), & x \in (0, 1), \\ u(0) = u(1) = 0. \end{cases} \quad (30)$$

In this section we consider the higher-order FE discretization of (30). Through the theory of block GLT sequences we show that the corresponding sequence of (normalized) FE discretization matrices enjoys a spectral distribution described by a $(p - k) \times (p - k)$ matrix-valued function, where p and k represent, respectively, the degree and the smoothness of the piecewise polynomial functions involved in the FE approximation. Note that this result represents a remarkable argument in support of [28, Conjecture 2].

4.1. FE Discretization

The weak form of (30) reads as follows: find $u \in H_0^1([0, 1])$ such that

$$\int_0^1 a(x)u'(x)w'(x)dx = \int_0^1 f(x)w(x)dx, \quad \forall w \in H_0^1([0, 1]).$$

In the FE method we fix a set of basis functions $\{\varphi_1, \dots, \varphi_N\} \subset H_0^1([0, 1])$ and we look for an approximation of the exact solution in the space $\mathcal{W} = \text{span}(\varphi_1, \dots, \varphi_N)$ by solving the following discrete problem: find $u_{\mathcal{W}} \in \mathcal{W}$ such that

$$\int_0^1 a(x)u'_{\mathcal{W}}(x)w'(x)dx = \int_0^1 f(x)w(x)dx, \quad \forall w \in \mathcal{W}.$$

Since $\{\varphi_1, \dots, \varphi_N\}$ is a basis of \mathcal{W} , we can write $u_{\mathcal{W}} = \sum_{j=1}^N u_j \varphi_j$ for a unique vector $\mathbf{u} = (u_1, \dots, u_N)^T$. By linearity, the computation of $u_{\mathcal{W}}$ (i.e., of \mathbf{u}) reduces to solving the linear system

$$A\mathbf{u} = \mathbf{f},$$

where $\mathbf{f} = (\int_0^1 f(x)\varphi_1(x)dx, \dots, \int_0^1 f(x)\varphi_N(x)dx)^T$ and A is the stiffness matrix,

$$A = \left[\int_0^1 a(x)\varphi_j'(x)\varphi_i'(x)dx \right]_{i,j=1}^N. \quad (31)$$

4.2. p -Degree C^k B-spline Basis Functions

Following the higher-order FE approach, the basis functions $\varphi_1, \dots, \varphi_N$ will be chosen as piecewise polynomials of degree $p \geq 1$. More precisely, for $p, n \geq 1$ and $0 \leq k \leq p - 1$, let $B_{1,[p,k]}, \dots, B_{n(p-k)+k+1,[p,k]} : \mathbb{R} \rightarrow \mathbb{R}$ be the B-splines of degree p and smoothness C^k defined on the knot sequence

$$\{\tau_1, \dots, \tau_{n(p-k)+p+k+2}\} = \left\{ \underbrace{0, \dots, 0}_{p+1}, \underbrace{\frac{1}{n}, \dots, \frac{1}{n}}_{p-k}, \underbrace{\frac{2}{n}, \dots, \frac{2}{n}}_{p-k}, \dots, \underbrace{\frac{n-1}{n}, \dots, \frac{n-1}{n}}_{p-k}, \underbrace{1, \dots, 1}_{p+1} \right\}. \quad (32)$$

We collect here a few properties of $B_{1,[p,k]}, \dots, B_{n(p-k)+k+1,[p,k]}$ that we shall use in this paper. For the formal definition of B-splines, as well as for the proof of the properties listed below, see [16,35].

- The support of the i th B-spline is given by

$$\text{supp}(B_{i,[p,k]}) = [\tau_i, \tau_{i+p+1}], \quad i = 1, \dots, n(p - k) + k + 1. \quad (33)$$

- Except for the first and the last one, all the other B-splines vanish on the boundary of $[0, 1]$, i.e.,

$$B_{i,[p,k]}(0) = B_{i,[p,k]}(1) = 0, \quad i = 2, \dots, n(p-k) + k. \quad (34)$$

- $\{B_{1,[p,k]}, \dots, B_{n(p-k)+k+1,[p,k]}\}$ is a basis for the space of piecewise polynomial functions of degree p and smoothness C^k , that is,

$$\mathcal{V}_{n,[p,k]} = \{v \in C^k([0, 1]) : v|_{[\frac{i}{n}, \frac{i+1}{n}]} \in \mathbb{P}_p \text{ for all } i = 0, \dots, n-1\},$$

where \mathbb{P}_p is the space of polynomials of degree $\leq p$. Moreover, $\{B_{2,[p,k]}, \dots, B_{n(p-k)+k,[p,k]}\}$ is a basis for the space

$$\mathcal{W}_{n,[p,k]} = \{w \in \mathcal{V}_{n,[p,k]} : w(0) = w(1) = 0\}.$$

- The B-splines form a non-negative partition of unity over $[0, 1]$:

$$B_{i,[p,k]} \geq 0 \text{ over } \mathbb{R}, \quad i = 1, \dots, n(p-k) + k + 1, \quad (35)$$

$$\sum_{i=1}^{n(p-k)+k+1} B_{i,[p,k]} = 1 \text{ over } [0, 1]. \quad (36)$$

- The derivatives of the B-splines satisfy

$$\sum_{i=1}^{n(p-k)+k+1} |B'_{i,[p,k]}| \leq c_p n \text{ over } [0, 1], \quad (37)$$

where c_p is a constant depending only on p . Note that the derivatives $B'_{i,[p,k]}$ may not be defined at some of the grid points $0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1$ in the case of C^0 smoothness ($k = 0$). In (37) it is assumed that the undefined values are excluded from the summation.

- All the B-splines, except for the first $k + 1$ and the last $k + 1$, are uniformly shifted-scaled versions of $p - k$ fixed reference functions $\beta_{1,[p,k]}, \dots, \beta_{p-k,[p,k]}$, namely the first $p - k$ B-splines defined on the reference knot sequence

$$\underbrace{0, \dots, 0}_{p-k}, \underbrace{1, \dots, 1}_{p-k}, \dots, \underbrace{\eta, \dots, \eta}_{p-k}, \quad \eta = \left\lceil \frac{p+1}{p-k} \right\rceil.$$

In formulas, setting

$$v = \left\lceil \frac{k+1}{p-k} \right\rceil, \quad (38)$$

for the B-splines $B_{k+2,[p,k]}, \dots, B_{k+1+(n-v)(p-k),[p,k]}$ we have

$$B_{k+1+(p-k)(r-1)+q,[p,k]}(x) = \beta_{q,[p,k]}(nx - r + 1), \quad r = 1, \dots, n - v, \quad q = 1, \dots, p - k. \quad (39)$$

We point out that the supports of the reference B-splines $\beta_{q,[p,k]}$ satisfy

$$\text{supp}(\beta_{1,[p,k]}) \subseteq \text{supp}(\beta_{2,[p,k]}) \subseteq \dots \subseteq \text{supp}(\beta_{p-k,[p,k]}) = [0, \eta].$$

Figures 1 and 2 show the graphs of the B-splines $B_{1,[p,k]}, \dots, B_{n(p-k)+k+1,[p,k]}$ for the degree $p = 3$ and the smoothness $k = 1$, and the graphs of the associated reference B-splines $\beta_{1,[p,k]}, \beta_{2,[p,k]}$.

The basis functions $\varphi_1, \dots, \varphi_N$ are defined as follows:

$$\varphi_i = B_{i+1,[p,k]}, \quad i = 1, \dots, n(p-k) + k - 1. \quad (40)$$

In particular, with the notations of Section 4.1, we have $N = n(p-k) + k - 1$ and $\mathcal{W} = \mathcal{W}_{n,[p,k]}$.

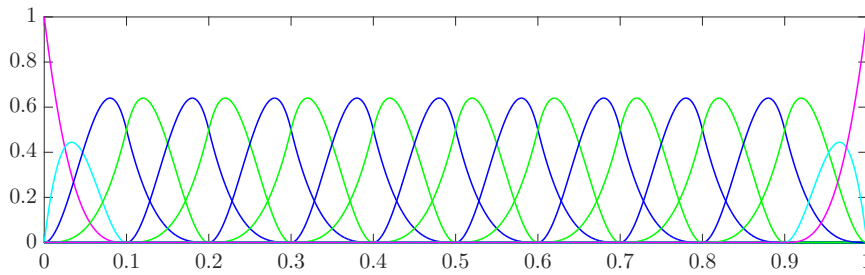


Figure 1. B-splines $B_{1,[p,k]}, \dots, B_{n(p-k)+k+1,[p,k]}$ for $p = 3$ and $k = 1$, with $n = 10$.

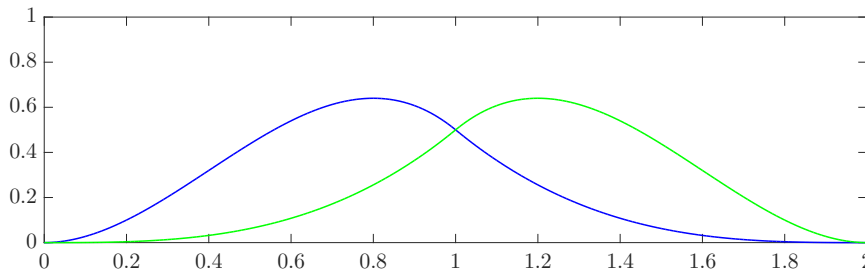


Figure 2. Reference B-splines $\beta_{1,[p,k]}, \beta_{2,[p,k]}$ for $p = 3$ and $k = 1$.

4.3. GLT Analysis of the Higher-Order FE Discretization Matrices

The stiffness matrix (31) resulting from the choice of the basis functions as in (40) will be denoted by $A_{n,[p,k]}(a)$,

$$A_{n,[p,k]}(a) = \left[\int_0^1 a(x) B'_{j+1,[p,k]}(x) B'_{i+1,[p,k]}(x) dx \right]_{i,j=1}^{n(p-k)+k-1}. \quad (41)$$

The main result of this section (Theorem 4) gives the spectral distribution of the normalized sequence $\{n^{-1}A_{n,[p,k]}(a)\}_n$. The proof of Theorem 4 is entirely based on the theory of block GLT sequences and it is therefore referred to as ‘‘GLT analysis’’. It also requires the following lemma, which provides an approximate construction of the matrix $A_{n,[p,k]}(1)$ corresponding to the constant-coefficient case where $a(x) = 1$ identically. In view of what follows, define the $(p-k) \times (p-k)$ blocks

$$K_{[p,k]}^{[\ell]} = \left[\int_{\mathbb{R}} \beta'_{j,[p,k]}(t) \beta'_{i,[p,k]}(t-\ell) dt \right]_{i,j=1}^{p-k}, \quad \ell \in \mathbb{Z}, \quad (42)$$

and the $(p-k) \times (p-k)$ matrix-valued function $\kappa_{[p,k]} : [-\pi, \pi] \rightarrow \mathbb{C}^{(p-k) \times (p-k)}$,

$$\kappa_{[p,k]}(\theta) = \sum_{\ell \in \mathbb{Z}} K_{[p,k]}^{[\ell]} e^{i\ell\theta} = K_{[p,k]}^{[0]} + \sum_{\ell > 0} \left(K_{[p,k]}^{[\ell]} e^{i\ell\theta} + (K_{[p,k]}^{[\ell]})^T e^{-i\ell\theta} \right). \quad (43)$$

Due to the compact support of the reference functions $\beta_{1,[p,k]}, \dots, \beta_{p-k,[p,k]}$, there is only a finite number of nonzero blocks $K_{[p,k]}^{[\ell]}$ and, consequently, the series in (43) is actually a finite sum.

Lemma 2. Let $p, n \geq 1$ and $0 \leq k \leq p-1$. Define $\tilde{A}_{n,[p,k]}(1)$ as the principal submatrix of $A_{n,[p,k]}(1)$ of size $(n-\nu)(p-k)$ corresponding to the indices $k+1, \dots, k+(n-\nu)(p-k)$, where $\nu = \lceil (k+1)/(p-k) \rceil$ as in (38). Then $\tilde{A}_{n,[p,k]}(1) = nT_{n-\nu}(\kappa_{[p,k]})$.

Proof. By (33) and (39), for all $r, R = 1, \dots, n - \nu$ and $q, Q = 1, \dots, p - k$ we have

$$\begin{aligned} (\tilde{A}_{n,[p,k]}(1))_{(p-k)(r-1)+q,(p-k)(R-1)+Q} &= \int_0^1 B'_{k+1+(p-k)(R-1)+Q,[p,k]}(x) B'_{k+1+(p-k)(r-1)+q,[p,k]}(x) dx \\ &= \int_{\mathbb{R}} B'_{k+1+(p-k)(R-1)+Q,[p,k]}(x) B'_{k+1+(p-k)(r-1)+q,[p,k]}(x) dx \\ &= n^2 \int_{\mathbb{R}} \beta'_{Q,[p,k]}(nx - R + 1) \beta'_{q,[p,k]}(nx - r + 1) dx \\ &= n \int_{\mathbb{R}} \beta'_{Q,[p,k]}(y) \beta'_{q,[p,k]}(y - r + R) dy \end{aligned}$$

and

$$(T_{n-\nu}(\kappa_{[p,k]}))_{(p-k)(r-1)+q,(p-k)(R-1)+Q} = (K_{[p,k]}^{[r-R]})_{q,Q} = \int_{\mathbb{R}} \beta'_{Q,[p,k]}(y) \beta'_{q,[p,k]}(y - r + R) dy,$$

which completes the proof. \square

Theorem 4. Let $a \in L^1([0, 1])$, $p \geq 1$ and $0 \leq k \leq p - 1$. Then $\{n^{-1}A_{n,[p,k]}(a)\}_n \sim_{\sigma,\lambda} a(x)\kappa_{[p,k]}(\theta)$.

Proof. The proof consists of four steps. Throughout this proof, we will use the following notations.

- $\nu = \lceil (k + 1)/(p - k) \rceil$ as in (38).
- For every square matrix A of size $n(p - k) + k - 1$ we denote by \tilde{A} the principal submatrix of A corresponding to the row and column indices $i, j = k + 1, \dots, k + (n - \nu)(p - k)$.
- $P_{n,[p,k]}$ is the $(n(p - k) + k - 1) \times (n - \nu)(p - k)$ matrix having $I_{(n-\nu)(p-k)}$ as the principal submatrix corresponding to the row and column indices $i, j = k + 1, \dots, k + (n - \nu)(p - k)$ and zeros elsewhere. Note that $P_{n,[p,k]}^T P_{n,[p,k]} = I_{(n-\nu)(p-k)}$ and $P_{n,[p,k]}^T A P_{n,[p,k]} = \tilde{A}$ for every square matrix A of size $n(p - k) + k - 1$.

Step 1. Consider the linear operator $A_{n,[p,k]}(\cdot) : L^1([0, 1]) \rightarrow \mathbb{R}^{(n(p-k)+k-1) \times (n(p-k)+k-1)}$,

$$A_{n,[p,k]}(g) = \left[\int_0^1 g(x) B'_{j+1,[p,k]}(x) B'_{i+1,[p,k]}(x) dx \right]_{i,j=1}^{n(p-k)+k-1}.$$

The next three steps are devoted to show that

$$\{P_{n,[p,k]}^T(n^{-1}A_{n,[p,k]}(g))P_{n,[p,k]}\}_n = \{n^{-1}\tilde{A}_{n,[p,k]}(g)\}_n \sim_{\text{GLT}} g(x)\kappa_{[p,k]}(\theta), \quad \forall g \in L^1([0, 1]). \quad (44)$$

Once this is done, the theorem is proved. Indeed, from (44) we immediately obtain the relation $\{P_{n,[p,k]}^T(n^{-1}A_{n,[p,k]}(a))P_{n,[p,k]}\}_n \sim_{\text{GLT}} a(x)\kappa_{[p,k]}(\theta)$. We infer that $\{P_{n,[p,k]}^T(n^{-1}A_{n,[p,k]}(a))P_{n,[p,k]}\}_n \sim_{\sigma,\lambda} a(x)\kappa_{[p,k]}(\theta)$ by **GLT 1** and $\{n^{-1}A_{n,[p,k]}(a)\}_n \sim_{\sigma,\lambda} a(x)\kappa_{[p,k]}(\theta)$ by **Theorem 1**.

Step 2. We first prove (44) in the constant-coefficient case where $g(x) = 1$ identically. In this case, by **Lemma 2** we have $n^{-1}\tilde{A}_{n,[p,k]}(1) = T_{n-\nu}(\kappa_{[p,k]})$. Hence, the desired relation $\{n^{-1}\tilde{A}_{n,[p,k]}(1)\}_n \sim_{\text{GLT}} \kappa_{[p,k]}(\theta)$ follows from **GLT 2**.

Step 3. Now we prove (44) in the case where $g \in C([0, 1])$. Let

$$Z_{n,[p,k]}(g) = n^{-1}\tilde{A}_{n,[p,k]}(g) - n^{-1}D_{n-\nu}(gI_{p-k})\tilde{A}_{n,[p,k]}(1).$$

By (32), (33) and (37), for all $r, R = 1, \dots, n - \nu$ and $q, Q = 1, \dots, p - k$ we have

$$\begin{aligned}
& |(nZ_{n,[p,k]}(g))_{(p-k)(r-1)+q,(p-k)(R-1)+Q}| \\
&= \left| (\tilde{A}_{n,[p,k]}(g))_{(p-k)(r-1)+q,(p-k)(R-1)+Q} - (D_{n-\nu}(gI_{p-k})\tilde{A}_{n,[p,k]}(\mathbf{1}))_{(p-k)(r-1)+q,(p-k)(R-1)+Q} \right| \\
&= \left| \int_0^1 \left[g(x) - g\left(\frac{r}{n-\nu}\right) \right] B'_{k+1+(p-k)(R-1)+Q,[p,k]}(x) B'_{k+1+(p-k)(r-1)+q,[p,k]}(x) dx \right| \\
&= \left| \int_{\tau_{k+1+(p-k)(r-1)+q}}^{\tau_{k+1+(p-k)(r-1)+q+p+1}} \left[g(x) - g\left(\frac{r}{n-\nu}\right) \right] B'_{k+1+(p-k)(R-1)+Q,[p,k]}(x) B'_{k+1+(p-k)(r-1)+q,[p,k]}(x) dx \right| \\
&\leq c_p^2 n^2 \int_{(r-1)/n}^{(r+p)/n} \left| g(x) - g\left(\frac{r}{n-\nu}\right) \right| dx \leq c_p^2 (p+1) n \omega_g\left(\frac{\nu+p}{n}\right),
\end{aligned}$$

where $\omega_g(\cdot)$ is the modulus of continuity of g and the last inequality is justified by the fact that the distance of the point $r/(n-\nu)$ from the interval $[(r-1)/n, (r+p)/n]$ is not larger than $(\nu+p)/n$. It follows that each entry of $Z_{n,[p,k]}(g)$ is bounded in modulus by $C_p \omega_g(1/n)$, where C_p is a constant depending only on p . Moreover, by (33), the matrix $Z_{n,[p,k]}(g)$ is banded with bandwidth bounded by a constant w_p depending only on p . Thus, by (29), $\|Z_{n,[p,k]}(g)\| \leq w_p C_p \omega_g(1/n) \rightarrow 0$ as $n \rightarrow \infty$, and so $\{Z_{n,[p,k]}(g)\}_n$ is zero-distributed by Proposition 1. Since

$$n^{-1} \tilde{A}_{n,[p,k]}(g) = n^{-1} D_{n-\nu}(gI_{p-k}) \tilde{A}_{n,[p,k]}(\mathbf{1}) + Z_{n,[p,k]}(g),$$

we conclude that $\{n^{-1} \tilde{A}_{n,[p,k]}(g)\}_n \sim_{\text{GLT}} g(x) \kappa_{[p,k]}(\theta)$ by **GLT 2**, **GLT 3** and Step 2.

Step 4. Finally, we prove (44) in the general case where $g \in L^1([0, 1])$. By the density of $C([0, 1])$ in $L^1([0, 1])$, there exist functions $g_m \in C([0, 1])$ such that $g_m \rightarrow g$ in $L^1([0, 1])$. By Step 3,

$$\{n^{-1} \tilde{A}_{n,[p,k]}(g_m)\}_n \sim_{\text{GLT}} g_m(x) \kappa_{[p,k]}(\theta). \quad (45)$$

Moreover,

$$g_m(x) \kappa_{[p,k]}(\theta) \rightarrow g(x) \kappa_{[p,k]}(\theta) \text{ in measure.} \quad (46)$$

We show that

$$\{n^{-1} \tilde{A}_{n,[p,k]}(g_m)\}_n \xrightarrow{\text{a.c.s.}} \{n^{-1} \tilde{A}_{n,[p,k]}(g)\}_n. \quad (47)$$

Once this is done, the thesis (44) follows immediately from **GLT 4**. To prove (47), we recall that

$$\|X\|_1 \leq \sum_{i,j=1}^N |x_{ij}|, \quad X \in \mathbb{C}^{N \times N}; \quad (48)$$

see, e.g., [26, Section 2.4.3]. By (37) we obtain

$$\begin{aligned}
\|\tilde{A}_{n,[p,k]}(g) - \tilde{A}_{n,[p,k]}(g_m)\|_1 &\leq \sum_{i,j=1}^{n(p-k)+k-1} \left| \int_0^1 [g(x) - g_m(x)] B'_{j+1,[p,k]}(x) B'_{i+1,[p,k]}(x) dx \right| \\
&\leq \int_0^1 |g(x) - g_m(x)| \sum_{i,j=1}^{n(p-k)+k-1} |B'_{j+1,[p,k]}(x)| |B'_{i+1,[p,k]}(x)| dx \\
&\leq c_p^2 n^2 \|g - g_m\|_{L^1}.
\end{aligned}$$

Thus, the a.c.s. convergence (47) follows from Proposition 2. \square

Remark 1. By following step by step the proof of Theorem 4, we can give an alternative (much simpler) proof of [6, Theorem A.6] based on the theory of block GLT sequences.

5. The Theory of Multilevel Block GLT Sequences

As illustrated in Sections 3 and 4, the theory of block GLT sequences allows the computation of the singular value and eigenvalue distribution of block structured matrices arising from the discretization of univariate DEs. In order to cope with multivariate DEs, i.e., PDEs, we need the multivariate version of the theory of block GLT sequences, also known as the theory of multilevel block GLT sequences. The present section is devoted to a careful presentation of this theory, which is obtained by combining the results of [29] with the necessary technicalities for tackling multidimensional problems [27].

Multi-Index Notation. The multi-index notation is an essential tool for dealing with sequences of matrices arising from the discretization of PDEs. A multi-index $\mathbf{i} \in \mathbb{Z}^d$, also called a d -index, is simply a (row) vector in \mathbb{Z}^d ; its components are denoted by i_1, \dots, i_d .

- $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots$ are the vectors of all zeros, all ones, all twos, \dots (their size will be clear from the context).
- For any d -index \mathbf{m} , we set $N(\mathbf{m}) = \prod_{j=1}^d m_j$ and we write $\mathbf{m} \rightarrow \infty$ to indicate that $\min(\mathbf{m}) \rightarrow \infty$.
- If \mathbf{h}, \mathbf{k} are d -indices, $\mathbf{h} \leq \mathbf{k}$ means that $h_r \leq k_r$ for all $r = 1, \dots, d$.
- If \mathbf{h}, \mathbf{k} are d -indices such that $\mathbf{h} \leq \mathbf{k}$, the multi-index range $\mathbf{h}, \dots, \mathbf{k}$ is the set $\{\mathbf{j} \in \mathbb{Z}^d : \mathbf{h} \leq \mathbf{j} \leq \mathbf{k}\}$. We assume for this set the standard lexicographic ordering:

$$\left[\dots \left[[(j_1, \dots, j_d)]_{j_d=h_d, \dots, k_d} \right]_{j_{d-1}=h_{d-1}, \dots, k_{d-1}} \dots \right]_{j_1=h_1, \dots, k_1}. \quad (49)$$

For instance, in the case $d = 2$ the ordering is the following: $(h_1, h_2), (h_1, h_2 + 1), \dots, (h_1, k_2), (h_1 + 1, h_2), (h_1 + 1, h_2 + 1), \dots, (h_1 + 1, k_2), \dots, (k_1, h_2), (k_1, h_2 + 1), \dots, (k_1, k_2)$.

- When a d -index \mathbf{j} varies over a multi-index range $\mathbf{h}, \dots, \mathbf{k}$ (this is sometimes written as $\mathbf{j} = \mathbf{h}, \dots, \mathbf{k}$), it is understood that \mathbf{j} varies from \mathbf{h} to \mathbf{k} following the specific ordering (49). For instance, if $\mathbf{m} \in \mathbb{N}^d$ and if we write $\mathbf{x} = [x_i]_{i=1}^m$, then \mathbf{x} is a vector of size $N(\mathbf{m})$ whose components x_i , $i = 1, \dots, m$, are ordered in accordance with (49): the first component is $x_1 = x_{(1, \dots, 1, 1)}$, the second component is $x_{(1, \dots, 1, 2)}$, and so on until the last component, which is $x_m = x_{(m_1, \dots, m_d)}$. Similarly, if $X = [x_{ij}]_{i,j=1}^m$, then X is a $N(\mathbf{m}) \times N(\mathbf{m})$ matrix whose components are indexed by two d -indices \mathbf{i}, \mathbf{j} , both varying from $\mathbf{1}$ to \mathbf{m} according to the lexicographic ordering (49).
- Given $\mathbf{h}, \mathbf{k} \in \mathbb{Z}^d$ with $\mathbf{h} \leq \mathbf{k}$, the notation $\sum_{\mathbf{j}=\mathbf{h}}^{\mathbf{k}}$ indicates the summation over all \mathbf{j} in $\mathbf{h}, \dots, \mathbf{k}$.
- Operations involving d -indices that have no meaning in the vector space \mathbb{Z}^d must be interpreted in the componentwise sense. For instance, $\mathbf{i}\mathbf{j} = (i_1 j_1, \dots, i_d j_d)$, $\mathbf{i}/\mathbf{j} = (i_1/j_1, \dots, i_d/j_d)$, etc.

Multilevel Block Matrix-Sequences. Given $d, s \geq 1$, a d -level s -block matrix-sequence (or simply a matrix-sequence if d and s can be inferred from the context or we do not need/want to specify them) is a sequence of matrices of the form $\{A_n\}_n$, where:

- n varies in some infinite subset of \mathbb{N} ;
- $\mathbf{n} = \mathbf{n}(n)$ is a d -index in \mathbb{N}^d which depends on n and satisfies $\mathbf{n} \rightarrow \infty$ as $n \rightarrow \infty$;
- A_n is a square matrix of size $N(\mathbf{n})^s$.

Multilevel Block Toeplitz Matrices. Given a function $f : [-\pi, \pi]^d \rightarrow \mathbb{C}^{s \times s}$ in $L^1([-\pi, \pi]^d)$, its Fourier coefficients are denoted by

$$f_{\mathbf{k}} = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} f(\boldsymbol{\theta}) e^{-i\mathbf{k} \cdot \boldsymbol{\theta}} d\boldsymbol{\theta} \in \mathbb{C}^{s \times s}, \quad \mathbf{k} \in \mathbb{Z}^d,$$

where $\mathbf{k} \cdot \boldsymbol{\theta} = k_1 \theta_1 + \dots + k_d \theta_d$ and the integrals are computed componentwise. For $\mathbf{n} \in \mathbb{N}^d$, the n th multilevel block Toeplitz matrix generated by f is defined as

$$T_n(f) = [f_{\mathbf{i}-\mathbf{j}}]_{\mathbf{i}, \mathbf{j}=1}^{\mathbf{n}} \in \mathbb{C}^{N(\mathbf{n})^s \times N(\mathbf{n})^s}.$$

It is not difficult to see that the map $f \mapsto T_n(f)$ is linear. Moreover, it can be shown that

$$T_n(f^*) = (T_n(f))^*, \quad (50)$$

where the transpose conjugate function f^* is defined by $f^*(\boldsymbol{\theta}) = (f(\boldsymbol{\theta}))^*$; in particular, all the matrices $T_n(f)$ are Hermitian whenever f is Hermitian a.e. We also recall that if $\mathbf{n} \in \mathbb{N}^d$ and $f_1, f_2, \dots, f_d : [-\pi, \pi] \rightarrow \mathbb{C}$ belong to $L^1([-\pi, \pi])$ then

$$T_{n_1}(f_1) \otimes T_{n_2}(f_2) \otimes \cdots \otimes T_{n_d}(f_d) = T_n(f), \quad (51)$$

where $f : [-\pi, \pi]^d \rightarrow \mathbb{C}$ is defined by $f(\boldsymbol{\theta}) = f(\theta_1)f(\theta_2) \cdots f(\theta_d)$; see, e.g., [27, Lemma 5.3].

Multilevel Block Diagonal Sampling Matrices. For $\mathbf{n} \in \mathbb{N}^d$ and $a : [0, 1]^d \rightarrow \mathbb{C}^{s \times s}$, we define the multilevel block diagonal sampling matrix $D_n(a)$ as the block diagonal matrix

$$D_n(a) = \text{diag}_{i=1, \dots, n} a \left(\frac{\mathbf{i}}{\mathbf{n}} \right) \in \mathbb{C}^{N(\mathbf{n})s \times N(\mathbf{n})s}.$$

Multilevel Block GLT Sequences. Let $d, s \geq 1$ be fixed positive integers. A d -level s -block GLT sequence (or simply a GLT sequence if d and s can be inferred from the context or we do not need/want to specify them) is a special d -level s -block matrix-sequence $\{A_n\}_n$ equipped with a measurable function $\kappa : [0, 1]^d \times [-\pi, \pi]^d \rightarrow \mathbb{C}^{s \times s}$, the so-called symbol. We use the notation $\{A_n\}_n \sim_{\text{GLT}} \kappa$ to indicate that $\{A_n\}_n$ is a GLT sequence with symbol κ . The symbol of a GLT sequence is unique in the sense that if $\{A_n\}_n \sim_{\text{GLT}} \kappa$ and $\{A_n\}_n \sim_{\text{GLT}} \zeta$ then $\kappa = \zeta$ a.e. in $[0, 1]^d \times [-\pi, \pi]^d$. The main properties of d -level s -block GLT sequences are listed below.

GLT 1. If $\{A_n\}_n \sim_{\text{GLT}} \kappa$ then $\{A_n\}_n \sim_{\sigma} \kappa$. If moreover each A_n is Hermitian then $\{A_n\}_n \sim_{\lambda} \kappa$.

GLT 2. We have:

- $\{T_n(f)\}_n \sim_{\text{GLT}} \kappa(\mathbf{x}, \boldsymbol{\theta}) = f(\boldsymbol{\theta})$ if $f : [-\pi, \pi]^d \rightarrow \mathbb{C}^{s \times s}$ is in $L^1([-\pi, \pi]^d)$;
- $\{D_n(a)\}_n \sim_{\text{GLT}} \kappa(\mathbf{x}, \boldsymbol{\theta}) = a(\mathbf{x})$ if $a : [0, 1]^d \rightarrow \mathbb{C}^{s \times s}$ is Riemann-integrable;
- $\{Z_n\}_n \sim_{\text{GLT}} \kappa(\mathbf{x}, \boldsymbol{\theta}) = O_s$ if and only if $\{Z_n\}_n \sim_{\sigma} 0$.

GLT 3. If $\{A_n\}_n \sim_{\text{GLT}} \kappa$ and $\{B_n\}_n \sim_{\text{GLT}} \zeta$ then:

- $\{A_n^*\}_n \sim_{\text{GLT}} \kappa^*$;
- $\{\alpha A_n + \beta B_n\}_n \sim_{\text{GLT}} \alpha \kappa + \beta \zeta$ for all $\alpha, \beta \in \mathbb{C}$;
- $\{A_n B_n\}_n \sim_{\text{GLT}} \kappa \zeta$;
- $\{A_n^\dagger\}_n \sim_{\text{GLT}} \kappa^{-1}$ provided that κ is invertible a.e.

GLT 4. $\{A_n\}_n \sim_{\text{GLT}} \kappa$ if and only if there exist GLT sequences $\{B_{n,m}\}_n \sim_{\text{GLT}} \kappa_m$ such that $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ and $\kappa_m \rightarrow \kappa$ in measure.

6. Discretizations of Systems of PDEs: The General GLT Approach

In this section we outline the main ideas of a multidimensional block GLT analysis for general discretizations of PDE systems. What we are going to present here is then a generalization of what we have seen in Section 3. We begin by proving a series of auxiliary results. In what follows, given $\mathbf{n} \in \mathbb{N}^d$ and $s \geq 1$, we denote by $\Pi_{n,s}$ the permutation matrix given by

$$\Pi_{n,s} = \begin{bmatrix} I_s \otimes \mathbf{e}_1^T \\ I_s \otimes \mathbf{e}_2^T \\ \vdots \\ I_s \otimes \mathbf{e}_n^T \end{bmatrix} = \sum_{k=1}^n \mathbf{e}_k \otimes I_s \otimes \mathbf{e}_k^T, \quad (52)$$

where \mathbf{e}_i , $i = 1, \dots, n$, are the vectors of the canonical basis of $\mathbb{R}^{N(n)}$, which, for convenience, are indexed by a d -index $\mathbf{i} = 1, \dots, n$ instead of a linear index $i = 1, \dots, N(n)$. Note that $\Pi_{n,2}$ coincides with the matrix Π_n in (20).

Lemma 3. Let $\mathbf{n} \in \mathbb{N}^d$, let $f_{ij} : [-\pi, \pi]^d \rightarrow \mathbb{C}$ be in $L^1([-\pi, \pi]^d)$ for $i, j = 1, \dots, s$, and set $f = [f_{ij}]_{i,j=1}^s$. The block matrix $T_n = [T_n(f_{ij})]_{i,j=1}^s$ is similar via the permutation (52) to the multilevel block Toeplitz matrix $T_n(f)$, that is, $\Pi_{n,s} T_n \Pi_{n,s}^T = T_n(f)$.

Proof. Let E_{ij} be the $s \times s$ matrix having 1 in position (i, j) and 0 elsewhere. Since $T_n = \sum_{i,j=1}^s E_{ij} \otimes T_n(f_{ij})$ and $T_n(f) = \sum_{i,j=1}^s T_n(f_{ij} E_{ij})$ by the linearity of the map $T_n(\cdot)$, it is enough to show that

$$\Pi_{n,s}(E \otimes T_n(g))\Pi_{n,s}^T = T_n(gE), \quad \forall g \in L^1([-\pi, \pi]^d), \quad \forall E \in \mathbb{C}^{s \times s}.$$

By (5) and (6),

$$\begin{aligned} \Pi_{n,s}(E \otimes T_n(g))\Pi_{n,s}^T &= \left[\sum_{k=1}^n \mathbf{e}_k \otimes I_s \otimes \mathbf{e}_k^T \right] (E \otimes T_n(g)) \left[\sum_{\ell=1}^n \mathbf{e}_\ell^T \otimes I_s \otimes \mathbf{e}_\ell \right] \\ &= \sum_{k,\ell=1}^n (\mathbf{e}_k \otimes I_s \otimes \mathbf{e}_k^T) (E \otimes T_n(g)) (\mathbf{e}_\ell^T \otimes I_s \otimes \mathbf{e}_\ell) \\ &= \sum_{k,\ell=1}^n \mathbf{e}_k \mathbf{e}_\ell^T \otimes E \otimes \mathbf{e}_k^T T_n(g) \mathbf{e}_\ell = \sum_{k,\ell=1}^n \mathbf{e}_k \mathbf{e}_\ell^T \otimes (T_n(g))_{k\ell} E = T_n(gE), \end{aligned}$$

as required. \square

Lemma 4. Let $\mathbf{n} \in \mathbb{N}^d$, let $a_{ij} : [0, 1]^d \rightarrow \mathbb{C}$ for $i, j = 1, \dots, s$, and set $a = [a_{ij}]_{i,j=1}^s$. The block matrix $D_n = [D_n(a_{ij})]_{i,j=1}^s$ is similar via the permutation (52) to the multilevel block diagonal sampling matrix $D_n(a)$, that is, $\Pi_{n,s} D_n \Pi_{n,s}^T = D_n(a)$.

Proof. With obvious adaptations, it is the same as the proof of Lemma 3. \square

We recall that a d -variate trigonometric polynomial is a finite linear combination of the d -variate Fourier frequencies $e^{i\mathbf{k} \cdot \boldsymbol{\theta}}$, $\mathbf{k} \in \mathbb{Z}^d$.

Theorem 5. For $i, j = 1, \dots, s$, let $\{A_{n,ij}\}_n$ be a d -level 1-block GLT sequence with symbol $\kappa_{ij} : [0, 1]^d \times [-\pi, \pi]^d \rightarrow \mathbb{C}$. Set $A_n = [A_{n,ij}]_{i,j=1}^s$ and $\kappa = [\kappa_{ij}]_{i,j=1}^s$. Then, the matrix-sequence $\{\Pi_{n,s} A_n \Pi_{n,s}^T\}_n$ is a d -level s -block GLT sequence with symbol κ .

Proof. The proof consists of the following two steps.

Step 1. We first prove the theorem under the additional assumption that $A_{n,ij}$ is of the form

$$A_{n,ij} = \sum_{\ell=1}^{L_{ij}} D_n(a_{\ell,ij}) T_n(f_{\ell,ij}), \quad (53)$$

where $L_{ij} \in \mathbb{N}$, $a_{\ell,ij} : [0, 1]^d \rightarrow \mathbb{C}$ is Riemann-integrable, and $f_{\ell,ij} : [-\pi, \pi]^d \rightarrow \mathbb{C}$ belongs to $L^1([-\pi, \pi]^d)$. Note that the symbol of $\{A_{n,ij}\}_n$ is

$$\kappa_{ij}(\mathbf{x}, \boldsymbol{\theta}) = \sum_{\ell=1}^{L_{ij}} a_{\ell,ij}(\mathbf{x}) f_{\ell,ij}(\boldsymbol{\theta}).$$

By setting $L = \max_{i,j=1,\dots,s} L_{ij}$ and by adding zero matrices of the form $D_n(0)T_n(0)$ in the summation (53) whenever $L_{ij} < L$, we can assume, without loss of generality, that

$$A_{n,ij} = \sum_{\ell=1}^L D_n(a_{\ell,ij})T_n(f_{\ell,ij}),$$

$$\kappa_{ij}(\mathbf{x}, \boldsymbol{\theta}) = \sum_{\ell=1}^L a_{\ell,ij}(\mathbf{x})f_{\ell,ij}(\boldsymbol{\theta}),$$

with L independent of i, j . Let E_{ij} be the $s \times s$ matrix having 1 in position (i, j) and 0 elsewhere. Then,

$$\begin{aligned} \Pi_{n,s}A_n\Pi_{n,s}^T &= \sum_{\ell=1}^L \Pi_{n,s} \left[D_n(a_{\ell,ij})T_n(f_{\ell,ij}) \right]_{i,j=1}^s \Pi_{n,s}^T \\ &= \sum_{\ell=1}^L \Pi_{n,s} \left[\sum_{i,j=1}^s (E_{ij} \otimes D_n(a_{\ell,ij}))(E_{ij} \otimes T_n(f_{\ell,ij})) \right] \Pi_{n,s}^T \\ &= \sum_{\ell=1}^L \sum_{i,j=1}^s \Pi_{n,s}(E_{ij} \otimes D_n(a_{\ell,ij}))\Pi_{n,s}^T \Pi_{n,s}(E_{ij} \otimes T_n(f_{\ell,ij}))\Pi_{n,s}^T. \end{aligned}$$

By Lemmas 3 and 4,

$$\begin{aligned} \Pi_{n,s}(E_{ij} \otimes D_n(a_{\ell,ij}))\Pi_{n,s}^T &= D_n(a_{\ell,ij}E_{ij}), \\ \Pi_{n,s}(E_{ij} \otimes T_n(f_{\ell,ij}))\Pi_{n,s}^T &= T_n(f_{\ell,ij}E_{ij}). \end{aligned}$$

It follows that

$$\Pi_{n,s}A_n\Pi_{n,s}^T = \sum_{\ell=1}^L \sum_{i,j=1}^s D_n(a_{\ell,ij}E_{ij})T_n(f_{\ell,ij}E_{ij}).$$

Thus, by **GLT 2** and **GLT 3**, $\{\Pi_{n,s}A_n\Pi_{n,s}^T\}_n$ is a d -level s -block GLT sequence with symbol

$$\kappa(\mathbf{x}, \boldsymbol{\theta}) = \sum_{\ell=1}^L \sum_{i,j=1}^s a_{\ell,ij}(\mathbf{x})f_{\ell,ij}(\boldsymbol{\theta})E_{ij} = [\kappa_{ij}(\mathbf{x}, \boldsymbol{\theta})]_{i,j=1}^s.$$

Step 2. We now prove the theorem in its full generality. Since $\{A_{n,ij}\}_n \sim_{\text{GLT}} \kappa_{ij}$, by [27, Corollary 7.2] there exist functions $a_{\ell,ij}^{(m)}, f_{\ell,ij}^{(m)}$, $\ell = 1, \dots, L_{ij}^{(m)}$, such that

- $a_{\ell,ij}^{(m)} \in C^\infty([0, 1]^d)$ and $f_{\ell,ij}^{(m)}$ is a d -variate trigonometric polynomial,
- $\kappa_{ij}^{(m)}(\mathbf{x}, \boldsymbol{\theta}) = \sum_{\ell=1}^{L_{ij}^{(m)}} a_{\ell,ij}^{(m)}(\mathbf{x})f_{\ell,ij}^{(m)}(\boldsymbol{\theta}) \rightarrow \kappa_{ij}(\mathbf{x}, \boldsymbol{\theta})$ a.e.,
- $\{A_{n,ij}^{(m)} = \sum_{\ell=1}^{L_{ij}^{(m)}} D_n(a_{\ell,ij}^{(m)})T_n(f_{\ell,ij}^{(m)})\}_n \xrightarrow{\text{a.c.s.}} \{A_{n,ij}\}_n$.

Set $A_n^{(m)} = [A_{n,ij}^{(m)}]_{i,j=1}^s$ and $\kappa^{(m)}(\mathbf{x}, \boldsymbol{\theta}) = [\kappa_{ij}^{(m)}(\mathbf{x}, \boldsymbol{\theta})]_{i,j=1}^s$. We have:

- $\{\Pi_{n,s}A_n^{(m)}\Pi_{n,s}^T\}_n \sim_{\text{GLT}} \kappa^{(m)}$ by Step 1;
- $\kappa^{(m)} \rightarrow \kappa$ a.e. (and hence also in measure);
- $\{\Pi_{n,s}A_n^{(m)}\Pi_{n,s}^T\}_n \xrightarrow{\text{a.c.s.}} \{\Pi_{n,s}A_n\Pi_{n,s}^T\}_n$ because $\{A_n^{(m)}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ by Lemma 1.

We conclude that $\{\Pi_{n,s}A_n\Pi_{n,s}^T\}_n \sim_{\text{GLT}} \kappa$ by **GLT 4**. \square

Now, suppose we have a system of linear PDEs of the form

$$\begin{cases} \sum_{j=1}^s \mathcal{L}_{1j} u_j(\mathbf{x}) = f_1(\mathbf{x}), \\ \sum_{j=1}^s \mathcal{L}_{2j} u_j(\mathbf{x}) = f_2(\mathbf{x}), \\ \vdots \\ \sum_{j=1}^s \mathcal{L}_{sj} u_j(\mathbf{x}) = f_s(\mathbf{x}), \end{cases} \quad (54)$$

where $\mathbf{x} \in (0, 1)^d$. The matrices A_n resulting from any standard discretization of (54) are parameterized by a d -index $\mathbf{n} = (n_1, \dots, n_d)$, where n_i is related to the discretization step h_i in the i th direction, and $n_i \rightarrow \infty$ if and only if $h_i \rightarrow 0$ (usually, $h_i \approx 1/n_i$). By choosing each n_i as a function of a unique discretization parameter $n \in \mathbb{N}$, as it normally happens in practice where the most natural choice is $n_i = n$ for all $i = 1, \dots, d$, we see that $\mathbf{n} = \mathbf{n}(n)$ and, consequently, $\{A_n\}_n$ is a (d -level) matrix-sequence. Moreover, it turns out that, after a suitable normalization that we ignore in this discussion,¹ A_n has the following block structure:

$$A_n = [A_{n,ij}]_{i,j=1}^s, \quad (55)$$

where $A_{n,ij}$ is the (normalized) matrix coming from the discretization of the differential operator \mathcal{L}_{ij} . In the simplest case where the operators \mathcal{L}_{ij} have constant coefficients and we use equispaced grids in each direction, the matrix $A_{n,ij}$ takes the form

$$A_{n,ij} = T_n(f_{ij}) + Z_{n,ij},$$

where f_{ij} is a d -variate trigonometric polynomial, while the perturbation $Z_{n,ij}$ is usually a low-rank correction due to boundary conditions and, in any case, we have $\{Z_{n,ij}\}_n \sim_\sigma 0$. Hence,

$$\{A_{n,ij}\}_n \sim_{\text{GLT}} f_{ij}$$

by **GLT 2** and **GLT 3**, and it follows from Theorem 5 that

$$\{\Pi_{n,s} A_n \Pi_{n,s}^T\}_n \sim_{\text{GLT}} [f_{ij}]_{i,j=1}^s.$$

In the case where the operators \mathcal{L}_{ij} have variable coefficients, the matrix $A_{n,ij}$ usually takes the form

$$A_{n,ij} = \sum_{\ell=1}^{L_{ij}} D_n(a_{\ell,ij}) T_n(f_{\ell,ij}) + Z_{n,ij},$$

where $L_{ij} \in \mathbb{N}$, $f_{\ell,ij}$ is a d -variate trigonometric polynomial, $\{Z_{n,ij}\}_n \sim_\sigma 0$, and the functions $a_{\ell,ij} : [0, 1]^d \rightarrow \mathbb{R}$, $\ell = 1, \dots, L_{ij}$, are related to the coefficients of \mathcal{L}_{ij} .² Hence,

$$\{A_{n,ij}\}_n \sim_{\text{GLT}} \kappa_{ij}(\mathbf{x}, \boldsymbol{\theta}) = \sum_{\ell=1}^{L_{ij}} a_{\ell,ij}(\mathbf{x}) f_{\ell,ij}(\boldsymbol{\theta})$$

by **GLT 2** and **GLT 3**, and it follows from Theorem 5 that

$$\{\Pi_{n,s} A_n \Pi_{n,s}^T\}_n \sim_{\text{GLT}} [\kappa_{ij}]_{i,j=1}^s.$$

¹ The normalization we are talking about is the analog of the normalization that we have seen in Section 3, which allowed us to pass from the matrix A_n in (12) to the matrix B_n in (14).

² For example, in Section 3, while proving (21), we have seen that $K_n(a_{11})$, which plays there the same role as the matrix $A_{n,11}$ here, is equal to $D_n(a_{11}) T_n(2 - 2 \cos \theta) + Z_n$ for some zero-distributed sequence $\{Z_n\}_n$.

7. B-spline IgA Discretization of a Variational Problem for the Curl-Curl Operator

For any function $\mathbf{u}(x_1, x_2) = [u_1(x_1, x_2), u_2(x_1, x_2)]^T$, defined over some open set $\Omega \subseteq \mathbb{R}^2$ and taking values in \mathbb{R}^2 , the curl operator is formally defined as follows:

$$(\nabla \times \mathbf{u})(x_1, x_2) = \frac{\partial u_2}{\partial x_1}(x_1, x_2) - \frac{\partial u_1}{\partial x_2}(x_1, x_2), \quad (x_1, x_2) \in \Omega.$$

Clearly, this definition has meaning when the components u_1, u_2 belong to $H^1(\Omega)$, so that their partial derivatives exist in the Sobolev sense. Now, let $\Omega = (0, 1)^2$, set

$$\begin{aligned} (L^2(\Omega))^2 &= \{\mathbf{u} : \Omega \rightarrow \mathbb{R}^2 : u_1, u_2 \in L^2(\Omega)\}, \\ H(\text{curl}, \Omega) &= \{\mathbf{u} \in (L^2(\Omega))^2 : \nabla \times \mathbf{u} \text{ exists in the Sobolev sense, } \nabla \times \mathbf{u} \in L^2(\Omega)\}, \end{aligned}$$

and consider the following variational problem: find $\mathbf{u} \in H(\text{curl}, \Omega)$ such that

$$(\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in H(\text{curl}, \Omega), \quad (56)$$

where $\mathbf{f}(x_1, x_2) = [f_1(x_1, x_2), f_2(x_1, x_2)]^T$ is a vector field in $(L^2(\Omega))^2$ and

$$\begin{aligned} (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) &= \int_{\Omega} (\nabla \times \mathbf{u})(x_1, x_2) (\nabla \times \mathbf{v})(x_1, x_2) \, dx_1 dx_2, \\ (\mathbf{f}, \mathbf{v}) &= \int_{\Omega} [f_1(x_1, x_2)v_1(x_1, x_2) + f_2(x_1, x_2)v_2(x_1, x_2)] \, dx_1 dx_2. \end{aligned}$$

Variational problems of the form (56) arise in important applications, such as time harmonic Maxwell's equations and magnetostatic problems. In this section we consider a so-called compatible B-spline IgA discretization of (56); see [15] for details. We show that the corresponding sequence of discretization matrices enjoys a spectral distribution described by a 2×2 matrix-valued function whose determinant is zero everywhere. The results of this section have already been obtained in [31], but the derivation presented here is entirely based on the theory of multilevel block GLT sequences and turns out to be simpler and more lucid than that in [31]. For simplicity, throughout this section the B-splines $B_{i,[p,p-1]}$, $i = 1, \dots, n + p$, and the associated reference B-spline $\beta_{1,[p,p-1]}$, will be denoted by $B_{i,[p]}$, $i = 1, \dots, n + p$, and $\beta_{[p]}$, respectively. The function $\beta_{[p]}$ is the so-called cardinal B-spline of degree p over the knot sequence $\{0, 1, \dots, p + 1\}$. In view of what follows, we recall from [16] and [23, Lemma 4] that the cardinal B-spline $\beta_{[q]}$ is defined for all degrees $q \geq 0$, belongs to $C^{q-1}(\mathbb{R})$, and satisfies the following properties:

$$\text{supp}(\beta_{[q]}) = [0, q + 1] \quad (57)$$

for $q \geq 1$,

$$\beta'_{[q]}(t) = \beta_{[q-1]}(t) - \beta_{[q-1]}(t - 1), \quad (58)$$

for $t \in \mathbb{R}$ and $q \geq 1$, and

$$\int_{\mathbb{R}} \beta_{[q_1]}^{(r_1)}(\tau) \beta_{[q_2]}^{(r_2)}(\tau + t) \, d\tau = (-1)^{r_1} \beta_{[q_1+q_2+1]}^{(r_1+r_2)}(q_1 + 1 + t) = (-1)^{r_2} \beta_{[q_1+q_2+1]}^{(r_1+r_2)}(q_2 + 1 - t) \quad (59)$$

for $t \in \mathbb{R}$ and $q_1, q_2, r_1, r_2 \geq 0$. Moreover, property (39) in the case $k = p - 1$ simplifies to

$$B_{i,[p]}(x) = \beta_{[p]}(nx - i + p + 1), \quad i = p + 1, \dots, n. \quad (60)$$

7.1. Compatible B-Spline IgA Discretization

Let $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$, let $p \geq 2$, and define the space

$$\mathcal{V}_{\mathbf{n},[p]}(\text{curl}, \Omega) = \text{span} \left\{ \begin{array}{l} \left[\begin{array}{l} B_{i_1,[p-1]}(x_1) B_{i_2,[p]}(x_2) \\ B_{j_1,[p]}(x_1) B_{j_2,[p-1]}(x_2) \end{array} \right] : \begin{array}{l} i_1 = 1, \dots, n_1 + p - 1, \quad i_2 = 1, \dots, n_2 + p, \\ j_1 = 1, \dots, n_1 + p, \quad j_2 = 1, \dots, n_2 + p - 1 \end{array} \end{array} \right\}. \quad (61)$$

Following a compatible B-spline approach [15], we look for an approximation of the solution in the space $\mathcal{V}_{\mathbf{n},[p]}(\text{curl}, \Omega)$ by solving the following discrete problem: find $\mathbf{u}_{\mathcal{V}} \in \mathcal{V}_{\mathbf{n},[p]}(\text{curl}, \Omega)$ such that

$$(\nabla \times \mathbf{u}_{\mathcal{V}}, \nabla \times \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{V}_{\mathbf{n},[p]}(\text{curl}, \Omega).$$

After choosing a suitable ordering on the basis functions of $\mathcal{V}_{\mathbf{n},[p]}(\text{curl}, \Omega)$ displayed in (61), by linearity the computation of $\mathbf{u}_{\mathcal{V}}$ reduces to solving a linear system whose coefficient matrix is given by

$$A_{\mathbf{n},[p]} = \begin{bmatrix} A_{\mathbf{n},[p],11} & A_{\mathbf{n},[p],12} \\ A_{\mathbf{n},[p],21} & A_{\mathbf{n},[p],22} \end{bmatrix} = \begin{bmatrix} M_{n_1,[p-1]} \otimes K_{n_2,[p]} & -H_{n_1,[p]} \otimes (H_{n_2,[p]})^T \\ -(H_{n_1,[p]})^T \otimes H_{n_2,[p]} & K_{n_1,[p]} \otimes M_{n_2,[p-1]} \end{bmatrix},$$

where

$$\begin{aligned} (M_{n,[p-1]})_{ij} &= \int_0^1 B_{j,[p-1]}(x) B_{i,[p-1]}(x) dx, \quad i, j = 1, \dots, n + p - 1, \\ (K_{n,[p]})_{ij} &= \int_0^1 B'_{j,[p]}(x) B'_{i,[p]}(x) dx, \quad i, j = 1, \dots, n + p, \\ (H_{n,[p]})_{ij} &= \int_0^1 B'_{j,[p]}(x) B_{i,[p-1]}(x) dx, \quad i = 1, \dots, n + p - 1, \quad j = 1, \dots, n + p. \end{aligned}$$

Note that $M_{n,[p-1]}$ is a square matrix of size $n + p - 1$, $K_{n,[p]}$ is a square matrix of size $n + p$, while $H_{n,[p]}$ is a rectangular matrix of size $(n + p - 1) \times (n + p)$.

7.2. GLT Analysis of the B-Spline IgA Discretization Matrices

In the main result of this section (Theorem 6), assuming that $\mathbf{n} = n\mathbf{v}$ for a fixed vector \mathbf{v} , we show that the spectral distribution of the sequence $\{A_{\mathbf{n},[p]}\}_n$ is described by a 2×2 matrix-valued function whose determinant is zero everywhere (Remark 2). In order to prove Theorem 6, some preliminary work is necessary. We first note that, in view of the application of Theorem 5, the matrix $A_{\mathbf{n},[p]}$ has an unpleasant feature: the anti-diagonal blocks $A_{\mathbf{n},[p],12}$ and $A_{\mathbf{n},[p],21}$ are not square and, moreover, the square diagonal blocks $A_{\mathbf{n},[p],11}$ and $A_{\mathbf{n},[p],22}$ do not have the same size whenever $n_1 \neq n_2$. Let us then introduce the nicer matrix

$$\tilde{A}_{\mathbf{n},[p]} = \begin{bmatrix} \tilde{A}_{\mathbf{n},[p],11} & \tilde{A}_{\mathbf{n},[p],12} \\ \tilde{A}_{\mathbf{n},[p],21} & \tilde{A}_{\mathbf{n},[p],22} \end{bmatrix} = \begin{bmatrix} \tilde{M}_{n_1,[p-1]} \otimes K_{n_2,[p]} & -\tilde{H}_{n_1,[p]} \otimes (\tilde{H}_{n_2,[p]})^T \\ -(\tilde{H}_{n_1,[p]})^T \otimes \tilde{H}_{n_2,[p]} & K_{n_1,[p]} \otimes \tilde{M}_{n_2,[p-1]} \end{bmatrix},$$

where $\tilde{M}_{n,[p-1]}$ and $\tilde{H}_{n,[p]}$ are square matrices of size $n + p$ given by

$$\tilde{M}_{n,[p-1]} = \left[\begin{array}{ccc|c} & & & 0 \\ & M_{n,[p-1]} & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & 0 \end{array} \right], \quad \tilde{H}_{n,[p]} = \left[\begin{array}{ccc|c} & & & H_{n,[p]} \\ \hline 0 & \dots & 0 & \end{array} \right].$$

Each block $\tilde{A}_{n,[p],ij}$ of the matrix $\tilde{A}_{n,[p]}$ is now a square block of size $(n_1 + p)(n_2 + p)$. Moreover,

$$M_{n,[p-1]} = P_{n,[p]} \tilde{M}_{n,[p-1]} (P_{n,[p]})^T, \quad H_{n,[p]} = P_{n,[p]} \tilde{H}_{n,[p]},$$

where the matrix

$$P_{n,[p]} = \begin{bmatrix} & & 0 \\ & I_{n+p-1} & \vdots \\ & & 0 \end{bmatrix}$$

satisfies $P_{n,[p]}(P_{n,[p]})^T = I_{n+p-1}$. By (5) and (6),

$$\begin{aligned} A_{n,[p],11} &= (P_{n_1,[p]} \otimes I_{n_2+p}) \tilde{A}_{n,[p],11} (P_{n_1,[p]} \otimes I_{n_2+p})^T, \\ A_{n,[p],12} &= (P_{n_1,[p]} \otimes I_{n_2+p}) \tilde{A}_{n,[p],12} (I_{n_1+p} \otimes P_{n_2,[p]})^T, \\ A_{n,[p],21} &= (I_{n_1+p} \otimes P_{n_2,[p]}) \tilde{A}_{n,[p],21} (P_{n_1,[p]} \otimes I_{n_2+p})^T, \\ A_{n,[p],22} &= (I_{n_1+p} \otimes P_{n_2,[p]}) \tilde{A}_{n,[p],22} (I_{n_1+p} \otimes P_{n_2,[p]})^T, \end{aligned}$$

and so

$$A_{n,[p]} = P_{n,[p]} \tilde{A}_{n,[p]} (P_{n,[p]})^T, \quad P_{n,[p]} = \begin{bmatrix} P_{n_1,[p]} \otimes I_{n_2+p} & \\ & I_{n_1+p} \otimes P_{n_2,[p]} \end{bmatrix}. \quad (62)$$

In view of the application of Theorem 1, we note that

$$P_{n,[p]} \in \mathbb{R}^{[(n_1+p-1)(n_2+p)+(n_1+p)(n_2+p-1)] \times 2(n_1+p)(n_2+p)}, \quad (63)$$

$$P_{n,[p]}(P_{n,[p]})^T = I_{(n_1+p-1)(n_2+p)+(n_1+p)(n_2+p-1)}, \quad (64)$$

$$\lim_{n \rightarrow \infty} \frac{(n_1 + p - 1)(n_2 + p) + (n_1 + p)(n_2 + p - 1)}{2(n_1 + p)(n_2 + p)} = \lim_{n \rightarrow \infty} \left[\frac{n_1 + p - 1}{2(n_1 + p)} + \frac{n_2 + p - 1}{2(n_2 + p)} \right] = 1. \quad (65)$$

Lemma 5. Let $p \geq 2$ and $n \geq 1$. Then

$$n^{-1}K_{n,[p]} = T_{n+p}(f_p) + Q_{n,[p]}, \quad \text{rank}(Q_{n,[p]}) \leq 4p, \quad (66)$$

$$\tilde{H}_{n,[p]} = T_{n+p}(g_p) + R_{n,[p]}, \quad \text{rank}(R_{n,[p]}) \leq 4p, \quad (67)$$

$$n\tilde{M}_{n,[p-1]} = T_{n+p}(h_p) + S_{n,[p]}, \quad \text{rank}(S_{n,[p]}) \leq 4p, \quad (68)$$

where

$$f_p(\theta) = \sum_{k \in \mathbb{Z}} -\beta''_{[2p+1]}(p+1-k)e^{ik\theta}, \quad (69)$$

$$g_p(\theta) = \sum_{k \in \mathbb{Z}} -\beta'_{[2p]}(p-k)e^{ik\theta}, \quad (70)$$

$$h_p(\theta) = \sum_{k \in \mathbb{Z}} \beta_{[2p-1]}(p-k)e^{ik\theta}, \quad (71)$$

and we note the three series are actually finite sums because of (57).

Proof. For every $i, j = p+1, \dots, n$, since $[-i+p+1, n-i+p+1] \supseteq [0, p+1] = \text{supp}(\beta_{[p]})$ and $[-i+p, n-i+p] \supseteq [0, p] = \text{supp}(\beta_{[p-1]})$, by (59) and (60) we obtain

$$\begin{aligned} (K_{n,[p]})_{ij} &= \int_0^1 B'_{j,[p]}(x)B'_{i,[p]}(x)dx = n^2 \int_0^1 \beta'_{[p]}(nx-j+p+1)\beta'_{[p]}(nx-i+p+1)dx \\ &= n \int_{-i+p+1}^{n-i+p+1} \beta'_{[p]}(\tau+i-j)\beta'_{[p]}(\tau)d\tau = n \int_{\mathbb{R}} \beta'_{[p]}(\tau)\beta'_{[p]}(\tau+i-j)d\tau \end{aligned}$$

$$\begin{aligned}
&= -n\beta''_{[2p+1]}(p+1+i-j) = -n\beta''_{[2p+1]}(p+1-i+j), \\
(\tilde{H}_{n,[p]})_{ij} &= \int_0^1 B'_{j,[p]}(x)B_{i,[p-1]}(x)dx = n \int_0^1 \beta'_{[p]}(nx-j+p+1)\beta_{[p-1]}(nx-i+p)dx \\
&= \int_{-i+p}^{n-i+p} \beta'_{[p]}(\tau+i-j+1)\beta_{[p-1]}(\tau)d\tau = \int_{\mathbb{R}} \beta_{[p-1]}(\tau)\beta'_{[p]}(\tau+i-j+1)d\tau \\
&= \beta'_{[2p]}(p+i-j+1) = -\beta'_{[2p]}(p-i+j), \\
(\tilde{M}_{n,[p-1]})_{ij} &= \int_0^1 B_{j,[p-1]}(x)B_{i,[p-1]}(x)dx = \int_0^1 \beta_{[p-1]}(nx-j+p)\beta_{[p-1]}(nx-i+p)dx \\
&= n^{-1} \int_{-i+p}^{n-i+p} \beta_{[p-1]}(\tau+i-j)\beta_{[p-1]}(\tau)d\tau = n^{-1} \int_{\mathbb{R}} \beta_{[p-1]}(\tau)\beta_{[p-1]}(\tau+i-j)d\tau \\
&= n^{-1}\beta_{[2p-1]}(p+i-j) = n^{-1}\beta_{[2p-1]}(p-i+j).
\end{aligned}$$

Thus,

$$[(n^{-1}K_{n,[p]})_{ij}]_{i,j=p+1}^n = [-\beta''_{[2p+1]}(p+1-i+j)]_{i,j=p+1}^n = T_{n-p}(f_p), \quad (72)$$

$$[(\tilde{H}_{n,[p]})_{ij}]_{i,j=p+1}^n = [-\beta'_{[2p]}(p-i+j)]_{i,j=p+1}^n = T_{n-p}(g_p), \quad (73)$$

$$[(n\tilde{M}_{n,[p-1]})_{ij}]_{i,j=p+1}^n = [\beta_{[2p-1]}(p-i+j)]_{i,j=p+1}^n = T_{n-p}(h_p). \quad (74)$$

It follows from (72) that the principal submatrix of $n^{-1}K_n^{[p]} - T_{n+p}(f_p)$ corresponding to the row and column indices $i, j = p+1, \dots, n$ is the zero matrix, which implies (66). Similarly, (73) and (74) imply (67) and (68), respectively. \square

Theorem 6. Let $p \geq 2$, let $\nu = (\nu_1, \nu_2) \in \mathbb{Q}^2$ be a vector with positive components, and assume that $\mathbf{n} = n\nu$ (it is understood that n varies in the infinite subset of \mathbb{N} such that $\mathbf{n} = n\nu \in \mathbb{N}^2$). Then

$$\{A_{\mathbf{n},[p]}\}_n \sim_{\sigma,\lambda} \kappa(\boldsymbol{\theta}) = \begin{bmatrix} \frac{\nu_2}{\nu_1} h_p(\theta_1) f_p(\theta_2) & -g_p(\theta_1) \overline{g_p(\theta_2)} \\ -\overline{g_p(\theta_1)} g_p(\theta_2) & \frac{\nu_1}{\nu_2} f_p(\theta_1) h_p(\theta_2) \end{bmatrix}.$$

Proof. The thesis follows immediately from Theorem 1 and (62)–(65) as soon as we have proved that

$$\{\tilde{A}_{\mathbf{n},[p]}\}_n \sim_{\sigma,\lambda} \kappa(\boldsymbol{\theta}). \quad (75)$$

We show that

$$\{\tilde{A}_{\mathbf{n},[p],ij}\}_n \sim_{\text{GLT}} \kappa_{ij}(\boldsymbol{\theta}), \quad i, j = 1, 2. \quad (76)$$

Once this is done, the thesis (75) follows immediately from Theorem 5 and GLT 1 as the matrix $\tilde{A}_{\mathbf{n},[p]}$ is symmetric. Actually, we only prove (76) for $(i, j) = (1, 2)$ because the proof for the other pairs of indices (i, j) is conceptually the same. Setting $\mathbf{p} = (p, p)$ and keeping in mind the assumption $\mathbf{n} = n\nu$, by Lemma 5, (4), (50) and (51) we have

$$\begin{aligned}
\tilde{A}_{\mathbf{n},[p],12} &= -\tilde{H}_{n_1,[p]} \otimes (\tilde{H}_{n_2,[p]})^T = -(T_{n_1+p}(g_p) + R_{n_1,[p]}) \otimes (T_{n_2+p}(g_p) + R_{n_2,[p]})^T \\
&= -(T_{n_1+p}(g_p) + R_{n_1,[p]}) \otimes (T_{n_2+p}(\overline{g_p}) + (R_{n_2,[p]})^T) \\
&= -(T_{n_1+p}(g_p(\theta_1)\overline{g_p(\theta_2)}) + T_{n_1+p}(g_p) \otimes (R_{n_2,[p]})^T + R_{n_1,[p]} \otimes (\tilde{H}_{n_2,[p]})^T) \\
&= T_{n_1+p}(\kappa_{12}) + V_{\mathbf{n},[p]},
\end{aligned}$$

where $\text{rank}(V_{\mathbf{n},[p]}) \leq 4p(n_1+p) + 4p(n_2+p)$. Thus, $\{V_{\mathbf{n},[p]}\}_n \sim_{\sigma} 0$ by Proposition 1, and (76) for $(i, j) = (1, 2)$ follows from GLT 2 and GLT 3. \square

Remark 2. Using (58), it is not difficult to see that the functions $f_p(\theta)$ and $g_p(\theta)$ in (69) and (70) can be expressed in terms of $h_p(\theta)$ as follows:

$$f_p(\theta) = (2 - 2 \cos \theta)h_p(\theta), \quad g_p(\theta) = (e^{-i\theta} - 1)h_p(\theta).$$

Therefore, the 2×2 matrix-valued function $\kappa(\boldsymbol{\theta})$ appearing in Theorem 6 can be simplified as follows:

$$\kappa(\boldsymbol{\theta}) = \frac{1}{v_1 v_2} h_p(\theta_1) h_p(\theta_2) \begin{bmatrix} v_2(e^{i\theta_2} - 1) \\ -v_1(e^{i\theta_1} - 1) \end{bmatrix} \begin{bmatrix} v_2(e^{-i\theta_2} - 1) & -v_1(e^{-i\theta_1} - 1) \end{bmatrix}.$$

In particular, $\det(\kappa(\boldsymbol{\theta})) = 0$ for all $\boldsymbol{\theta}$. According to the informal meaning behind the spectral distribution $\{A_{n,[p]}\}_n \sim_\lambda \kappa(\boldsymbol{\theta})$ reported in [28, Remark 1] or [21, Section 4], this means that, for large n , one half of the eigenvalues of $A_{n,[p]}$ are approximately zero and one half is given by a uniform sampling over $[-\pi, \pi]^d$ of

$$\text{trace}(\kappa(\boldsymbol{\theta})) = \frac{v_2}{n_1} h_p(\theta_1) h_p(\theta_2) (4 - 2 \cos \theta_1 - 2 \cos \theta_2).$$

8. Linear Elasticity Equations Discretized by the Modified Taylor–Hood FE Q1isoQ1 Stable Pair

In this section we focus on a specific formulation of the linear elasticity equations, which is related to the glacial isostatic adjustment (GIA) model; see, e.g., [42]. The GIA model is used to describe the response of the Earth to the redistribution of mass due to alternating glaciation and deglaciation periods. The considered elasticity problem is the following bidimensional system of PDEs,

$$\begin{cases} -2\alpha \Delta \mathbf{u} + \alpha (\nabla \times \mathbf{u})^2 - \nabla(\mathbf{b} \cdot \mathbf{u}) + \alpha \nabla p = \mathbf{f}, \\ \alpha \nabla \cdot \mathbf{u} - \rho p = 0, \end{cases} \quad (77)$$

which is defined over a rectangular domain $\Omega \subset \mathbb{R}^2$ and is accompanied by appropriate boundary conditions. In (77), $\mathbf{u} = [u_1(x_1, x_2), u_2(x_1, x_2)]^T$ is a vector containing the displacements u_1 and u_2 in the x_1 and x_2 directions, respectively; p is the pressure; $\mathbf{b} = [b_1, b_2]^T$ is a constant advection vector; and $\rho = \alpha^2 / \lambda$, where α and λ are the so-called Lamé coefficients, which are assumed to be constant.

Note that problem (77) is of the form (54). Hence, we expect that a standard discretization of (77) yields a matrix having a block structure of the form (55). Indeed, if we discretize (77) by the modified Taylor–Hood Q1isoQ1 FE stable pair [14, p. 167] on a uniform mesh with stepsize $\frac{1}{n}$ in each direction x_i , the computation of the numerical solution reduces to solving a linear system whose coefficient matrix is a two-by-two block matrix given by

$$A_n = \begin{bmatrix} K_n & H_n^T \\ H_n & -\rho M_n \end{bmatrix},$$

where $\mathbf{n} = (n, n)$. Here, the pivot block K_n is a two-by-two block matrix, M_n is the so-called mass matrix, and the blocks $H_n = [H_{n,1}, H_{n,2}]$ and H_n^T correspond to discrete divergence and gradient operators, respectively; note that both H_n and H_n^T are block rectangular matrices. In what follows, we compute the symbol of each block of A_n .

Symbol of $\{K_n\}_n$. The matrix K_n can be symmetric positive definite (if $\mathbf{b} = \mathbf{0}$) or nonsymmetric. Denote $K_n = K_n^{sym} + K_n^{nonsym}$, where K_n^{nonsym} is the discrete advection counterpart and K_n^{sym} is the symmetric matrix corresponding to the discrete second-order operator. Neglecting the boundary effects, the Q1isoQ1 FE discretization leads to

$$(K_n^{sym})_{ij} = T_n(f_{ij}^{sym}), \quad i, j = 1, 2,$$

where $T_n(f_{ij}^{sym})$ is a 2-level Toeplitz matrix generated by f_{ij}^{sym} , with

$$\begin{aligned} f_{11}^{sym}(\theta_1, \theta_2) &= 4 - 2 \cos \theta_1 (1 + \cos \theta_2), \\ f_{12}^{sym}(\theta_1, \theta_2) &= f_{21}^{sym}(\theta_1, \theta_2) = \sin \theta_1 \sin \theta_2, \\ f_{22}^{sym}(\theta_1, \theta_2) &= 4 - 2 \cos \theta_2 (1 + \cos \theta_1); \end{aligned}$$

see [19] for details. Analogously, for K_n^{nonsym} we have

$$f^{nonsym}(\theta_1, \theta_2) = -4i \begin{bmatrix} b_1 \sin \theta_1 (2 + \cos \theta_2) & b_2 \sin \theta_1 (2 + \cos \theta_2) \\ b_1 \sin \theta_2 (2 + \cos \theta_1) & b_2 \sin \theta_2 (2 + \cos \theta_1) \end{bmatrix}.$$

Therefore, by Lemma 3, both K_n^{sym} and K_n^{nonsym} are similar via a proper permutation matrix to 2-level block Toeplitz matrices associated with the 2×2 matrix-valued functions $f^{sym} = [f_{ij}^{sym}]_{i,j=1}^2$ and $f^{nonsym} = [f_{ij}^{nonsym}]_{i,j=1}^2$, respectively. Moreover, by GLT 1,

$$\begin{aligned} \{K_n^{sym}\}_n &\sim_{\sigma, \lambda} f^{sym}(\theta_1, \theta_2), \\ \{K_n^{nonsym}\}_n &\sim_{\sigma} f^{nonsym}(\theta_1, \theta_2). \end{aligned}$$

As a consequence, thanks to GLT 3,

$$\{K_n\}_n \sim_{\sigma} f^K(\theta_1, \theta_2), \quad f^K(\theta_1, \theta_2) = f^{sym}(\theta_1, \theta_2) + f^{nonsym}(\theta_1, \theta_2).$$

If we include the boundary effects, the Q1isoQ1 FE scheme for discretizing the original problem yields a K_n which, up to a permutation, is given by $T_n(f^K) + E_n$, where E_n is a low-rank perturbation. Thus, by Theorem 5, the symbol of $\{K_n\}_n$ remains the same, regardless of the boundary conditions.

Symbol of $\{n^2 M_n\}_n$. A direct check shows that M_n is block-tridiagonal and each block has tridiagonal structure. More precisely, $n^2 M_n$ is a symmetric 2-level Toeplitz matrix associated with the following bivariate scalar function

$$f^M(\theta_1, \theta_2) = 4(2 + \cos \theta_1)(2 + \cos \theta_2),$$

and hence, by GLT 1,

$$\{n^2 M_n\}_n \sim_{\sigma, \lambda} f^M(\theta_1, \theta_2).$$

As for the matrix K_n , also in this case the symbol does not change when we include the boundary conditions.

Symbol of the anti-diagonal part of A_n . Due to the rectangular nature of the blocks H_n, H_n^T , we cannot compute their symbols directly. Our strategy is therefore to embed them into larger matrices and to derive their symbols by applying Theorem 1 on the anti-diagonal part of A_n , which is denoted by J_n in what follows. We begin by rewriting $H_{n,1}, H_{n,2}$ as the *downsampling* of larger square matrices $\tilde{H}_{n,1}, \tilde{H}_{n,2}$, namely

$$H_{n,k} = \tilde{H}_{n,k} C, \quad k = 1, 2,$$

where $\tilde{H}_{n,k}$ are five-diagonal block matrices such that each block is itself five-diagonal, while C is called *cutting* matrix and is a special projection matrix that deletes columns block-wise and within the blocks of $\tilde{H}_{n,k}$. More in detail, $n\tilde{H}_{n,1}, n\tilde{H}_{n,2}$ are 2-level Toeplitz matrices associated with the following bivariate complex-valued functions:

$$\begin{aligned} f^{\tilde{H}_1}(\theta_1, \theta_2) &= -4i\phi(\theta_1)\psi(\theta_2), \\ f^{\tilde{H}_2}(\theta_1, \theta_2) &= -4i\psi(\theta_1)\phi(\theta_2), \end{aligned}$$

where $\phi(\theta) = 2 \sin \theta + \sin(2\theta)$ and $\psi(\theta) = 5 + 6 \cos \theta + \cos(2\theta)$, while for $\tilde{H}_{n,k}$, $k = 1, 2$, with odd matrix-size equal to N , C has the following form

$$C = \hat{C} \otimes \hat{C}, \quad \hat{C} = \begin{bmatrix} 1 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 1 \end{bmatrix}_{N \times \frac{N-1}{2}}$$

and is such that $C^T C = I$. Now, by **GLT 1**,

$$\{n\tilde{H}_{n,k}\}_n \sim_{\sigma} f^{\tilde{H}_k}(\theta_1, \theta_2). \quad (78)$$

Therefore, if we denote by \tilde{J}_n the anti-diagonal two-by-two block matrix with $\tilde{J}_{n,21} = \tilde{H}_n$, $\tilde{J}_{n,12} = \tilde{H}_n^T$, and $\tilde{H}_n = [\tilde{H}_{n,1}, \tilde{H}_{n,2}]$, and we use relation (78) combined with Theorem 5 and **GLT 1**, we obtain

$$\{n\tilde{J}_n\}_n \sim_{\sigma, \lambda} f^{\tilde{J}}(\theta_1, \theta_2), \quad f^{\tilde{J}} = \begin{bmatrix} 0 & 0 & \overline{f^{\tilde{H}_1}(\theta_1, \theta_2)} \\ 0 & 0 & \overline{f^{\tilde{H}_2}(\theta_1, \theta_2)} \\ f^{\tilde{H}_1}(\theta_1, \theta_2) & f^{\tilde{H}_2}(\theta_1, \theta_2) & 0 \end{bmatrix}.$$

As a consequence, by observing that

$$J_n = S^T \tilde{J}_n S, \quad S = \begin{bmatrix} C & O & O \\ O & C & O \\ O & O & I \end{bmatrix},$$

and by applying Theorem 1, we arrive at

$$\{nJ_n\}_n \sim_{\sigma, \lambda} f^{\tilde{J}}(\theta_1, \theta_2).$$

9. Conclusions

We have illustrated through specific examples the application potential of the theory of block GLT sequences and of its multivariate version, thus bringing to completion the purely theoretical work [29]. It should be said, however, that the theory of GLT sequences is still incomplete. In particular, besides filling in the details of the theory of multilevel block GLT sequences,³ it will be necessary to develop the theory of the so-called reduced GLT sequences, as explained in [26, Chapter 11].

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³ The results of Section 5 have been obtained as a combination of the results in [27,29], but formal proofs of them are still missing and will be the subject of a future paper.

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